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A set of adaptive observers for a class of MIMO nonlinear systems

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Abstract— **A globally exponentially convergent adaptive observer is proposed for a class of uniformly observable nonlinear systems in order to jointly estimate missing states and unknown constant parameters. The gain of this observer involves a design function that has to satisfy some mild conditions which are given. Different expressions of such a function are proposed. Of particular interest, it is shown that adaptive high gain like observers and adaptive sliding mode like observers can be derived by considering particular expressions of the design function. Simulation results are presented in order to illustrate the performance of the proposed observer.**

Keywords: Nonlinear system, High gain observer, Sliding mode, Adaptive observer, Persistent excitation.

I. INTRODUCTION

Joint estimation of missing states and unknown constant parameters using adaptive observers has been widely investigated and still constitutes the subject of many control research groups. Early results were related to linear systems and can be found in [12], [11] while more recent results are reported in [20], [19]. Since the eighties, many results on nonlinear systems have became available. For example, adaptive observers have been proposed for a class of nonlinear systems which can be linearized with a change of coordinates up to output injection in [1], [14], [15], [13]. More general results on nonlinear systems have been reported in [16], [4], [2]. The underlying approaches are however not constructive from the applicability point of view. In a relatively recent work [18], the authors proposed an adaptive observer for a class of single output uniformly observable nonlinear systems. This consists in an extension of the approach initially proposed in [19] for MIMO linear time-varying systems. The main advantage of this approach lies in both design and implementation simplicities.

In this paper, one proposes to extend this approach to a large class of uniformly observable nonlinear systems. To this end, one shall firstly consider the same class of systems as in [10] for which the authors proposed a high gain observer for the state estimation. Then, one shall combine the approach adopted in [10] with those proposed in [20] and [19] in order to design an adaptive nonlinear observer. The main characteristics of the proposed observer lie in its simplicity and its capability to give rise to different observers among which adaptive high gain like observers and adaptive sliding mode like observers. Indeed, the gain of the state estimation as well as that of the parameter adaptation involve a design function that has to satisfy some mild conditions which are given. Different expressions of the design function are proposed and it is shown that adaptive high gain like observers [3], [9], [10], [7] and adaptive sliding mode like observers [17], [5], [6], [8] can be derived by considering particular expressions of the design function. Of particular interest, the tuning of the observer is achieved through the choice of a single parameter.

This paper is organized as follows. The next section is devoted to the problem formulation. In section 3, the observer design will be detailed and the observer equations are given. Section 4 is devoted to a full convergence analysis. In section 5, different expressions of the design function are given to emphasize the versatility of the proposed adaptive observer. The performances of the adaptive observer are highlight in section 6 through an illustrative example.

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II. PROBLEM FORMULATION

Consider MIMO systems of the form:

$$\begin{cases} \dot{x} = f(u,x) + \Psi(u,x)\rho \\ y = f^0(u,x^1) \end{cases} \tag{1}$$

$$x = \begin{pmatrix} x^1 \\ x^2 \\ \vdots \\ x^q \end{pmatrix}; f(u,x) = \begin{pmatrix} f^1(u,x^1,x^2) \\ f^2(u,x^1,x^2,x^3) \\ \vdots \\ f^{q-1}(u,x) \\ f^q(u,x) \end{pmatrix}$$

$$\rho = \begin{pmatrix} \rho_1 \\ \rho_2 \\ \vdots \\ \rho_m \end{pmatrix}; \Psi^T(u,x) = \begin{pmatrix} \Psi_1^T(u,x) \\ \Psi_2^T(u,x) \\ \vdots \\ \Psi_m^T(u,x) \end{pmatrix};$$

$$\Psi_j(u,x) = \begin{pmatrix} \Psi_j^1(u,x^1) \\ \Psi_j^2(u,x^1,x^2) \\ \vdots \\ \Psi_j^{q-1}(u,x^1,\ldots,x^{q-1}) \\ \Psi_j^q(u,x) \end{pmatrix}; \tag{2}$$

the output $y = f^0(u,x^1) \in \mathbb{R}^p$; the state $x \in \mathbb{R}^n$ with $x^k \in \mathbb{R}^{n_k}$, $k = 1,\ldots,q$ and $p = n_0 \geq n_1 \geq n_2 \geq \ldots \geq n_q$, $\sum_{k=1}^{q} n_k = n$; the input $u(t) \in \mathcal{U}$ the set of bounded absolutely continuous functions with bounded derivatives from $\mathbb{R}^+$ into U a compact subset of $\mathbb{R}^s$; $f(u,x) \in \mathbb{R}^n$ with $f^k(u,x) \in \mathbb{R}^{n_k}$; $\rho \in \mathbb{R}^m$ is a vector of unknown constant parameters,

$\rho_i \in \mathbb{R}$, $i = 1, \ldots, m$; $\Psi(u,x)$ is a $n \times m$ matrix and each $\Psi_j(u,x) \in \mathbb{R}^n$, $j = 1, \ldots, m$, denotes its j^{th} column with $\Psi_j^k(u,x) \in \mathbb{R}^{n_k}$, $k = 1, \ldots, q$. Our objective consists in designing adaptive observers to simultaneously estimate the state and the unknown parameters. Such a design requires some assumptions which will be stated in due courses. At this step, one assumes the following:

(A1) For $0 \le k \le q-1$; the function $x^{k+1} \mapsto f^k(u, x^1, \ldots, x^k, x^{k+1})$ is one to one from $\mathbb{R}^{n_{k+1}}$ into $\mathbb{R}^{n_k}$. Moreover, $\exists \alpha, \beta > 0$ such that $\forall k \in \{0, \ldots, q-1\}$, $\forall x \in \mathbb{R}^n$ and $\forall u \in U$, one has:

$$0 < \alpha^2 I_{n_{k+1}} \le \left(\frac{\partial f^k}{\partial x^{k+1}}(u,x)\right)^T \frac{\partial f^k}{\partial x^{k+1}}(u,x) \le \beta^2 I_{n_{k+1}}$$

(A2) The matrix $\Psi(u(t), x(t))$ is uniformly bounded.

In the case where the vector ρ is known, it has been shown [10] that system (1) under Assumption (A1) characterizes a subclass of U-uniformly observable systems. Moreover, a high gain observer has been proposed for system (1) under some lipschitz conditions commonly used in the high gain observer synthesis. In the sequel, one shall combine the observer design approach proposed in [10], with those proposed in [18] and [20], in order to propose an adaptive observer for system (1). This observer allows to jointly estimate the missing states as well as the unknown parameters. A main characteristic of the proposed observer lies in the structure of its gain which involves a design function that has to satisfy some mild conditions given in the sequel. One shall exhibit many expressions that satisfy these conditions and shall show that some of these expressions give rise to adaptive versions of high gain and sliding mode like observers.

III. Observer design

One shall firstly introduce a state transformation that puts system (1) under an appropriate form for the observer synthesis. Then, the equations of the proposed observer will be derived in the new coordinates before being given in the original ones.

A. State transformation

Consider the following change of coordinates:

$$\Phi : \mathbb{R}^n \longrightarrow \mathbb{R}^{n_0 q}$$
$$x = \begin{pmatrix} x^1 \\ x^2 \\ \vdots \\ x^q \end{pmatrix} \mapsto z = \begin{pmatrix} z^1 \\ z^2 \\ \vdots \\ z^q \end{pmatrix} = \Phi(u,x) =$$
$$\begin{pmatrix} f^0(u,x^1) \\ \frac{\partial f^0}{\partial x^1}(u,x^1) f^1(u,x^1,x^2) \\ \frac{\partial f^0}{\partial x^1}(u,x^1)\frac{\partial f^1}{\partial x^2}(u,x^1,x^2) f^2(u,x^1,x^2,x^3) \\ \vdots \\ \left(\prod_{k=0}^{q-2} \frac{\partial f^k}{\partial x^{k+1}}(u,x)\right) f^{q-1}(u,x) \end{pmatrix}$$

where $z^k \in \mathbb{R}^{n_0}$, $k = 1, \ldots, q$. According to Assumption (A1), the map Φ is one to one. Let Φ^c denote its converse. Before deriving the dynamics of z, let us introduce the following notations :

• $\Lambda(u,x)$ is the diagonal matrix:

$$\Lambda(u,x) = diag\left(\frac{\partial f^0}{\partial x^1}(u,x), \frac{\partial f^0}{\partial x^1}(u,x)\frac{\partial f^1}{\partial x^2}(u,x), \ldots, \prod_{k=0}^{q-1} \frac{\partial f^k}{\partial x^{k+1}}(u,x)\right) \quad (3)$$

According to Assumption (A1), $\Lambda(u,x)$ is left invertible. Let $\Lambda^+(u,x)$ denotes its left inverse, one can easily check that:

$$\Lambda(u,x)f(u,x) = Az + G(u,x) \text{ or equivalently}$$
$$f(u,x) = \Lambda^+(u,x)Az + \Lambda^+(u,x)G(u,x) \quad (4)$$

where the $n_0 q \times n_0 q$ square matrix A and $G(u,x) \in \mathbb{R}^{n_0 q}$ are respectively given by:

$$A = \begin{bmatrix} 0 & I_{n_0 q} \\ 0 & 0 \end{bmatrix} \quad (5)$$

and $G(u,x) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \left(\prod_{k=0}^{q-1} \frac{\partial f^k}{\partial x^{k+1}}(u,x)\right) f^q(u,x) \end{pmatrix}$.

The dynamics of z can be derived as follows:

$$\begin{aligned} \dot z(t) &= \frac{\partial \Phi}{\partial x}(u,x)\dot x(t) + \frac{\partial \Phi}{\partial u}(u,x)\dot u(t) \\ &= \frac{\partial \Phi}{\partial x}(u,x)\left(f(u,x) + \Psi(u,x)\rho\right) + \frac{\partial \Phi}{\partial u}(u,x)\dot u(t) \\ &= Az + G(u,x) + \Lambda(u,x)\Psi(u,x)\rho \\ &+ \left(\frac{\partial \Phi}{\partial x}(u,x) - \Lambda(u,x)\right) \\ &\quad \left(\Lambda^+(u,x)Az + \Lambda^+(u,x)G(u,x) + \Psi(u,x)\rho\right) \\ &+ \frac{\partial \Phi}{\partial u}(u,x)\dot u(t) \end{aligned} \quad (6)$$

according to (4).

For sake of clarity, one shall use the following notations in the sequel:

$$\begin{aligned} \Theta(u,z) &\triangleq \left(\frac{\partial \Phi}{\partial x}(u,x) - \Lambda(u,x)\right)\Lambda^+(u,x) \\ Q(u,z) &\triangleq \Lambda(u,x)\Psi(u,x) \\ \varphi(u,z) &\triangleq \Theta(u,z)Az + G(u,x) \\ \psi(u,\dot u,\rho,z) &\triangleq \varphi(u,z) + \Pi(u,z)\rho \\ &+ \frac{\partial \Phi}{\partial u}(u,\Phi^c(z))\dot u(t) \end{aligned} \quad (7)$$

From the fact that Θ is diagonal with zeros on its main diagonal, one can easily show that $\varphi(u,z)$ and each column of the matrix $\Pi(u,z)$ have triangular structures with respect to z. This property is then also valid for $\psi(u,\dot u,\rho,z)$.

Using the adopted notations, system (1) can be written in the new coordinates z as follows:

$$\left\{ \begin{array}{l} \dot{z} = Az + Q(u,z)\rho + \psi(u,\dot{u},\rho,z) \\ y = Cz = z^1 \end{array} \right. \quad (8)$$

where

$$C = [I_{n_0}, 0_{n_0}, \ldots, 0_{n_0}] \quad (9)$$

is a $n_0 \times n_0 q$ matrix.

B. Observer synthesis

Before giving our candidate observer, one introduces the following notations.
1) let Δ_θ be the block diagonal matrix defined by:

$$\Delta_\theta = diag\left[I_{n_0}, \frac{1}{\theta}I_{n_0}, \ldots, \frac{1}{\theta^{q-1}}I_{n_0}\right] \quad (10)$$

where $\theta > 0$ is a real number.
2) Let Ω_θ be a the following $m \times m$ diagonal matrix:

$$\Omega_\theta = diag\left[1, \frac{1}{\theta^{\nu_1}}, \ldots, \frac{1}{\theta^{\nu_{m-1}}}\right] \quad (11)$$

where the ν_k's, $k = 1, \ldots, m-1$ are positive integers which are chosen such that each term of the matrix $\Delta_\theta Q(u,z)\Omega_\theta^{-1} \triangleq \Delta_\theta \Lambda(u,\Phi^c(z))\Psi(u,\Phi^c(z))\Omega_\theta^{-1}$ is a polynomial in $\frac{1}{\theta}$ (see e.g. [18]).
3) Let S be the unique solution of the algebraic Lyapunov equation :

$$S + A^T S + SA - C^T C = 0 \quad (12)$$

where A and C are respectively given by equations (5) and (9). One can show that S is symmetric positive definite and in particular one has : $S^{-1}C^T = \begin{bmatrix} C_q^1 I_{n_1} \\ C_q^2 I_{n_1} \\ \vdots \\ C_q^q I_{n_1} \end{bmatrix}$.
4) For any $\xi \in \mathbb{R}^{n_0 q}$, Let $\Upsilon_\xi(t)$ be a $n_0 q \times m$ matrix satisfying the following Ordinary Differential Equation (ODE):

$$\begin{array}{rcl} \dot{\Upsilon}_\xi(t) & = & \theta\left((A - S^{-1}C^T C)\right)\Upsilon_\xi(t) \\ & & +\theta\Delta_\theta Q(u(t),\xi(t))\Omega_\theta^{-1} \end{array} \quad (13)$$

5) Let $P(t)$ be the $m \times m$ symmetric matrix governed by the following differential equation:

$$\dot{P}(t) = -\theta P(t)\Upsilon_{\hat{z}}^T(t)C^T C\Upsilon_{\hat{\xi}}(t)P(t) + \theta P(t) \quad (14)$$

where $P(t_0) \in \mathbb{R}^m \times \mathbb{R}^m$ is chosen symmetric positive definite and the matrix $\Upsilon_{\hat{\xi}}(t)$ is governed by (13).

6) $\forall \xi = \begin{pmatrix} \xi^1 \\ \vdots \\ \xi^q \end{pmatrix} \in \mathbb{R}^{n_0 q}$ with $\xi^k \in \mathbb{R}^{n_0}$, $k = 1, \ldots, q$, set $\bar{\xi} = \Delta_\theta \xi$ and let $K(\xi) \triangleq K(\xi^1) \in \mathbb{R}^{n_0 q}$ be a vector of smooth functions satisfying:

$$\begin{array}{rcl} \forall \xi \in \mathbb{R}^{n_0 q} : \bar{\xi}^T K(\xi) & \geq & \frac{1}{2}\xi^T C^T C \xi \quad (15) \\ \exists \sigma > 0; \forall \xi \in \mathbb{R}^{n_0 q} : \|K(\xi^1)\| & \leq & \sigma\|\xi^1\| \quad (16) \end{array}$$

where the matrices Δ_θ and C are respectively given by (10) and (9).

The observer synthesis needs the following additional assumptions :
(A3) The functions $\Phi(u,\Phi^c(z))$, $Q(u,z)$, $\varphi(u,z)$, $\Pi(u,z)$ and $\frac{\partial \Phi}{\partial u}(u,\Phi^c(z))$ are globally Lipschitz with respect to z uniformly in u.

(A4) For any $\xi \in \mathbb{R}^{n_0 q}$, the matrix $C\Upsilon_\xi(t)$ is persistently exciting.

Notice that under Assumptions Assumptions (A1) and (A3), $Q(u,\xi)$ is bounded. Since the matrix $(A - S^{-1}C^T C)$ is Hurwitz and each term of the matrix $\Delta_\theta Q(u(.),\xi(.))\Omega_\theta^{-1}$ is polynomial in $1/\theta$, one can conclude that for $\theta \geq 1$ the matrix $\Upsilon_\xi(t)$ is bounded and corresponding bounds do not depend on θ. Furthermore, one can show that under assumption (A4), the matrix $P(t)$ governed by (14) is symmetric positive definite and that it is bounded from above and from below and corresponding bounds do not depend on θ.

A candidate adaptive observer for system (8) is:

$$\begin{array}{rcl} \dot{\hat{z}}(t) & = & A\hat{z} + Q(u,\hat{z})\hat{\rho}(t) + \psi(u,\dot{u},\hat{\rho},\hat{z}) \\ & - & \theta\Delta_\theta^{-1}\left(S^{-1} + \Upsilon_{\hat{z}}(t)P(t)\Upsilon_{\hat{z}}^T(t)\right)K(\tilde{z}^1) \\ & - & \frac{\partial \Phi}{\partial x}(u,\Phi^c(\hat{z})) \\ & & \left(\Lambda^+(u,\Phi^c(\hat{z})) - \left(\frac{\partial \Phi}{\partial x}(u,\Phi^c(\hat{z}))\right)^+\right) \\ & & \theta\Delta_\theta^{-1}\left(S^{-1} + \Upsilon_{\hat{z}}(t)P(t)\Upsilon_{\hat{z}}^T(t)\right)K(\tilde{z}^1) \\ \dot{\hat{\rho}}(t) & = & -\theta^2\Omega_\theta^{-1}P(t)\Upsilon_{\hat{z}}^T(t)K(\tilde{z}^1) \end{array} \quad (17)$$

where $\hat{z} = \begin{bmatrix} \hat{z}^1 \\ \hat{z}^2 \\ \vdots \\ \hat{z}^q \end{bmatrix} \in \mathbb{R}^n$ with $\hat{z}^k \in \mathbb{R}^{n_0}$, $k = 1, \ldots, q$; $\tilde{z} = \hat{z} - z$ where z is the unknown trajectory of system (8); $\hat{\rho} = \begin{bmatrix} \hat{\rho}_1 \\ \hat{\rho}_2 \\ \vdots \\ \hat{\rho}_m \end{bmatrix} \in \mathbb{R}^m$; S, C, Δ_θ, Ω_θ and $\Lambda(u,\Phi^c(\hat{z}))$ are respectively given by (12), (9), (10), (11) and (3); the matrices $\Upsilon_{\hat{z}}(t)$ and $P(t)$ are respectively governed by equation (13) (with $\xi \triangleq \hat{z}$) and equation (14); $K(\tilde{z}^1)$ is a rectangular matrix satisfying conditions (15) and (16); u and y are respectively the input and the output of system (8) and $\theta > 0$ is a real number.

Indeed, one states the following :

Theorem *3.1:* Assume that system (8) satisfies Assumptions (A1) to (A4). Then, system (17) is a globally exponentially convergent adaptive observer for system (8).
The proof of this theorem is detailed in section IV.

C. Observers equations in the original coordinates

Proceeding as in [7], one can easily show that observer (17) can be written in the original coordinates as follows:

$$
\begin{array}{rcl}
\dot{\hat{x}}(t) & = & f(u,\hat{x}) + \Psi(u,\hat{x})\hat{\rho}(t) - \theta\Lambda^{+}(u,\hat{x})\Delta_\theta^{-1} \\
& & \left(S^{-1} + \Upsilon_{\hat{x}}(t)P(t)\Upsilon_{\hat{x}}^T(t)\right)K(\tilde{y}) \\
\dot{\hat{\rho}}(t) & = & -\theta^2\Omega_\theta^{-1}P(t)\Upsilon_{\hat{x}}^T(t)K(\tilde{y}) \\
\dot{\Upsilon}_{\hat{x}}(t) & = & \theta(A - S^{-1}C^TC)\Upsilon_{\hat{x}}(t) \\
& + & \theta\Delta_\theta\Lambda(u,\hat{x})\Psi(u,\hat{x})\Omega_\theta^{-1} \\
\dot{P}(t) & = & -\theta\left(P(t)\Upsilon_{\hat{x}}^T(t)C^TC\Upsilon_{\hat{x}}(t)P(t) - P(t)\right)
\end{array} \quad (18)
$$

where $\hat{x} = \begin{pmatrix} \hat{x}^1 \\ \hat{x}^2 \\ \vdots \\ \hat{x}^q \end{pmatrix} \in \mathbb{R}^n$ with $\hat{x}^k \in \mathbb{R}^{n_k}$, $k = 1, \ldots, q$; $\hat{\rho} = \begin{pmatrix} \hat{\rho}_1 \\ \hat{\rho}_2 \\ \vdots \\ \hat{\rho}_m \end{pmatrix} \in \mathbb{R}^m$ with $\hat{\rho}_i \in \mathbb{R}$, $i = 1, \ldots, m$; $\Upsilon_{\hat{x}}(t) \triangleq \Upsilon_{\hat{z}}(t)$; $\tilde{y} = f^0(u,\hat{x}^1) - f^0(u,x)$ where x is the unknown trajectory of system (1) and the other variables keep the same meaning as before.

IV. Convergence analysis

Set $\tilde{z}(t) = \hat{z}(t) - z(t)$ and $\tilde{\rho}(t) = \hat{\rho}(t) - \rho$. Then,

$$
\begin{array}{rcl}
\dot{\tilde{z}} & = & A\tilde{z} - \theta\Delta_\theta^{-1}\left(S^{-1} + \Upsilon_{\hat{z}}(t)P(t)\Upsilon_{\hat{z}}^T(t)\right)K(\tilde{z}^1) \\
& + & Q(u,\hat{z})\tilde{\rho}(t) + (Q(u,\hat{z}) - Q(u,z))\rho \\
& + & \psi(u,\dot{u},\hat{\rho},\hat{z}) - \psi(u,\dot{u},\rho,z) \\
& - & \theta\Gamma(u,\hat{z})\Delta_\theta^{-1}(S^{-1} + \Upsilon_{\hat{z}}(t)P(t)\Upsilon_{\hat{z}}^T(t))K(\tilde{z}^1)
\end{array} \quad (19)
$$

$$
\dot{\tilde{\rho}} = -\theta^2\Omega_\theta^{-1}P(t)\Upsilon_{\hat{z}}^T(t)K(\tilde{z}^1) \quad (20)
$$

where $\Gamma(u,\hat{z}) = \dfrac{\partial\Phi}{\partial x}(u,\Phi^c(\hat{z}))\left(\Lambda^{+}(u,\Phi^c(\hat{z})) - \left(\dfrac{\partial\Phi}{\partial x}(u,\Phi^c(\hat{z}))\right)^{+}\right)$.

Notice that using Assumption (A1) and (A3), one can easily deduce that $\Gamma(u,\hat{z})$ is bounded. Furthermore, one can easily check the following identities: $\Delta_\theta A\Delta_\theta^{-1} = \theta A$ and $C\Delta_\theta = C$.

Set $\bar{z} = \Delta_\theta\tilde{z}$ and $\bar{\rho} = \dfrac{1}{\theta}\Omega_\theta\tilde{\rho}$, one obtains:

$$
\begin{array}{rcl}
\dot{\bar{z}} & = & \theta A\bar{z} - \theta\left(S^{-1} + \Upsilon_{\hat{z}}(t)P(t)\Upsilon_{\hat{z}}^T(t)\right)K(\tilde{z}^1) \\
& + & \theta\Delta_\theta Q(u,\hat{z})\Omega_\theta^{-1}\bar{\rho}(t) + \Delta_\theta(Q(u,\hat{z}) - Q(u,z))\rho \\
& + & \Delta_\theta\left(\psi(u,\dot{u},\hat{\rho},\hat{z}) - \psi(u,\dot{u},\rho,z)\right) \\
& - & \theta\Delta_\theta\Gamma(u,\hat{z})\Delta_\theta^{-1}\left(S^{-1} + \Upsilon_{\hat{z}}(t)P(t)\Upsilon_{\hat{z}}^T(t)\right)K(\tilde{z}^1)
\end{array} \quad (21)
$$

$$
\dot{\bar{\rho}}(t) = -\theta P(t)\Upsilon_{\hat{z}}^T(t)K(\tilde{z}^1) \quad (22)
$$

Substituting (22) in (21), one obtains:

$$
\begin{array}{rcl}
\dot{\bar{z}} & = & \theta A\bar{z} - \theta S^{-1}K(\tilde{z}^1) + \Upsilon_{\hat{z}}(t)\dot{\bar{\rho}}(t) \\
& + & \theta\Delta_\theta Q(u,\hat{z})\Omega_\theta^{-1}\bar{\rho}(t) + \Delta_\theta(Q(u,\hat{z}) - Q(u,z))\rho \\
& + & \Delta_\theta\left(\psi(u,\dot{u},\hat{\rho},\hat{z}) - \psi(u,\dot{u},\rho,z)\right) \\
& - & \theta\Delta_\theta\Gamma(u,\hat{z})\Delta_\theta^{-1} \\
& & \left(S^{-1} + \Upsilon_{\hat{z}}(t)P(t)\Upsilon_{\hat{z}}^T(t)\right)K(\tilde{z}^1)
\end{array} \quad (23)
$$

Now, define : $\eta(t) = \bar{z}(t) - \Upsilon_{\hat{z}}(t)\bar{\rho}(t)$ where the matrix $\Upsilon_{\hat{z}}(t) \in \mathbb{R}^{n_0 q\times m}$ is governed by equation (13) with $\xi \triangleq \hat{z}$. One can show that:

$$
\begin{array}{rcl}
\dot{\eta}(t) & = & \theta A\eta + \theta S^{-1}C^TC\Upsilon_{\hat{z}}(t)\bar{\rho} - \theta S^{-1}K(\tilde{z}^1) \\
& + & \Delta_\theta(Q(u,\hat{z}) - Q(u,z))\rho \\
& + & \Delta_\theta\left(\psi(u,\dot{u},\hat{\rho},\hat{z}) - \psi(u,\dot{u},\rho,z)\right) \\
& - & \theta\Delta_\theta\Gamma(u,\hat{z})\Delta_\theta^{-1} \\
& & \left(S^{-1} + \Upsilon_{\hat{z}}(t)P(t)\Upsilon_{\hat{z}}^T(t)\right)K(\tilde{z}^1)
\end{array} \quad (24)
$$

Set $V_1(\eta(t)) = \eta^T(t)S\eta(t)$, $V_2(\bar{\rho}(t)) = \bar{\rho}^T(t)P^{-1}(t)\bar{\rho}(t)$ where $P(t)$ is given by (14) and let $V(\eta(t),\bar{\rho}(t)) = V_1(\eta(t)) + V_2(\bar{\rho}(t))$ be the Lyapunov candidate function. One has:

$$
\begin{array}{rcl}
\dot{V}_1(t) & = & 2\theta\eta^TSA\eta + 2\theta\eta^TC^TC\Upsilon_{\hat{z}}\bar{\rho} - 2\theta\eta^TK(\tilde{z}^1) \\
& + & 2\eta^TS\Delta_\theta(Q(u,\hat{z}) - Q(u,z))\rho \\
& + & 2\eta^TS\Delta_\theta\left(\psi(u,\dot{u},\hat{\rho},\hat{z}) - \psi(u,\dot{u},\rho,z)\right) \\
& - & 2\eta^TS\Delta_\theta\Gamma(u,\hat{z})\Delta_\theta^{-1} \\
& & \left(\theta S^{-1}C^TC\bar{z} + \Upsilon_{\hat{z}}(t)P(t)\Upsilon_{\hat{z}}^T(t)K(\tilde{z}^1)\right)
\end{array} \quad (25)
$$

According to (15), one obtains:

$$
\begin{array}{rcl}
\eta^TK(\tilde{z}^1) & = & (\eta + \Upsilon_{\hat{z}}\bar{\rho})^TK(\tilde{z}^1) - (\Upsilon_{\hat{z}}\bar{\rho})^TK(\tilde{z}^1) \\
& = & (\Delta_\theta\tilde{z})^TK(\tilde{z}^1) - (\Upsilon_{\hat{z}}\bar{\rho})^TK(\tilde{z}^1) \\
& \geq & \frac{1}{2}\tilde{z}^TC^TC\tilde{z} - (\Upsilon_{\hat{z}}\bar{\rho})^TK(\tilde{z}^1) \\
& \geq & \frac{1}{2}\bar{z}^TC^TC\bar{z} - (\Upsilon_{\hat{z}}\bar{\rho})^TK(\tilde{z}^1) \\
& \geq & \frac{1}{2}\left(\eta^TC^TC\eta + (\Upsilon_{\hat{z}}\bar{\rho})^TC^TC\Upsilon_{\hat{z}}\bar{\rho}\right) \\
& + & \eta^TC^TC\Upsilon_{\hat{z}}\bar{\rho} - (\Upsilon_{\hat{z}}\bar{\rho})^TK(\tilde{z}^1)
\end{array} \quad (26)
$$

Using (12) and (26), inequation (25) becomes:

$$
\begin{array}{rcl}
\dot{V}_1(t) & = & -\theta V_1 - \theta\left(\Upsilon_{\hat{z}}\bar{\rho}\right)^TC^TC\Upsilon_{\hat{z}}\bar{\rho} + 2\theta\left(\Upsilon_{\hat{z}}\bar{\rho}\right)^TK(\tilde{z}^1) \\
& + & 2\|\eta\|\|S\|\left(\|\Delta_\theta(Q(u,\hat{z}) - Q(u,z))\rho\|\right. \\
& + & \left.\|\Delta_\theta\left(\psi(u,\dot{u},\hat{\rho},\hat{z}) - \psi(u,\dot{u},\rho,z)\|\right)\right) \\
& + & 2\|\eta\|\|S\|\|\theta\Delta_\theta\Gamma(u,\hat{z})\Delta_\theta^{-1}\| \\
& & \left(\|S^{-1}\| + \|\Upsilon_{\hat{z}}(t)P(t)\Upsilon_{\hat{z}}^T(t)\|\right)\|K(\tilde{z}^1)\|
\end{array} \quad (27)
$$

As the matrix $\Gamma(u,\hat{z})$ is lower triangular with zeros on the main diagonal and is bounded, one can easily show that $\theta\Delta_\theta\Gamma(u,\hat{z})\Delta_\theta^{-1}$ is bounded for $\theta \geq 1$ and the corresponding upper bound does not depend on θ. Similarly, it is obvious that $\Upsilon_{\hat{z}}(t)P(t)\Upsilon_{\hat{z}}^T(t)$ is bounded since $\Upsilon(t)$ and $P(t)$ are so. According to the Lipschitz condition on $Q(u,z)$ and since each term of this matrix has a triangular structure, one can easily show that:

$$\|\Delta_\theta(Q(u,\hat{z})-Q(u,z))\rho\| \leq k_1\|\rho\|\|\bar{z}\| \leq c_1\|\eta\|+c_2\|\bar{\rho}\| \tag{28}$$

for some constants k_1, c_1 and c_2 which do not depend on θ. Let us now examine the boundedness of $\|\Delta_\theta\left(\psi(u,\dot{u},\hat{\rho},\hat{z})-\psi(u,\dot{u},\rho,z)\right)\|$. To this end, one shall consider all the terms which intervene in the expression of (7). Indeed, according to the triangular structures of φ and $\frac{\partial\Phi}{\partial u}$ and to the Lipschitz condition on these functions, one has:

$$\|\Delta_\theta(\varphi(u,\hat{z})-\varphi(u,z))\rho\| \leq k_2\|\bar{z}\| \leq \gamma_3\|\eta\|+\gamma_4\|\bar{\rho}\| \tag{29}$$

$$\begin{aligned}&\|\Delta_\theta\left(\frac{\partial\Phi}{\partial u}(u,\Phi^c(\hat{z}))-\frac{\partial\Phi}{\partial u}(u,\Phi^c(\hat{z}))\right)\dot{u}(t)\| \leq \\ &k_3\|\dot{u}(t)\|\|\bar{z}\| \leq \gamma_5\|\eta\|+\gamma_6\|\bar{\rho}\|\end{aligned} \tag{30}$$

where k_2, k_3, γ_i, $i=2\ldots,6$ are positive constants which do not depend on θ. Furthermore, one has:

$$\begin{aligned}&\|\Delta_\theta(\Pi(u,\hat{z})\hat{\rho}-\Pi(u,z)\rho)\| = \\ &\|\Delta_\theta\Pi(u,\hat{z})\tilde{\rho}+\Delta_\theta(\Pi(u,\hat{z})-\Pi(u,z))\rho\| \\ \leq\ &\|\theta\Delta_\theta\Pi(u,\hat{z})\Omega_\theta^{-1}\bar{\rho}\|+\|\Delta_\theta(\Pi(u,\hat{z})-\Pi(u,z))\rho\| \\ \leq\ &\|\theta\Delta_\theta\Pi(u,\hat{z})\Omega_\theta^{-1}\bar{\rho}\|+k_4\bar{z}\end{aligned} \tag{31}$$

for some positive real $k_4>0$, according to the lipschitz assumption on Π. Moreover, one has:

$$\begin{aligned}\theta\Delta_\theta\Pi(u,\hat{z})\Omega_\theta^{-1} &= \theta\Delta_\theta\Theta(u,\hat{z})Q(u,\hat{z})\Omega_\theta^{-1} \\ &= \left(\theta\Delta_\theta\Theta(u,\hat{z})\Delta_\theta^{-1}\right)\left(\Delta_\theta Q(u,\hat{z})\Omega_\theta^{-1}\right)\end{aligned}$$

On one hand, the matrix $\Theta(u,\hat{z})$ is lower triangular with zeros on its main diagonal and it is bounded. On the other hand, each term of the matrix $\Delta_\theta Q(u,\hat{z})\Omega_\theta^{-1}$ is a polynomial in $1/\theta$. It follows that the matrices $\theta\Delta_\theta\Theta(u,\hat{z})\Delta_\theta^{-1}$ and $\Delta_\theta Q(u,\hat{z})\Omega_\theta^{-1}$ are bounded and corresponding bounds do not depend on θ for $\theta \geq 1$. As a result, inequality (31) becomes:

$$\begin{aligned}\|\Delta_\theta(\Pi(u,\hat{z})\hat{\rho}-\Pi(u,z)\rho)\| &\leq k_5\|\bar{\rho}\|+k_4\bar{z} \\ &\leq \gamma_7\|\eta\|+\gamma_8\|\bar{\rho}\|\end{aligned} \tag{32}$$

for some reals $k_4,k_5,\gamma_7,\gamma_8>0$ which do not depend on θ. Finally, it is obvious that $\|\bar{z}\| \leq \|\eta\|+\|\Upsilon(t)\|\|\bar{\rho}\|$.

According to the previous developments, inequality (27) can be written as follows:

$$\begin{aligned}\dot{V}_1(t) &\leq -\theta V_1-\theta\left(\Upsilon_{\hat{z}}\bar{\rho}\right)^T C^TC\Upsilon_{\hat{z}}\bar{\rho}+2\theta\left(\Upsilon_{\hat{z}}\bar{\rho}\right)^T K(\tilde{z}^1) \\ &+ c_1\|\eta\|^2+c_2\|\eta\|\|\bar{\rho}\| \\ &\leq -\theta V_1-\theta\left(\Upsilon_{\hat{z}}\bar{\rho}\right)^T C^TC\Upsilon_{\hat{z}}\bar{\rho}+2\theta\left(\Upsilon_{\hat{z}}\bar{\rho}\right)^T K(\tilde{z}^1) \\ &+ k_1V_1+k_2\sqrt{V}_1\sqrt{V}_2\end{aligned} \tag{33}$$

for some constant real number $c_1,c_2,k_1,k_2>0$ which do not depend on θ.
Otherwise, one can show that:

$$\dot{V}_2(t) = -\theta V_2-2\theta\bar{\rho}^T\ \Upsilon_{\hat{z}}^T K(\tilde{z}^1)+\theta\bar{\rho}^T\Upsilon_{\hat{z}}^T(t)C^TC\Upsilon_{\hat{z}}(t)\bar{\rho}$$

Hence, $\dot{V}(t) \leq -(\theta-k_1)V_1-\theta V_2+k_2\sqrt{V}_1\sqrt{V}_2$.
Finally, set $V_1^\star=(\theta-k_1)V_1$ and $V_2^\star=\theta V_2$, one has

$$\begin{aligned}\dot{V}(t) &\leq -(V_1^\star+V_2^\star)+\frac{k_2}{\sqrt{\theta(\theta-k_1)}}(V_1^\star+V_2^\star) \\ &= -(1-\frac{k_2}{\sqrt{\theta(\theta-k_1)}})(V_1^\star+V_2^\star)\end{aligned} \tag{34}$$

Now, choose θ such that $(1-\frac{k_2}{\sqrt{\theta(\theta-k_1)}})>0$, one can show that: $\dot{V}(t) \leq -(\theta-k_1-\frac{k_2}{\sqrt{\theta}})V(t)$. This ends the proof.

V. Some particular observers

Some particular expressions of the vector $K(\tilde{x}^1)$ that satisfy conditions (15) and (16) shall be given and discussed in this section. These expressions will be given in the new coordinates z in order to easily check conditions (15) and (16) as well as in the original coordinates x in order to easily recognize the structure of the resulting observers.

A. Adaptive high gain observers

Consider the following expression of $K(\tilde{z}^1)$:

$$\begin{aligned}K_{HG}(\tilde{z}^1) &= kC^TC\tilde{z}=kC^T\tilde{z}^1=kC^T\tilde{y} \\ &= kC^T(f^0(u,\hat{x}^1)-f^0(u,x^1))\end{aligned} \tag{35}$$

where $k \geq \frac{1}{2}$ is a real number. One can easily check that expression (35) satisfies conditions (15) and (16). Notice that when no parameter needs to be estimate, it is easy to see that the so obtained observer is of the high gain variety (see e.g. [9], [10], [7]). More specifically, the proposed observer with $K(\tilde{z}^1)$ specialized as in (35) is in fact an adaptive version of the well known high gain state observer.

B. Adaptive sliding mode like observers

At first glance, the following vector seems to be a potential candidate for the expression of $K(\tilde{z}^1)$:

$$\begin{aligned}K(\tilde{z}^1) &= kC^TCsign(\tilde{z})=kC^Tsign(\tilde{z}^1) \\ &= kC^Tsign(f^0(u,\hat{x}^1)-f^0(u,x^1))\end{aligned} \tag{36}$$

where $k>0$ is a real number and 'sign' is the usual sign function. Indeed, condition (15) is trivially satisfied by (36). Similarly, for bounded input bounded output systems, condition (16) is also satisfied for relatively high values of k. However, expression (36) cannot be used due the discontinuity of sign function. Indeed, such discontinuity hampers the applicability of the Lyapunov approach used throughout the proof. In order to overcome this difficulty, one shall use continuous functions which have similar properties that those of the sign function. Of practical importance, these functions are widely used when implementing sliding mode observers. Indeed, consider the following function:

$$K_{Tanh}(\tilde{z})=kC^TTanh(f^0(u,\hat{x}^1)-f^0(u,x^1)) \tag{37}$$

where $Tanh$ denotes the hyperbolic tangent function and $k>0$ is a real number. It is easy to see that conditions (15) and (16) are satisfied for relatively high values of k.

Similarly to the hyperbolic tangent function, one can easily check that the inverse tangent function $K_{ArcTan}(\tilde{y})$, the hyperbolic sine function $K_{Sinh}(\tilde{y})$, etc., also constitute valid expressions for $K(\tilde{z}^1)$. Besides, one can consider new valid expressions for $K(\tilde{z}^1)$, for example by adding $K_{Tanh}(\tilde{z})$ to $K_{HG}(\tilde{z})$.

VI. EXAMPLE

Les us consider an academic example with $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$ and the functions f and Ψ are given by: $f(u,x) = \begin{pmatrix} x_2(1+x_2^2) - x_1^3 \\ -0.02x_2^3 \end{pmatrix}$ and $\Psi(u,x) = \begin{pmatrix} 3sin(19t) & 0 \\ 0 & 5\dfrac{sin(15t)}{1+3x_2^2} \end{pmatrix}$. The output $y = \begin{pmatrix} y_1 = (0.5 - sin(5t)) \\ y_2 = sin(5t) \end{pmatrix}$. Our objective consists in estimating three unknown parameters ρ_i, $i = 1, 2$ as well as the states x_1 and x_2. Many observers under form (18) have been synthesized by considering different expressions of the function $K(\tilde{y})$ such as $K_{Tanh}(\tilde{y})$, $K_{ArcTan}(\tilde{y})$, $K_{TSinh}(\tilde{y})$, etc. The obtained results were very similar and one only presents here those obtained with $K_{HG}(\tilde{y})$. A jump has been simulated for each parameter at time equal to 20. Simulation was carried out under the following initial conditions: $x_1 = 30$; $x_2 = 20$; $\hat{x}_1 = 25$; $\hat{x}_2 = 25$. The initial parameter estimates have been arbitrarily set to zero. The value of the design parameter θ was set to 10. The obtained results are given in figures 1 and 2. They clearly show the adaptation alertness as well as exponential convergence of the proposed adaptive observer.

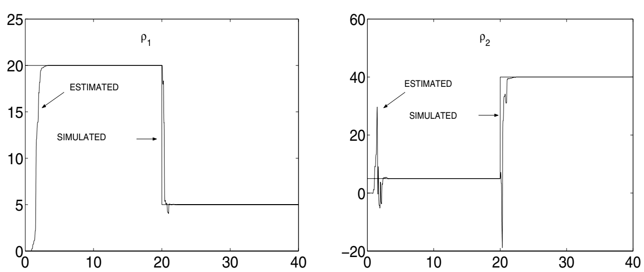

Fig. 1. Comparison of the parameter estimates with their true values

VII. CONCLUSION

A canonical form for a class of uniformly observable nonlinear systems has been investigated from the adaptive observer design point of view. A globally exponentially convergent adaptive observer has then been proposed for this class of systems. The gain of the parameter adaptation of the proposed observer involves a design function satisfying some mild conditions that have been given. Of fundamental importance, the global exponential convergence of the observer was shown to be guaranteed under the well known persistent

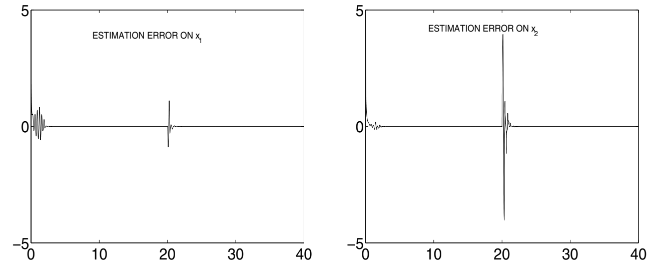

Fig. 2. Estimation errors on x_1 and x_2

excitation condition. Simulation results have been reported to emphasize the performance of the proposed observer.

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