Proceedings of the
44th IEEE Conference on Decision and Control, and
the European Control Conference 2005
Seville, Spain, December 12-15, 2005

Complementary filter design on the special orthogonal group $SO(3)$

Robert Mahony
Department of Engineering, Australian National University, ACT, 0200, AUSTRALIA.
mahony@ieee.org

Tarek Hamel
I3S-CNRS
Nice-Sophia Antipolis, France
thamel@i3s.unice.fr

Jean-Michel Pflimlin
LAAS - CNRS
Toulouse, France
pflimlin@laas.fr

***Abstract*— This paper considers the problem of obtaining high quality attitude extraction and gyros bias estimation from typical low cost inertial measurement units for applications in control of unmanned aerial vehicles. Two different non-linear complementary filters are proposed: *Direct complementary filter* and *Passive non-linear complementary filter*. Both filters evolve explicitly on the special orthogonal group $SO(3)$ and can be expressed in quaternion form for easy implementation. An extension to the passive complementary filter is proposed to provide adaptive gyro bias estimation.**

I. INTRODUCTION

A fundamental problem in autonomous control of flight vehicles is estimation of vehicle attitude. Due to the fast time-scales associated with the attitude stabilisation loop, degradation of attitude estimates can lead quickly to instability and cataclysmic failure of the system. Historically, applications in unmanned aerial vehicles have addressed this issue by using high quality sensor systems, for example the YAMAHA RMAX radio controlled helicopters use high quality optical gyros for their stability augmentation systems. However, military specification inertial measurement units are expensive, often subject to export restrictions and not suitable for commercial applications. More recently, the focus on new low cost aerial robotic systems, has lead to a strong interest in attitude estimation algorithms. Traditional linear Kalman filter techniques [6] (including EKF techniques [2], [4]) have proved extremely difficult to apply robustly to applications with low quality sensor systems [5]. The inherent non-linearity of the system and non-Guassian noise encountered in practice can lead to very poor behaviour of such filters. Many UAV projects use a simple complementary filter design to estimate attitude states [5]. Such filters are very robust and easy to implement and tune, however, the classical implementation is for a single-input integration [1], [5]. Work in the nineties used the quaternion parameterisation of $SO(3)$ to derive filters for state estimation [7]. Following this work there has been considerable interest in filtering and control design for systems on $SO(3)$ working in the quaternion representation. In the last few years some authors have tackled the joint estimation and control problem in this framework [9]. These filters turn out to be very closely related to the complementary filters in common use in practice.

In this paper we study the design of complementary filters on $SO(3)$ in a general setting. We consider the question of measurements, error criterion and filter design from first principals with attention to the Lie group structure of $SO(3)$. From this work we propose two non-linear filters: a *direct complementary filter* and *passive complementary filter*. The direct filter appears to be the filter that has been proposed by authors working purely in the quaternion representation of $SO(3)$ [9]. The passive filter has several practical advantages over the direct filter associated with implementation and low-sensitivity to noise. We provide a derivation of the passive filter in the quaternion representation and a short review of classical complementary filters for completeness.

II. FILTER DESIGN AND ERROR CRITERIA FOR ESTIMATION ON $SO(3)$

In this section, a detailed analysis of the natural error coordinates and cost functions are presented for an estimation problem of the dynamics of a rigid body evolving on $SO(3)$. The following frames of reference are defined

- $\mathcal{A}$ denotes an inertial (fixed) frame of reference.
- $\mathcal{B}$ denotes a body fixed frame of reference.
- $\mathcal{E}$ denotes the estimator frame of reference.

Let $R = {}^{A}_{B}R$ denote the attitude of the body-fixed frame relative to the inertial frame. Let $\Omega = {}^{B}\Omega$ denote the angular velocity of the body-fixed frame expressed in the body fixed frame $\mathcal{B}$.

The system considered is the kinematics

$$\dot{R} = R\Omega_\times \tag{1}$$

where $\Omega_\times$ denotes the skew-symmetric (or anti-symmetric) matrix such that $\Omega_\times a = \Omega \times a$ for all vectors a. The inverse operation that takes an anti-symmetric matrix to its associated vector is denoted vex, $\Omega = \text{vex}(\Omega_\times)$. The kinematics can also be written directly in terms of the quaternion representation of $SO(3)$ by

$$\dot{q} = \frac{1}{2} q \otimes \mathbf{p}(\Omega) \quad (2)$$

where $q = \{(s, v) \in \mathbb{R} \times \mathbb{R}^3 | \ s^2 + |v|^2 = 1\}$ represents the unit quaternion and $\mathbf{p}(\Omega)$ represents the pure quaternion associated to Ω, $\mathbf{p}(\Omega) = (0, \Omega)'$. The quaternion multiplication is denoted $q_1 \otimes q_2 = [(s_1 s_2 - v_1^T v_2), (s_1 v_2 + s_2 v_1 + v_1 \times v_2)]$.

The goal of developing an estimator is to provide a smooth estimate $\hat{R}(t) \in SO(3)$ of a state $R(t)$ that is evolving due to some external input based on a set of measurements. The measurements available from a typical inertial measurement unit are 3-axis rate gyro, acceleration and magnetometer measurements. The rate gyro measurements are measured in the body fixed frame,

$$\Omega_y = \Omega + b(t) + \mu_\Omega$$

where Ω is the true body velocity, $b(t)$ denotes a slowly time-varying bias and μ_Ω denotes a noise process. The bias destroys any low frequency information from the gyro readings, however, the high frequency information in Ω_y is reasonably good. The acceleration and magnetometer readings are combined together to provide an algebraic estimate of the attitude rotation R_y. Although this is already an estimate of the desired rotation, it is highly unreliable at high frequency due to its dependence on the system dynamics (that perturb the gravitational field estimate with dynamic effects) and its high sensitivity to noise due to the magnetometer noise and the algebraic reconstruction of R_y. The goal of a complementary filter is to fuse the low-frequency measurement R_y with the high-frequency rate measurement Ω_y, to produce a good estimate $\hat{R}$ of the attitude, valid over the frequency domain.

Let $\hat{R}$ denote an estimate of the body fixed rotation matrix R. The rotation $\hat{R}$ can itself be considered coordinates for the estimator frame of reference $\mathcal{E}$ and is also the transformation matrix between frames

$$\hat{R} = {}^{A}_{E}\hat{R} : \mathcal{E} \to \mathcal{A}.$$

The natural estimation error to use is the rotation

$$\tilde{R} = \hat{R}^T R : \mathcal{B} \to \mathcal{E} \quad (3)$$

The rotation $\tilde{R}$ may be thought of as the attitude of the body-fixed frame with respect to the estimator frame. The goal of the filter design is to find kinematics for $\hat{R}(t) \in SO(3)$ such that $\tilde{R}(t) \to I$ uniformly.

Let π_a, π_s denote, respectively, the anti-symmetric and symmetric projection operators in matrix space:

$$\pi_a(\tilde{R}) = \frac{1}{2}(\tilde{R} - \tilde{R}^T), \qquad \pi_s(\tilde{R}) = \frac{1}{2}(\tilde{R} + \tilde{R}^T).$$

Thus,

$$\tilde{R} = \pi_a(\tilde{R}) + \pi_s(\tilde{R}),\ \pi_a(\tilde{R})^T = -\pi_a(\tilde{R}),\ \pi_s(\tilde{R})^T = \pi_s(\tilde{R}).$$

Let (θ, a) denote the angle-axis coordinates of $\tilde{R}$. One has [3]:

$$\tilde{R} = \exp(\theta a_\times), \qquad \log(\tilde{R}) = \theta a_\times$$
$$\cos(\theta) = \frac{1}{2}(\text{tr}(\tilde{R}) - 1), \qquad a_\times = \frac{1}{\sin(\theta)} \pi_a(\tilde{R})$$

The cost function considered is

$$E_t := \|I_3 - \tilde{R}\|^2 = \frac{1}{2}\text{tr}(I_3 - \tilde{R}) \quad (4)$$

To see that this error measures an absolute difference between $\tilde{R}$ and I_3 note that

$$E_t := \frac{1}{2}\text{tr}(I - \tilde{R}) = (1 - \cos(\theta)) = 2\sin(\theta/2)^2. \quad (5)$$

Thus, driving Eq. 4 to zero ensures that $\theta \to 0$. We term this error the *trace error* or *quaternion error*. The second terminology comes from observing that if $\tilde{q} = (\tilde{s}, \tilde{v})$ is the quaternion related to $\tilde{R}$ then $E_t = 2|\tilde{v}|^2 = 2(1 - \tilde{s}^2)$. Much of the earlier work on non-linear filtering and control design on $SO(3)$ has used the error criteria E_t and the quaternion representation. In this document, we will work almost exclusively directly on the matrix representation of $SO(3)$, although, equivalent derivations in the quaternion framework are provided for completeness.

III. Non-linear complementary filter design on $SO(3)$

In this section, a general framework for non-linear estimator for the rotation $R(t)$ is proposed. The present section focuses on the case where exact measurements of $R(t)$ and $\Omega(t)$ are available. Practical implementation for measured values is considered at the end of the section.

A. Theoretical development of complementary filter on $SO(3)$.

The goal of attitude estimation is to provide a set of dynamics for an estimate $\hat{R}(t) \in SO(3)$ to drive the error rotation (Eq. 3) $\tilde{R}(t) \to I_3$ given measurements Ω^y and R_y of the angular velocity and true rotation respectively. It is useful to begin by considering the case where

$$R_y = R, \quad \text{and} \quad \Omega^y = \Omega.$$

In principle, it is not necessary to use a filter in this case, since the measurements are exact. It is instructive, however, to study the dynamics of a filter $\hat{R}(t)$ that is driven by exact information without the complexities of dealing with sensor noise characteristics, biases and partial state measurements. Knowing that, for $R \in SO(3)$ and for any vector $a \in \mathbb{R}^3$, we have:

$$(Ra)_\times = R a_\times R^T,$$

the kinematics of the true system are given by

$$\dot{R} = R\Omega_\times = (R\Omega)_\times R$$

where $\Omega \in \mathcal{B}$ and $(R\Omega) \in \mathcal{A}$. The dynamics of the filter $\hat{R}$ are required to evolve on $SO(3)$ and should have a form similar to the kinematics of R. Note that when $\hat{R} = R$ then the estimate must evolve according to the kinematics of R. Thus, we propose kinematics of the form

$$\dot{\hat{R}} = (R\Omega + \hat{R}\omega(\hat{R}, R))_\times \hat{R} \tag{6}$$

where $\omega(\hat{R}, R) \in \mathcal{E}$ is a correction term that is zero when $\hat{R} = R$. Note that the kinematics for $\hat{R}$ are expressed with respect to the inertial frame and the forward kinematics of the true system are given in the inertial frame. In particular, if no correction term is used, $\omega \equiv 0$, then

$$\begin{aligned}\dot{\tilde{R}} =& \hat{R}^T (R\Omega)_\times^T R + \hat{R}^T (R\Omega)_\times R \\ =& \hat{R}^T \left(-(R\Omega)_\times + (R\Omega)_\times\right) R = 0\end{aligned} \tag{7}$$

The kinematics of $\tilde{R}$ are given by

$$\begin{aligned}\dot{\tilde{R}} &= \hat{R}^T (R\Omega + \hat{R}\omega)_\times^T R + \hat{R}^T (R\Omega)_\times R = \hat{R}^T (\hat{R}\omega_\times \hat{R}^T)^T R \\ &= \omega_\times^T \tilde{R} = -\omega_\times \tilde{R}\end{aligned} \tag{8}$$

Since $\omega \in \mathcal{E}$ was specified in the estimator frame, the above equation corresponds to the natural form of the kinematics of $\tilde{R} : \mathcal{B} \to \mathcal{E}$.

The correction term ω in the filter dynamics Eq. 6 can be chosen using a straightforward Lyapunov argument. We choose

$$\omega = k_{est}\text{vex}(\pi_a(\tilde{R})) \Leftrightarrow \omega_\times = k_{est}\pi_a(\tilde{R}),\ k_{est} > 0. \tag{9}$$

Lemma 3.1: Consider the true system

$$\dot{R} = R\Omega_\times$$

and assume that R and Ω are measured. The *direct complementary filter* dynamics are specified by

$$\begin{aligned}\dot{\hat{R}} &= (R\Omega + \hat{R}\omega)_\times \hat{R}, \\ &= \left((R\Omega)_\times + k_{est}\hat{R}\pi_a(\tilde{R})\hat{R}^T\right)\hat{R}, \text{for } \hat{R}(0) = \hat{R}_0.\end{aligned} \tag{10}$$

Then

$$\dot{E}_t = -2k_{est}\cos^2(\theta/2)E_t$$

and for any initial condition $\hat{R}_0$ such that

$$\theta_0 = \frac{1}{\sqrt{2}}||\log(\tilde{R}_0)|| < \pi,$$

then $E_t \to 0$ and $\hat{R}(t) \to R(t)$ exponentially.

Proof: Deriving the Lyapunov function E_t subject to dynamics Eq. 6 yields

$$\begin{aligned}\dot{E}_t &= -\frac{1}{2}\text{tr}(\dot{\tilde{R}}) = -\frac{1}{2}\text{tr}\left(\omega_\times^T \tilde{R}\right) \\ &= -\frac{1}{2}\text{tr}(\omega_\times^T \tilde{R}) = -\frac{1}{2}\text{tr}\left[\omega_\times^T (\pi_s(\tilde{R}) + \pi_a(\tilde{R}))\right].\end{aligned}$$

knowing that, for any any symmetric matrix A_s and any skew-symmetric A_a,

$$\text{tr}(A_s A_a) = 0,$$

the derivative of the cost function E_t becomes:

$$\dot{E}_t = -\frac{1}{2}\text{tr}(\omega_\times^T \pi_a(\tilde{R}))$$

Using the chosen expression of the correction term (Eq. 9), one has

$$\dot{E}_t = -\frac{k_{est}}{2}||\pi_a(\tilde{R})||_F^2.$$

(where $||\cdot||_F$ denotes the Frobenius norm). Substituting for $\sin(\theta)a_\times = \pi_a(\tilde{R})$ gives

$$\begin{aligned}\dot{E}_t &= -\frac{k_{est}}{2}\sin^2(\theta)||a_\times||^2 = -k_{est}\sin^2(\theta) = \\ &= -4k_{est}\sin^2(\theta/2)\cos^2(\theta/2) = -2k_{est}\cos^2(\theta/2)E_t.\end{aligned}$$

For $\theta_0 < \pi$ then E_t is exponentially decreasing to zero. Lyapunov's direct method completes the proof. ■

We term the filter Eq. 10 a *complementary filter on* $SO(3)$ since it recaptures the basic block diagram structure of a classical complementary filter (see Appendix A). A representative block diagram for the filter is shown in Figure 1. Consider a comparison of the terms in this block diagram with the block diagram of a classical complementary filter, Figure 8.

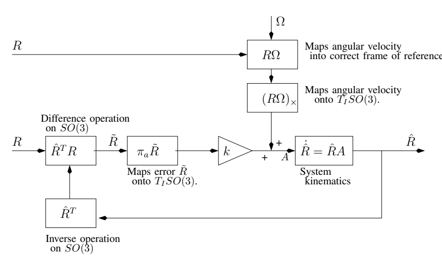

Fig. 1. Block diagram of the non-linear filter on $SO(3)$.

The '$\hat{R}^T$' operation is an inverse operation on $SO(3)$ and is equivalent to a '$-$' operation on Euclidean space. The '$\hat{R}^T R$' operation is equivalent to generating the error term '$y - \hat{x}$' in the classical complementary filter (cf. §). The two operations $\pi_a\tilde{R}$ and $(R\Omega)_\times$ are maps from error space and velocity space into the tangent space of $SO(3)$, an operation that is unnecessary on Euclidean space due to the identification $T_x\mathbb{R}^n \equiv \mathbb{R}^n$. The kinematic model is a first order system that is essentially an integrator.

An important aspect of the proposed filter is the premultiplication of Ω by the rotation R to ensure that the velocity is in the correct frame of reference. This comes from the fact that the measured angular velocity lies in the body-fixed frame whereas the filter requires an angular velocity estimate in the inertial frame. The requirement for the transformation R is a consequence of the measurement, not directly of the geometry. If an angular velocity measurement was obtained directly in the inertial frame, for example, such as would be supplied by an external vision system, then this transformation would be unnecessary.

B. Implementing a complementary filter on $SO(3)$

Given the theoretical form of the filter Eq. 10 it is important to consider how such a filter may be implemented. In practice, the various inputs to the filter are replaced by measurements and filtered estimates of the states. There are three key inputs

1) The reference rotation R that generates the error term $\hat{R}^T R$. Based on an understanding of complementary filter design (cf. Appendix) it is natural to use the estimate R_y to generate the error term.
2) A direct measurement of the angular velocity $\Omega_y \approx \Omega$ is available and used to drive the feedforward term in the filter.
3) Finally, an estimate of the rotation R must be used to map the body-fixed-frame velocity back into the inertial frame. There are two choices for this crucial choice:

Direct complementary filter: Using the measured rotation R_y as a feed-forward estimate of the $\mathcal{B} \to \mathcal{A}$ transformation is the most common approach taken in the literature [7], [9], [8]. A block diagram of this filter design is shown in Figure 2. This approach has the advantage that it does not introduce any additional feedback loop in the filter dynamics. The error in the feed-forward estimate of the velocity will enter as bounded noise in the filter implementation and the overall filter will inherit strong stability properties from the Lyapunov analysis in Lemma 3.1. It has the disadvantage that the measurement of R_y is typically corrupted by high frequency noise and this noise is transmitted to the output via the $\mathcal{B} \to \mathcal{A}$.

Passive complementary filter: Using the estimated rotation $\hat{R}$ to feedback an estimate of the $\mathcal{B} \to \mathcal{A}$ transformation is a natural alternative to using R_y, Figure 3. The authors do not know of a reference where a theoretical analysis of this approach has been discussed in the literature, although, it is certain that many authors will have used this idea in practice. The advantage is that the estimate $\hat{R}$ is likely to be a better estimate of the true rotation than R_y if the filter remains well behaved.

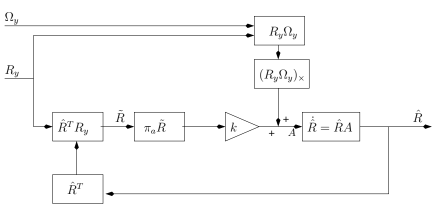

Fig. 2. Block diagram of the direct complementary filter on $SO(3)$.

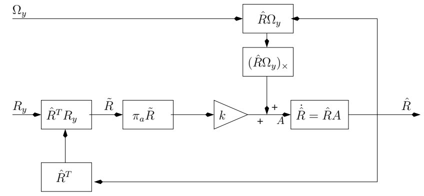

Fig. 3. Block diagram of the passive complementary filter on $SO(3)$.

C. Passive Complementary Filter Design

In this section we provide an analysis of the passive complementary filter.

It is interesting to note that using the feedback estimate $\hat{R}$ as the estimate for transforming the filter design leads to filter kinematics in the estimator frame of reference

$$\dot{\hat{R}} = \left(\hat{R}\Omega + \hat{R}\omega\right)_\times \hat{R} = \hat{R}\left(\hat{R}\Omega_\times \hat{R}^T + k_{est}\hat{R}\pi_a(\tilde{R})\hat{R}^T\right)\hat{R}$$
$$= \hat{R}\left(\Omega_\times + k_{est}\pi_a(\tilde{R})\right) = \hat{R}\left(\Omega + \omega\right)_\times \tag{11}$$

In terms of the frames of reference, the factorisation relates to the fact that the use of $\hat{R}$ as a feedback estimation of the transformation corresponds to assuming that the velocity is measured directly in the estimator frame of reference. This due to invariance of the anti-symmetric projection operator under the adjoint map

$$\mathrm{Ad}_{\tilde{R}}\pi_a(\tilde{R}) = \tilde{R}\pi_a(\tilde{R})\tilde{R}^T = \pi_a(\tilde{R})$$

we think of $\pi_a(\tilde{R}) \in \mathcal{B}$. Consequently, the equivalent error correction in the estimator frame is the same skew-symmetric matrix. This is natural as

$$\pi_a(\tilde{R}) = \frac{\sin(\theta)}{\theta}\log(\tilde{R})$$

and $\log$ is invariant under the Ad operator [3]. Therefore, the filter may be rewritten naturally in the estimator frame of reference. This structure is important since it considerably simplifies the implementation of the filter as shown in Figure 4.

Lemma 3.2: Consider the true system

$$\dot{R} = R\Omega_\times$$

and assume that R and Ω are measured. Using the *passive complementary filter* dynamics specified by Eq. 11. Then

$$\dot{E}_t = -2k_{est}\cos^2(\theta/2)E_t$$

and for any initial condition $\hat{R}_0$ such that

$$\theta_0 = \frac{1}{\sqrt{2}}||\log(\tilde{R}_0)|| < \pi,$$

then $E_t \to 0$ and $\hat{R}(t) \to R(t)$ exponentially.

Proof: Analogously to the proof of lemma 3.1, the trace error criterion $E_t = \frac{1}{2}\mathrm{tr}(I - \tilde{R})$ (Eq. 4) is used.

Deriving Eq. 4 subject to dynamics Eq. 11 yields

$$\begin{aligned}\dot{E}_t &= -\frac{1}{2}\mathrm{tr}(\dot{\tilde{R}}) = -\frac{1}{2}\mathrm{tr}(-(\Omega+\omega)_\times \tilde{R} + \tilde{R}\Omega_\times) \\ &= -\frac{1}{2}\mathrm{tr}([\tilde{R}, \Omega_\times]) - \frac{1}{2}\mathrm{tr}(\omega_\times^T \tilde{R})\end{aligned}$$

Note that trace of a Lie bracket is zero, $\mathrm{tr}([\tilde{R}, \Omega_\times]) = 0$.

Analogously to the proof of lemma 3.1, using the chosen expression for the correction term (Eq. 9) and the fact that $\mathrm{tr}(\omega_\times^T \pi_s(\tilde{R})) = 0$, one obtains

$$\dot{E}_t = -\frac{1}{2}k_{est}||\pi_a(\tilde{R})||^2 = -2k_{est}\cos^2(\theta/2)E_t.$$

For $\theta_0 < \pi$ then E_q is exponentially decreasing to zero. Lyapunov's direct method completes the proof. ■

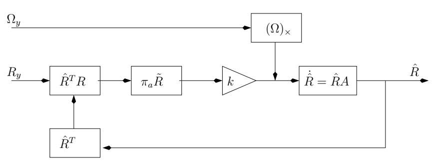

Fig. 4. Block diagram of the non-linear filter using feedback ($\hat{R}$) estimation of body-fixed-frame velocity and expressed in the estimator frame of reference.

D. Simulation results

The proposed designs of direct and passive filters have been simulated. The initial condition of orientation matrix is the identity matrix ($R(0) = I_3$). The chosen initial deviation $\tilde{R}$ for both filters corresponds to a deviation of $\frac{\pi}{2}$ around the axis e_2. The orientation velocity has been chosen constant of 0.3rad/s around the axis e_3 ($\Omega = 0.3e_3$). Figure 5 shows that although trajectories are different, both errors of estimation have, at a given time, the same deviation with respect to the direction e_3. In other words, they are on the same circle at the same time.

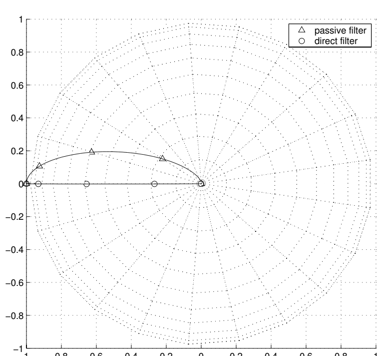

Fig. 5. Trajectories of $\tilde{R}e_3$ for direct and passive complementary filters on the plan $\{e_1, e_2\}$.

IV. Attitude extraction and gyros bias estimation from Passive Complementary Filter

In this section we propose to redesign the passive and complementary filter Eq. 11 for state attitude estimation on $SO(3)$ in the presence of constant gyro bias. The redesign approach consists in considering a dynamic estimator for the bias b.

Theorem 4.1: Consider the true system

$$\dot{R} = R\Omega_\times$$

along with measurements

$$\begin{aligned} R_y &\approx R, && \text{valid for low frequencies,} \\ \Omega_y &\approx \Omega + b && \text{for constant bias } b.\end{aligned}$$

For a correction term ω given by Eq. 9, the passive complementary filter dynamics are specified by

$$\dot{\hat{R}} = \hat{R}(\Omega^y - \hat{b} + \omega)_\times \qquad (12)$$

along with the following dynamics for the estimate $\hat{b}$

$$\dot{\hat{b}} = -k_b \mathrm{vex}(\pi_a(\tilde{R})), \quad k_b > 0 \qquad (13)$$

Consider the following Lyapunov function,

$$V = E_t + \frac{1}{2k_b}|\tilde{b}|^2$$

Then for any initial condition $\hat{R}_0$ such that

$$\theta_0 = \frac{1}{\sqrt{2}}||\log(\tilde{R}_0)|| < \pi \text{ and that } k_b > \frac{2\tilde{b}(0)}{2 - E(0)},$$

then $E_t \to 0$ and $\hat{R}(t) \to R(t)$ uniformly and $\tilde{b} \to 0$ converges to zero.

Proof: To prove the stability of the proposed observer, we recall the expression of candidate Lyapunov function V:

$$V = \frac{1}{2}\mathrm{tr}(I_3 - \tilde{R}) + \frac{1}{2k_b}|\tilde{b}|^2$$

Differentiating V, one gets:

$$\begin{aligned}\dot{V} &= -\frac{1}{2}\mathrm{tr}(\dot{\tilde{R}}) - \frac{1}{k_b}\tilde{b}^T\dot{\hat{b}} \\ &= -\frac{1}{2}\mathrm{tr}([\tilde{R}, \Omega_\times]) + \frac{1}{2}\mathrm{tr}((\omega + \tilde{b})_\times \tilde{R}) - \frac{1}{k_b}\tilde{b}^T\dot{\hat{b}}\end{aligned}$$

Knowing that trace of a Lie bracket is zero ($\mathrm{tr}[\tilde{R}, \Omega_\times] = 0$), by simple verification that $\tilde{b}^T\dot{\hat{b}} = \mathrm{tr}(\tilde{b}_\times^T \dot{\hat{b}}_\times)$ and that $\mathrm{tr}((\omega + \tilde{b})_\times \pi_s(\tilde{(R)})) = 0$, the derivative of the candidate Lyapunov function becomes:

$$\begin{aligned}\dot{V} &= -\,\mathrm{tr}((\omega + \tilde{b})_\times^T \pi_a(\tilde{R})) - \frac{1}{k_b}\mathrm{tr}(\tilde{b}_\times^T \dot{\hat{b}}_\times) \\ &= -\,\mathrm{tr}(\omega_\times^T \pi_a(\tilde{R})) - \mathrm{tr}[\tilde{b}_\times^T(\pi_a(\tilde{R}) + \frac{1}{k_b}\dot{\hat{b}}_\times)]\end{aligned}$$

Introducing now, the expressions of ω (Eq. 9) and of $\dot{\hat{b}}$ (Eq. 13), the derivative of the Lyapunov function V becomes:

$$\dot{V} = -k_{est}\text{tr}(\pi_a(\tilde{R})^T\pi_a(\tilde{R})) = -k_{est}||\pi_a(\tilde{R})||^2 \quad (14)$$

From the last equation where the Lyapunov function derivative is negative semi-definite, standard Lyapunov argument concludes that $\pi_a(\tilde{R})$ tends to zero. Consequently $\tilde{R}$ converges towards $\tilde{R}^T$ and therefore $\tilde{R}$ tends to I_3 and E_t converges to zero.

To address the problem of gyroscope bias error estimation we appeal to LaSalle principle. The invariant set is contained in the set defined by the condition $\pi_a\tilde{R} = 0$. Recalling Eq. 13, it follows that $\dot{\hat{b}} = 0$ on the invariant set and therefore the gyros bias estimation error converges to a constant value $\tilde{b}^\infty$. Recalling the derivative of $\tilde{R}$, one gets:

$$\dot{\tilde{R}} = -(\Omega + \tilde{b} + \omega)_\times\tilde{R} + \tilde{R}\Omega_\times$$

Using the fact $\tilde{R} \equiv I_3$ and that $\omega \equiv 0$, it yields:

$$\tilde{b}^\infty = 0$$

From the above discussion $\tilde{b}$ converges asymptotically towards zero. ■

Remark 4.2: It is interesting to note that attitude extraction and gyros bias estimation from direct complementary filter leads to the following kinematics:

$$\begin{aligned}\dot{\hat{R}} &= \hat{R}(\tilde{R}(\Omega^y - \hat{b}) + \omega)_\times \\ &= \hat{R}(\tilde{R}(\Omega^y - \hat{b}) + k_{\text{est}}\text{vex}(\pi_a(\tilde{R})))_\times \end{aligned} \quad (15)$$

along with dynamics

$$\dot{\hat{b}} = -k_b\tilde{R}^T\text{vex}(\pi_a(\tilde{R}))$$

for the estimate $\hat{b}$. However, as $\tilde{R}^T\text{vex}(\pi_a(\tilde{R})) = \text{vex}(\tilde{R}^T\pi_a(\tilde{R})\tilde{R}) = \text{vex}(\pi_a(\tilde{R}))$, expression of gyros bias adaptive filter remains unchanged:

$$\dot{\hat{b}} = -k_b\text{vex}(\pi_a(\tilde{R})) \quad (16)$$

△

A. A Quaternion-Based derivation of the passive complementary filter

As discussed in the introduction section much of the existing literature in estimation and control has been developed on the quaternion group rather than directly on the Lie group $SO(3)$. Although the two groups are homomorphic, it is not easy to detect the passivity structure used to derive the passive complementary filter when working in the quaternion representation. The authors feel that this explains the fact that existing works have not used the passive complementary filter Eq. 11. In this section we will provide the passive complementary filter in terms of unit quaternion formulation.

Recall the unit quaternion kinematics given by Eq. 2 along with measurements: $q_y \approx q$ (valid for low frequencies) and $\Omega_y \approx \Omega + b$ (for constant bias b). Define the unit quaternion error $\tilde{q}$ between the estimated and actual unit quaternions:

$$\tilde{q} = \hat{q}^{-1} \otimes q = \begin{bmatrix} \tilde{s} \\ \tilde{v} \end{bmatrix}$$

Recall the observer based quaternion proposed by Thienel and Sanner [9] which represents one of recent development in this area:

$$\dot{\hat{q}} = \frac{1}{2}\hat{q} \otimes \mathbf{p}(\tilde{R}(\Omega_y - \hat{b} + k_{est}\text{sgn}(\tilde{s})\tilde{v}))$$

along with the following dynamics for the estimate $\hat{b}$.

$$\dot{\hat{b}} = -k_b\text{sgn}(\tilde{s})\tilde{v}$$

If we replace the term $\text{sgn}(\tilde{s})$ in the expression of the observer by $2\tilde{s}$, the filter of the reference [9] is equivalent to the proposed direct complementary filter designed on $SO(3)$. Indeed, we have

$$2\tilde{s}\tilde{v} = 2\cos(\theta/2)\sin(\theta/2)a = \frac{1}{2}(\sin\theta)a = \text{vex}(\pi_a(\tilde{R}))$$

Consequently, the observer expresses as:

$$\dot{\hat{q}} = \frac{1}{2}\hat{q} \otimes \mathbf{p}(\tilde{R}(\Omega_y - \hat{b}) + k_{est}\tilde{R}\text{vex}(\pi_a(\tilde{R})))$$

Knowing that $\tilde{R}\text{vex}(\pi_a(\tilde{R})) = \text{vex}(\pi_a(\tilde{R}))$, the final expression of the Thienel and Sanner's observer could be rewritten as follows:

$$\dot{\hat{q}} = \frac{1}{2}\hat{q} \otimes \mathbf{p}(\tilde{R}(\Omega_y - \hat{b}) + k_{est}\text{vex}(\pi_a(\tilde{R})))$$

along with the following dynamics for the estimate $\hat{b}$

$$\dot{\hat{b}} = -k_b\text{vex}(\pi_a(\tilde{R}))$$

which is the equivalent quaternion formulation of the direct complementary filter designed on the orthogonal group $SO(3)$.

From the above discussion and formulation, the passive complementary filter in terms of unit quaternion representation can be expressed as follows:

$$\dot{\hat{q}} = \frac{1}{2}\hat{q} \otimes \mathbf{p}(\Omega_y - \hat{b} + \omega) \quad (17)$$

where $\omega = k_{est}\tilde{s}\tilde{v}$ and $\dot{\hat{b}} = -k_b\tilde{s}\tilde{v}$.

B. Experimental results

In this section, we present experimental results to evaluate the performance of the proposed filters.

Experiments have been done on a real platform to illustrate the convergence of the 3 Euler angles as well as the gyro bias estimates. The platform was equipped with an IMU and provided magnetic field measurements. The platform simulates the movement of a flying vehicle with an orientation trajectory known *a priori*. From the platform, we could extract the real values of the three Euler angles to compare them with the estimates provided by direct and passive complementary filters. The frequency of inertial sensors data acquisition is set to 25 Hz.

Due to the discrete data acquisition and in order to preserve the evolution of the estimated matrix $\hat{R}$ on the $SO(3)$ manifold, the proposed observer has been implemented in the experiments in discrete time using exact integration of passive complementary filters dynamics (Eq. 11) and Euler integration of the bias estimation dynamics in Eq. 13. More precisely, if we denote the sample time by T, rewrite $\dot{\tilde{R}} = \tilde{R}\tilde{\Omega}_\times$ and assume that $\tilde{\Omega}(t) = \tilde{\Omega}^k$ for $t \in [kT, (k+1)T]$, then we have the discrete time model of the observer:

$$\tilde{R}^{k+1} = A^k \tilde{R}^k, \quad \hat{b}^{k+1} = b^k - T k_b \pi_a(\tilde{R}^k)$$

where $A^k = \exp(\tilde{\Omega}^k_\times)$ has closed form solution given by Rodrigue's formula (see reference [3]):

$$A^k_k = I_3 - \tilde{\Omega}^k_\times \frac{\sin(|\tilde{\Omega}^k|T)}{|\tilde{\Omega}^k|} + (\tilde{\Omega}^k_\times)^2 \frac{1-\cos(|\tilde{\Omega}^k|T)}{|\tilde{\Omega}^k|^2} \quad (18)$$

Note that exact integration of direct complementary filter is impossible. This due the actual dependance of the orientation velocity Ω (see Eq. 10) but in the experiment we have assumed that the term $\tilde{R}\Omega$ is constant in the interval $t \in [kT, (k+1)T]$.

For the experiments the gains of the the proposed filters has been chosen as follows: $k_{\text{est}} = 1\text{rd.s}^{-1}$ and $k_b = 0.3\text{rd.s}^{-1}$

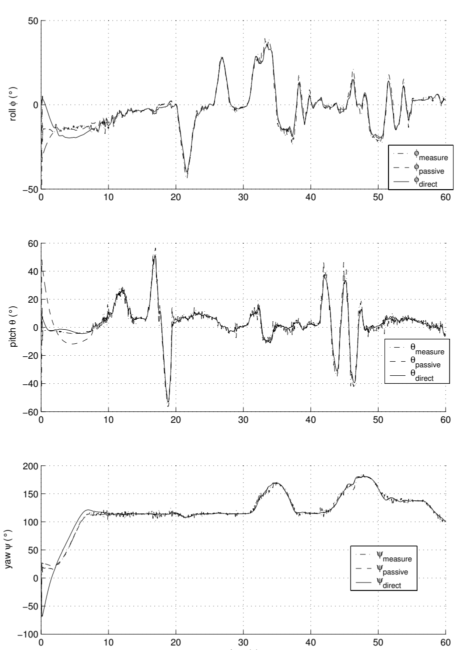

Fig. 6. Euler angles from direct and passive complementary filters

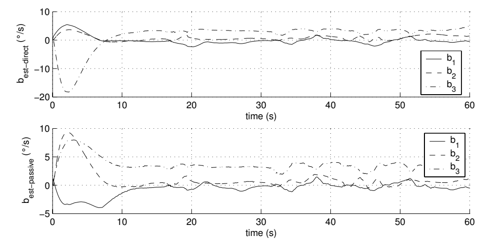

Fig. 7. Bias estimation from direct and passive complementary filters

V. Concluding remarks

In this paper we have discussed the problem of orientation extraction and gyro bias estimation from direct and passive complementary filters developed directly in the special orthogonal group SO(3). Some advantages of passive complementary filter have been presented and discussed with respect the direct version of the filter. Experimental results have been provided as complement of the theoretical approach.

References

[1] A.J. Baerveldt and R. Klang. A low-cost and low-weight attitude estimation system for an autonomous helicopter. pages 391–395, 1997.

[2] J. L. Marins, X. Yun, E. R. Backmann, R. B. McGhee, and M.J. Zyda. An extended kalman filter for quaternion-based orientation estimation using marg sensors. In *IEEE/RSJ International Conference on Intelligent Robots and Systems*, pages 2003–2011, 2001.

[3] R.M. Murray, Z. Li, and S. Sastry. *A mathematical introduction to robotic manipulation*. CRC Press, 1994.

[4] H. Rehbinder and X. Hu. Drift-free attitude estimation for accelerated rigid bodies. *Automatica*, pages 653–659, April 2004.

[5] J. M. Roberts, P. I. Corke, and G. Buskey. Low-cost flight control system for small autonomous helicopter. In *Australian Conference on Robotics and Automation, Auckland, 27-29 Novembre*, pages 71–76, 2002.

[6] M. Jun S.I. Roumeliotis and G.S. Sukhatme. State estimation via sensor modeling for helicopter control using an indirect kalman filter. In *IEEE/RSJ International Conference on Intelligent Robots and Systems*, pages 1346–1353, 1999.

[7] S. Salcudean. A globally convergent angular velocity observer for rigid body motion. *IEEE Transactions on Automatic Control*, 46, no 12:1493–1497, 1991.

[8] A. Tayebi and S. McGilvray. Attitude stabilization of a four-rotor aerial robot: Theory and experiments. *To appear in IEEE Transactions on Control Systems Technology*, 2005.

[9] J. Thienel and R.M. Sanner. A coupled nonlinear spacecraft attitude controller and observer with an unknown constant gyro bias and gyro noise. *IEEE Transactions on Automatic Control*, 48, no 11:2011–2014, 2003.

Appendix

A Review of Complementary Filtering: Complementary filters provide a means of fusing low bandwidth position measurements with high band width rate measurements for first order kinematic systems.

A. Classical Complementary Filters

Consider a simple first order integrator

$$\dot{x} = u \quad (19)$$

with typical measurement characteristics

$$y_x = L(s)x + \mu_x, \quad y_u = u + \mu_u + b(t) \tag{20}$$

where $L(s)$ is low pass filter associated with sensor characteristics, μ represents noise in both measurements and $b(t)$ is a deterministic perturbation that is dominated by low-frequency content. Normally the low pass filter $L(s) \approx 1$ over the frequency range on which the measurement y_x is of interest. A demonstrative example is a single axis attitude estimator with a tilt meter providing a direct measurement of attitude (filtered by the natural low pass dynamics of the tilt meter) and y_u provided by a rate gyro with associated noise μ_u and bias $b(t)$.

The measurements y_x and y_u can be fused into an estimate $\hat{x}$ of the state x via a filter

$$\hat{x} = F_1(s)y_x + F_2(s)\frac{y_u}{s}$$

where F_1 is low pass and F_2 is high-pass. Note that the rate measurement y_u is first integrated to get a position estimate and then filtered by F_2 to retain only the high frequency prediction, the low frequency component of y_u/s is highly unreliable due to the integration of $\mu_u + b(t)$. A filter of the above form is termed a *complementary filter* if

$$F_1(s) + F_2(s) = 1. \tag{21}$$

That is if $F_1(s)$ and $F_2(s)$ are complementary sensitivity transfer functions then the overall filter is termed complementary. In practice, the above condition ensures that the cross over frequency of the low pass filter $F_1(s)$ corresponds to the cross over frequency of the high pass filter, and the whole frequency domain is evenly represented in the filter response.

There is a systems interpretation of a complementary filter that provides an excellent structure for implementation in a practical system. Consider the block diagram in Figure 8. The output $\hat{x}$ can be written

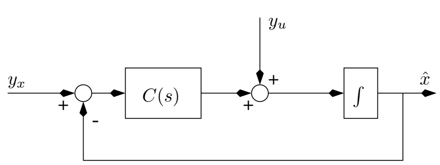

Fig. 8. Block diagram of a classical complementary filter.

$$\hat{x}(s) = \frac{C(s)}{s + C(s)}y_x(s) + \frac{s}{C(s) + s}\frac{y_u(s)}{s}$$
$$= T(s)y_x(s) + S(s)\frac{y_u(s)}{s}$$

where $S(s)$ is the sensitivity function of the closed-loop system and $T(s)$ is the complementary sensitivity. It follows that $T(s) + S(s) = 1$. Thus, a complementary filter design is achieved by using classical control design techniques to design the compensator $C(s) = A(s)/B(s)$ and then implementing the simple linear system

$$sB(s)\hat{x} = B(s)y_u + A(s)(y_x - \hat{x}).$$

The most common complementary filter considered involves a proportional feedback $C(s) = k$. In this case the closed-loop dynamics are given by

$$\dot{\hat{x}} = y_u + k(y_x - \hat{x}). \tag{22}$$

The frequency domain complementary filters (cf. Eq. 21) associated with the proportional feedback complementary filter are $F_1(s) = \frac{k}{s+k}$ and $F_2(s) = \frac{s}{s+k}$. Note that the crossover frequency for the filter is at $k$rad/s. Design of the gain k is based on analyzing the low pass characteristics of y_x and the low frequency noise characteristics of y_u to choose the best crossover frequency to tradeoff between the two measurements. If the rate measurement bias, $b(t) = b_0$, is a constant then it is possible to add an integrator into the compensator

$$C(s) = k + \frac{1}{\lambda s} \tag{23}$$

to reject a constant input disturbance from the closed-loop system.

It is useful to also consider a constructive Lyapunov analysis of closed-loop system Eq. 22 for measurements

$$y_u = u + b_0 + \mu_u, \quad y_x = x + \mu_x$$

for b_0 a constant. Applying the PI compensator, Eq. 23, one obtains state space filter dynamics

$$\dot{\hat{x}} = y_u - \hat{b} + k(y_x - \hat{x}), \quad \dot{\hat{b}} = \frac{-(y_x - \hat{x})}{\lambda}$$

The negative sign in the integrator state is introduced to indicate that the state $\hat{b}$ will cancel the bias in y_u. Consider the Lyapunov function

$$\mathcal{L} = \frac{1}{2}|x - \hat{x}|^2 + \frac{\lambda}{2}|b_0 - \hat{b}|^2$$

Abusing notation for the noise processes, and using $\tilde{x} = (x - \hat{x})$, and $\tilde{b} = (b_0 - \hat{b})$, one has

$$\frac{d}{dt}\mathcal{L} = -k|\tilde{x}|^2 - \mu_u\tilde{x} + \mu_x(\tilde{b} - k\tilde{x})$$

In the absence of noise one may apply Lyapunov's direct method to prove convergence of the state estimate. LaSalles principal of invariance may be used to show that $\hat{b} \to b_0$, however, Lyapunov theory itself does not ensure exponential convergence of the system. When the underlying system is linear, then the linear form of the feedback and adaptation law ensure that the closed-loop system is linear and stability implies exponential stability.