Proceedings of the
44th IEEE Conference on Decision and Control, and
the European Control Conference 2005
Seville, Spain, December 12-15, 2005

On the PE stabilization of time-varying systems: open questions and preliminary answers

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Abstract— **We address the following question: given a double integrator and a linear control that stabilizes it exponentially, is it possible to use the *same* control input in the case that the control input is multiplied by a time-varying term? Such question has many interesting motivations and generalizations: 1) we can pose the same problem for an input gain that depends on the state and time; 2) the stabilization –with the same method– of chains of integrators of higher order than two is fundamentally more complex and has applications in the stabilization of driftless systems; 3) the popular backstepping method stabilization method for systems with non-invertible input terms. The purpose of this note is two-fold: we present some open questions that we believe are significant in time-varying stabilization and present some preliminary answers for a simple, yet challenging case-study. Our solutions are stated in terms of persistency of excitation.**

I. Open questions and motivations

A. Linear time-varying systems

Let us consider the system $\dot{x} = u$ with $x \in \mathbb{R}$. It is evident that $u = u^*$ with $u^* = -x$ stabilizes the system exponentially. What can be concluded for the integrator with time-varying gain,

$$\dot{x} = g(t)u\,? \tag{1}$$

Which conditions need to be imposed on $t \mapsto g(t)$ so that (1) in closed loop with the *same* u^* be exponentially stable?

The answer to this question can be found in the literature on identification and adaptive control. For instance, from the seminal paper [11] we know that for the system

$$\dot{x} = -P(t)x \tag{2}$$

with $x \in \mathbb{R}^n$, $P \geq 0$ piecewise continuous bounded, and with bounded derivative, it is *necessary and sufficient*, for global exponential stability, that P also be *persistently exciting* (PE), *i.e.*, that there exist $\mu > 0$ and $T > 0$ such that

$$\int_t^{t+T} \xi^\top P(\tau)\xi \geq \mu \tag{3}$$

for all unitary vectors $\xi \in \mathbb{R}^n$ and all $t \geq 0$.

The immediate conclusion for the system of interest, *i.e.* (1), is that $u^* = -x$ remains a globally exponentially stabilizing control law if $g(\cdot)$ is non-negative, globally Lipschitz, locally integrable and PE. An interpretation of the stabilization mechanism can be given, in this case, in terms of an "average". Roughly speaking, one can dare say that even though it is not the control action u^* that enters the system for each t, this "ideal" control does drive the system "*in average*". For illustration, let $g(t) := \sin(t)^2$ then, the control action $u = -\sin(t)^2 x$ which, in average[2] corresponds to $u = -\frac{1}{2}x$ is tantamount to applying $u^* = -x$, modulo a gain-scale that only affects the rate of convergence but not the stabilization property of u^*.

Of course the previous naive thinking relies largely on the fact that we are dealing with a scalar system. Consider the higher-order integrator $\dot{x}^{(n)} = u$; in the state-space it takes the form:

$$\dot{x}_1 = x_2 \tag{4a}$$
$$\vdots$$
$$\dot{x}_i = x_{i+1} \tag{4b}$$
$$\vdots$$
$$\dot{x}_n = u\,. \tag{4c}$$

It is evident that, for a proper choice of k_i, we have that the control $u = u^*$ with $u^* = -\sum_{i=1}^n k_i x_i$ renders the closed-loop system globally exponentially stable. Consider the following:

Question 1 Does $u = g(t)u^*$ with g, $\dot{g}$ continuous and bounded and g PE, stabilize (4) ?

Intuitively one may think that the global exponential stability (GES) of the closed loop is guaranteed, at least, for

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[2]We have taken as average of $g(t)$, the function $\frac{1}{T}\int_0^T g(t)dt$, applied to $\sin(t)$ with $T = \pi$.

a more restricted choice of k_i and a particular class of PE functions g.

While for the case of $n = 1$ the answer is positive, the general case of $n > 1$ is fundamentally different and, in view of the available tools from the literature of adaptive control, a proof (or disproof) of the conjecture above is far from evident.

An extension of Question 1 concerns the analysis of chain-form systems; more precisely when $\dot{x}_i = \phi(t)x_{i+1}$ for all $i < n$ and $\dot{x}_n = u$ with $\phi(t)$ bounded and with bounded (continuous) derivative. This problem is not of pure theoretical interest but finds motivations in the control of nonholonomic systems, both in trajectory tracking and set-point stabilization. See the thorough discussions in [5], [7], [13]. For instance, it has been shown in [7] that such system in closed loop with $u = -k_1\phi(t)x_1 - k_2x_2 - k_3\phi(t)x_3 - \ldots$, and an appropriate choice of the gains k_i, is uniformly globally exponentially stable provided that $\phi(t)$ is PE, bounded and with bounded derivative. Implicitly, a similar condition has been used in [13] to show (non-uniform) asymptotic stability. Further generalizations following similar guidelines have been obtained for chains of nonholonomic integrators in [8].

Further natural extensions of these stabilization problems for chains of integrators concern the stabilization of systems with drift. Consider the system

$$\dot{x} = Ax + B(t)u\,, \tag{5}$$

where A is marginally stable, and the system $\dot{x} = Ax + \mathcal{B}u$ with $\mathcal{B}$ constant . One may pose the question: under which conditions on $B(t)$ and possibly on the pair $(A, \mathcal{B})$, does the control $u = u^*$ with $u^* = -Kx$ such that $\dot{x} = (A - \mathcal{B}K)x$ is GES, stabilize (5) exponentially. In the particular case that A is skewsymmetric, the answer seems at hand in the form of a PE condition on B; for instance, we may impose that $P(t) := B(t)K$ satisfy all the conditions from [11] and follow a similar reasoning.

Consider now the following problem.

Question 2 Let A be *unstable*, is (5) globally exponentially stabilizable by linear time-invariant feedback $u^* = -B(t)^\top Kx$, provided that $B(t)$ is PE? If not in general, is it true at least for particular choices of the gain K and PE functions B ?

In this note we address Question 2 for the particular case of the double integrator, cf. Section II-A.

B. Nonlinear time-varying systems

1) Nonlinear drift

An interesting nonlinear example that may be viewed as a generalization of system (5) appears in the control of spacecrafts with magnetic actuators, where the dynamic system has the form:

$$\dot{\omega} = S(\omega)\omega + g(t)u\,,$$

where S designates the following matrix:

$$S = \begin{bmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ -\omega_1 & \omega_2 & 0 \end{bmatrix}$$

and $g(t)$ is a time-varying matrix which is rank deficient (*i.e.* $\mathrm{rank}\{g(t)\} < 3$) for any fixed t but it is PE. The global exponential stabilization problem for the spacecraft with magnetic actuators was solved in [2] using PE arguments.

2) Nonlinear input gain

Let us consider again the simple integrator $\dot{x} = u$. Assume now that the integrator has an input gain that depends on the state and time, *i.e.*, consider the one-dimensional driftless system

$$\dot{x} = \phi(t, x)u \tag{6}$$

where $\phi : \mathbb{R}_{\geq 0} \times \mathbb{R} \to \mathbb{R}_{\geq 0}$ is locally Lipschitz in x uniformly in t and piecewise continuous in t. A natural question is whether the control input $u^* = -x$, which stabilizes the pure integrator exponentially, still stabilizes the "integrator" (6). Further, is this true for other stabilizing controls $u^*(t, x)$?

Following up the results from [10] we obtain that, when $F(t, x) := \phi(t, x)u^*(t, x)$ is continuous and locally Lipschitz in x uniformly in t, any control input $u = u^*(t, x)$ that stabilizes $\dot{x} = u$ also stabilizes *uniformly* asymptotically the system (6) only if $F(t, x)$ is Uδ-PE with respect to x; we recall the definition of the latter below. Sufficiency also holds, under certain regularity conditions, if $\dot{x} = F(t, x)$ is uniformly globally stable (UGS).

For the system $\dot{x} = F(t, x)$ define $x =: \mathrm{col}[x_1, x_2]$, correspondingly $n =: n_1 + n_2$, and the set $\mathcal{D}_1 := (\mathbb{R}^{n_1} \setminus \{0\}) \times \mathbb{R}^{n_2}$.

Definition 1 A function $\phi(\cdot, \cdot)$ where $t \mapsto \phi(t, x)$ is locally integrable, is said to be uniformly δ-persistently exciting (Uδ-PE) with respect to x_1 if for each $x \in \mathcal{D}_1$ there exist $\delta > 0$, $T > 0$ and $\mu > 0$ s.t. $\forall t \in \mathbb{R}_{\geq 0}$,

$$|z - x| \leq \delta \quad \Longrightarrow \quad \int_t^{t+T} |\phi(\tau, z)|\, d\tau \geq \mu\,. \tag{7}$$

□

Thus, if we impose that $\phi \geq 0$, from the results in the cited references we conclude that, in particular, the control $u^* = -x$ makes (6) uniformly globally asymptotically stable (UGAS) if and only if $\phi(t,x)$ is Uδ-PE with respect to x. We stress that in this case, *exponential* stability is out of reach —see [9], that is, the property that one had for the system with constant input gain is lost when the input gain is nonlinear time-varying.

Question 3 Let us consider now a system with drift and an additive input, *i.e.*

$$\dot{x} = f(t,x) + u, \qquad x \in \mathbb{R}^n \tag{8}$$

which is stabilizable by the obvious feedback $u^*(t,x) = -x - f(t,x)$. Does (8) remain (at least) UGAS under the same feedback, when the control is multiplied by a non-invertible nonlinear gain $g(t,x)$?

A motivation to address Question 3 is that the system (8) or, more generally,

$$\dot{x} = f(t,x) + g(t,x)u \tag{9}$$

with $g(t,x)$ *invertible* for all t and x (hence, feedback linearizable) appears in numerous applications such as control of mechanical systems. For instance, the Lagrangian model has the form (9) where $g(t,x)$ corresponds to $[0 \;\; D^{-\top}]^\top$ and D is the positive definite inertia matrix. Hence one may apply, for instance, $u = g(t,x)^{-1}u^*(t,x)$ where $u^*(t,x)$ stabilizes (9). For the case when $g(t,x)$ is not invertible (non-feedback linearizable systems), one may ask the following:

Question 4 Given $u^*(t,x)$ such that $\dot{x} = f(t,x) + u^*(t,x)$ is UGAS, does the control $u = g(t,x)^\top u^*(t,x)$ with $g(t,x)$ Uδ-PE, stabilize (9)? If not true in general, which extra conditions on the Uδ-PE function $g(t,x)$ must be imposed?

It seems reasonable to conjecture that the answer to Question 4 is true for a particular subclass of Uδ-PE functions. This subclass should be determined by quantifying the "amount of PE" needed, *i.e.*, determining lower positive bounds for μ and T in (7). To illustrate this, consider the case $x \in \mathbb{R}$. Let $u^* = -f(t,x) - kx$ as suggested above; the control $u = g(t,x)u^*(t,x)$ applied to (9) yields

$$\dot{x} = -kg(t,x)^2x + f(t,x)[1 - g(t,x)^2]\,.$$

Assume that $g(t,x)$ is Uδ-PE, bounded and with continuous bounded first derivatives. Then, UGAS of $\dot{x} = -kg(t,x)^2x$ may be asserted. So we are interested in establishing conditions on k and the "degree of excitation" of g (size of μ) so that $\dot{x} = -kg(t,x)^2x$ remains robust with respect to the perturbation $K(t,x) := f(t,x)[1 - g(t,x)^2]$

A similar question arises for *partial* feedback linearizable systems. A particular situation in control practice where this is significant is the control of underactuated systems. Suppose that the architecture of the system is such that one has an extra "degree of freedom" in imposing a time-varying redistribution of the control signals. More explicitly, consider a Lagrangian system:

$$\begin{bmatrix}\dot{q}\\ \ddot{q}\end{bmatrix} = \begin{bmatrix}\dot{q}\\ D(q)^{-1}[-C(q,\dot{q}) - g(q)] + D(q)^{-1}S(t,q)u\end{bmatrix} \tag{10}$$

where $q \in \mathbb{R}^n$ and $\dot{q}$ are the generalized coordinates, D is the (invertible) inertia matrix, C corresponds to the Coriolis and centrifugal forces matrix, g represents the gravitational forces vectors, $u \in \mathbb{R}^n$ with $m < n$ entries different from zero and $S(t,q)$ is an input matrix that is rank deficient for each t and q. One may think of $S(t,q)$ as the implementation of an algorithm that decides, whenever physically possible, how to distribute the m available input torques through the n degrees of freedom, according to both a posture-independent (hence time-varying) schedule and a criterion depending on the system's configuration. In such case, does the control law $u = S(t,q)^\top D(q)^{-\top}u^*(t,x)$, where u^* stabilizes (10) in the case that $S \equiv I$, stabilize (10)?

The system (9) is also at the basis of the popular backstepping control method (cf. [4]) for systems in strict-feedback form:

$$\begin{aligned}
\dot{x}_1 &= f_1(t,x_1) + g_1(t,x_1)x_2 && \text{(11a)}\\
\dot{x}_2 &= f_2(t,x_1,x_2) + g_2(t,x_1,x_2)x_3 && \text{(11b)}\\
&\;\;\vdots\\
\dot{x}_i &= f_i(t,x_1,\ldots,x_i) + g_i(t,x_1,\ldots,x_i)x_{i+1} && \text{(11c)}\\
&\;\;\vdots\\
\dot{x}_n &= f_n(t,x_1,\ldots,x_n) + g_n(t,x_1,\ldots,x_n)u\,. && \text{(11d)}
\end{aligned}$$

Following a classical backstepping control, one would need to assume that each $g_i(t,x)$ is invertible; alternatively, one needs to study case by case, in order to design a control input u and the intermediary "virtual controls" v_i^*, such that each subsystem, at each step of integration, is UGAS. Thus, a natural generalization of Question 4 is the following:

Question 5 Assuming that each $g_i(t,x_1,\ldots,x_i)$ is invertible design, recursively, a control u^* and virtual controls v_i*, that uniformly asymptotically stabilize the origin of (11). Replacing g_i^{-1} with g_i, in each expression of the so-obtained u^* and v_i^*, define the new control law u and virtual controls v_i. Does this control still stabilize the origin under the condition that each $g_i(t,x_1,\ldots,x_i)$ is (sufficiently) Uδ-PE?

II. SOME ANSWERS FOR THE DOUBLE INTEGRATOR

Consider Question 2 for the double integrator, *i.e.*

$$\dot{x}_1 = x_2 \tag{12a}$$
$$\dot{x}_2 = \alpha(t)u\,. \tag{12b}$$

In other words, consider the equation with unstable drift,

$$\dot{x} = A_0 x + \alpha(t) b u$$

with

$$A_0 := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad b := \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

We assume that α is bounded, measurable, non-negative and PE that is, there exist $\alpha_M > 0$, μ and T such that

$$\int_t^{t+T} \alpha(\tau)d\tau \geq \mu\,, \qquad \forall\, t \geq 0 \tag{13}$$
$$|\alpha|_\infty := \sup_{t\geq 0} |\alpha(t)| \leq \alpha_M\,.$$

Under the conditions above, we design for the system (12) a control u that renders the closed-loop system uniformly globally exponentially stable. The first solution requires the boundedness of $\dot{\alpha}$ however, it follows a methodical backstepping procedure; for this reason, the first solution is a (modest) step forward in the solution of the problem posed in Question 5. The second solution does not involve a condition on the boundedness of $\dot{\alpha}$ however, it largely exploits the properties of the double integrator, *i.e.*, it is case-dependent.

We believe that both solutions are interesting in themselves and may open new trains of thought to find answers to the more general questions raised above.

A. Linear system

1) First solution

Proposition 1 Let $\alpha(t)$ be bounded, PE, differentiable and have a bounded derivative. Then the system (12) in closed loop with

$$u := -2\dot{\alpha}(t)x_1 - \alpha(t)x_2 - \alpha(t)[x_2 + \alpha(t)^2 x_1] \tag{14}$$

is uniformly globally exponentially stable.

Proof . The proof is constructive: we follow a classical backstepping procedure. To that end we first notice that, from (13)

$$\dot{z} = -\alpha(t)^2 z$$

is uniformly globally exponentially stable. Hence, by [3, Theorem 3.10], given $q > 0$, there exists a solution $p(t)$ of

$$\dot{p}(t) = 2p(t)\alpha(t)^2 - q \tag{15}$$

such that $p_M \geq p(t) \geq p_m > 0$. Consider the positive definite function $V_1(t,x) = x_1^2 p(t)$ and the virtual control $v = -\alpha(t)^2 x_1$ for (12a). The total derivative of $V_1(t,x)$ along the trajectories of $\dot{x}_1 = -\alpha(t)^2 x_1 + (x_2 + \alpha(t)^2 x_1)$, using (15), yields

$$\dot{V}_1(t,x) = -q x_1^2 + 2x_1 p(t)[\alpha(t)^2 x_1 + x_2]\,.$$

Consider now $V(t,x) = \frac{1}{2p_M^2} V_1(x,t) + (x_2 + \alpha(t)^2 x_1)^2 p(t)$. Since $p(t)$ and $\alpha(t)$ are bounded there exist positive constants γ_1 and γ_2 such that

$$\gamma_1 \|x\|^2 \leq V(x,t) \leq \gamma_2 \|x\|^2\,. \tag{16}$$

The total derivative of V along the trajectories of (12) yields

$$\begin{aligned}\dot{V}(t,x) = &-\frac{q}{2p_M^2}x_1^2 + \frac{1}{p_M^2}x_1 p(t)[x_2 + \alpha(t)^2 x_1] \\ &+ 2\alpha(t)p(t)[x_2 + \alpha(t)^2 x_1][u + 2\dot{\alpha}(t)x_1 + \alpha(t)x_2] \\ &+ [x_2 + \alpha(t)^2 x_1]^2 \dot{p}(t)\end{aligned}$$

Replacing u from (14) we obtain that

$$\dot{V} \leq -\frac{1}{2}\begin{bmatrix} x_1 \\ \alpha^2 x_1 + x_2 \end{bmatrix}^\top \begin{bmatrix} \frac{q}{p_M^2} & -\frac{p(t)}{p_M^2} \\ -\frac{p(t)}{p_M^2} & 2q \end{bmatrix} \begin{bmatrix} x_1 \\ \alpha^2 x_1 + x_2 \end{bmatrix}, \tag{17}$$

where the argument of α was omitted. This implies, in view of (16), that for any $q > \sqrt{2}$ there exist c_1 and $c_2 > 0$, independent of the initial conditions, such that

$$\int_{t_\circ}^{\infty} |x_1(t)|^2\, dt \leq c_1 \,|x_\circ|^2 \tag{18a}$$

$$\int_{t_\circ}^{\infty} \left|\alpha(t)^2 x_1(t) + x_2(t)\right|^2 dt \leq c_1 \,|x_\circ|^2 \tag{18b}$$

$$\max\left\{|x_1|_\infty\,,\, |x_2|_\infty\right\} \leq c_2 \,|x_\circ|\,. \tag{18c}$$

It also follows from (18) and the bound on α that

$$\int_{t_\circ}^{\infty} |x_2(t)|^2\, dt \leq 2c_1(1 + \alpha_M^4)\,|x_\circ|^2\,.$$

Indeed, by the identity $(a+b)^2 \leq 2(a^2+b^2)$, we have that

$$\begin{aligned} |x_2(t)|_2 &= \left|x_2(t) + \alpha(t)^2 x_1 - \alpha(t)^2 x_1(t)\right|_2 \\ &\leq 2\left|\alpha(t)^2 x_1(t)\right|_2 + 2\left|x_2(t) + \alpha(t)^2 x_1(t)\right|_2 \\ &\leq 2(1+\alpha_M^4)c_1\,|x_0|^2\,. \end{aligned}$$

Uniform global exponential stability of the origin follows from [12, Lemma 3]. ■

Remark 1 From the proof of [12, Lemma 3] it is evident that one can also derive an explicit exponential estimate. □

2) Second solution

The second solution does not require boundedness of $\dot{\alpha}$ but uses a normalized control input. The sufficient condition is expressed in terms of the richness of α.

Proposition 2 There exist $k_1, k_2 > 0$ such that, for any given $\alpha(t)$ sufficiently rich, *i.e.*, PE and such that[3]

$$\mu \geq 0.7\, T\alpha_M \,, \tag{19}$$

there exists $\varepsilon > 0$ such that the control law

$$u := -\frac{\alpha(t)}{\alpha(t)+\varepsilon}\,(k_1 x_1 + k_2 x_2) \tag{20}$$

renders the closed-loop system, with (12), UGES.

Remark 2 Conversely, given any $k_1, k_2 > 0$, there exist $\varepsilon > 0$ and a persistently exciting signal $\alpha(t)$ such that (12) in closed-loop with u as defined above is UGES. It suffices indeed to notice that any positive constant (and therefore PE) is convenient.

Proof of Proposition 2. Let k_1 and k_2 be two positive constants to be fixed later. The system (12) in closed loop with (20) can be written

$$\dot{x} = Ax + \frac{\varepsilon}{\alpha(t)+\varepsilon} bc^\top x \,, \tag{21}$$

where

$$A := \begin{pmatrix} 0 & 1 \\ -k_1 & -k_2 \end{pmatrix},\; b := \begin{pmatrix} 0 \\ 1 \end{pmatrix},\; c := \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}.$$

In the spirit of the discussion that succeeds Question 4, the rationale of the proof consists in quantifying the gains k_1, k_2, related to the richness of $\alpha(t)$ so that $\dot{x} = Ax$ is robust with respect to the "perturbation" $\frac{\varepsilon}{\alpha(t)+\varepsilon} bc^\top x$.

Since A is Hurwitz, there exists a constant positive symmetric matrix P such that

$$A^\top P + PA = -I \,. \tag{22}$$

Let $V(x) := x^\top P x$. Then, its total time derivative along the solutions of (21) yields

$$\begin{aligned} \dot{V} &= x^\top (A^\top P + PA)x + \frac{\varepsilon}{\alpha(t)+\varepsilon} x^\top (Pbc^\top + cb^\top P)x \\ &\leq -|x|^2 + \frac{2\varepsilon\lambda_M}{\alpha(t)+\varepsilon}|x|^2 \,, \end{aligned}$$

where

$$\lambda_M := \max\left\{|\lambda| \mid \lambda \in \mathrm{sp}(Pbc^\top + cb^\top P)\right\} \,.$$

Define

$$\begin{aligned} p_m &:= \min\left\{\lambda \mid \lambda \in \mathrm{sp}(P)\right\} \,, \\ p_M &:= \max\left\{\lambda \mid \lambda \in \mathrm{sp}(P)\right\} \,, \end{aligned}$$

we then get that $p_m|x|^2 \leq V(x) \leq p_M |x|^2$. Hence

$$\dot{V}(x) \leq -\left(\frac{1}{p_M} - \frac{2\lambda_M}{p_m}\frac{\varepsilon}{\alpha(t)+\varepsilon}\right)V(x) \,. \tag{23}$$

For any $x_\circ \in \mathbb{R}^2$ and any $t_\circ \in \mathbb{R}_{\geq 0}$, let $x(t, t_\circ, x_\circ)$ designate the trajectory[4] of (21) starting in $x_\circ$ at $t = t_\circ$, and use the shorthand notation $v(t)$ for $V(x(t, t_\circ, x_\circ))$. Integrating (23) we obtain that, for any $t \geq t_\circ$,

$$v(t+T) \leq v(t)\exp\left[\frac{2\lambda_M}{p_m}\left(-\frac{p_m T}{2\lambda_M p_M} + \nu(t)\right)\right] \tag{24}$$

where

$$\nu(t) := \int_t^{t+T} \frac{\varepsilon}{\alpha(\tau)+\varepsilon} d\tau \,, \quad \forall t \geq t_0 \,.$$

The rest of the proof consist in rendering negative the argument in the exponential. To that end, we design an upper bound for $\nu(t)$. Given any $\rho > 0$, define the following set

$$I_{t,\rho} := \{\tau \in [t; t+T] \mid \alpha(\tau) \geq \rho\} \,,$$

and let $\overline{I}_{t,\rho}$ designate its complementary in $[t; t+T]$. We then have that

$$\begin{aligned} \nu(t) &= \int_{I_{t,\rho}} \frac{\varepsilon\, d\tau}{\alpha(\tau)+\varepsilon} + \int_{\overline{I}_{t,\rho}} \frac{\varepsilon\, d\tau}{\alpha(\tau)+\varepsilon} \\ &\leq \frac{\varepsilon}{\rho+\varepsilon}\int_{I_{t,\rho}} d\tau + \int_{\overline{I}_{t,\rho}} d\tau \\ &\leq T - \frac{\rho}{\rho+\varepsilon}\mathrm{meas}(I_{t,\rho}) \,. \end{aligned}$$

Based on the proof of [6, Lemma 2] we can bound $\mathrm{meas}(I_{t,\rho})$ from below. Indeed, from (13), we have that

$$\int_{I_{t,\rho}} \alpha(\tau)d\tau + \int_{\overline{I}_{t,\rho}} \alpha(\tau)d\tau \geq \mu \,.$$

Hence

$$\mu - \int_{\overline{I}_{t,\rho}} \alpha(\tau)d\tau \leq \alpha_M \mathrm{meas}(I_{t,\rho}) \,.$$

But, for all $\tau \in \overline{I}_{t,\rho}$, we have that $\alpha(\tau) < \rho$, so

$$\mu - \rho\,(T - \mathrm{meas}(I_{t,\rho})) \leq \alpha_M \mathrm{meas}(I_{t,\rho}) \,,$$

which yields, for all $0 < \rho < \alpha_M$,

$$\mathrm{meas}(I_{t,\rho}) \geq \frac{\mu - \rho T}{\alpha_M - \rho} \,.$$

Therefore,

$$\nu(t) \leq T - \frac{\rho}{\rho+\varepsilon}\frac{\mu - \rho T}{\alpha_M - \rho} \,. \tag{25}$$

Notice that, as ε and ρ go to zero, $\frac{\rho}{\rho+\varepsilon}\frac{\mu-\rho T}{\alpha_M-\rho}$ tends to $\frac{\mu}{\alpha_M}$. In addition, since $\mu \leq T\alpha_M$, we have that

$$\frac{\rho}{\rho+\varepsilon}\frac{\mu - \rho T}{\alpha_M - \rho} \leq \frac{\mu}{\alpha_M} \,, \quad \forall \varepsilon, \rho > 0 \,.$$

[3] Notice that, in view of (13), we necessarily have that $T\alpha_M \geq \mu$.

[4] The trajectories are unique since the right-hand side term of (21) is locally Lipschitz in x and piecewise continuous in t.

Therefore, for any $\varepsilon_1 > 0$, there exists $\varepsilon, \rho > 0$ such that

$$\frac{\rho}{\rho+\varepsilon}\frac{\mu-\rho T}{\alpha_M-\rho} = \frac{\mu}{\alpha_M} - \varepsilon_1\,,$$

and we get, with (25), that

$$\nu(t) \le T - \frac{\mu}{\alpha_M} + \varepsilon_1\,. \tag{26}$$

On the other hand, after a few calculations, whose details are voluntarily omitted here, we have that the positive symmetric matrix

$$P = \frac{1}{2k_1k_2}\begin{pmatrix} k_1+k_1^2+k_2^2 & k_2 \\ k_2 & 1+k_1 \end{pmatrix}$$

satisfies (22). Its eigenvalues are

$$p_m = \frac{(1+k_1)^2+k_2^2-\sqrt{(k_1^2+k_2^2)^2+2(k_2^2-k_1^2)+1}}{4k_1k_2}$$

$$p_M = \frac{(1+k_1)^2+k_2^2+\sqrt{(k_1^2+k_2^2)^2+2(k_2^2-k_1^2)+1}}{4k_1k_2}\,.$$

In addition, the eigenvalues of $Pbc^\top cb^\top P$ are 0 and

$$\lambda_M^2 = \left(1+\frac{1}{2k_1}\right)^2\,.$$

From this, by picking for instance $k_1 = 1$ and $k_2 = 1/10$, it is easy to check that the quantity $\frac{p_m}{2\lambda_M p_M}$ is greater than 0.3. Hence, for any signal α satisfying (19), we have that

$$\frac{\mu}{T\alpha_M} > 1 - \frac{p_m}{2\lambda_M p_M}\,. \tag{27}$$

and therefore there exists a positive ε_2 such that

$$\frac{\mu}{\alpha_M} = T - \frac{p_m T}{2\lambda_M p_M} + \varepsilon_2\,. \tag{28}$$

Pick $\varepsilon_1 = \frac{\varepsilon_2}{2}$. We then get from (26) that

$$\nu(t) \le \frac{p_m T}{2\lambda_M p_M} - \frac{\varepsilon_2}{2}\,.$$

Consequently, we have that there exists $\varepsilon_3 > 0$ such that

$$\exp\left[\frac{2\lambda_M}{p_m}\left(-\frac{p_m T}{2\lambda_M p_M} + \nu(t)\right)\right] = 1 - \varepsilon_3\,,$$

and (24) yields

$$v(t+T) - v(t) \le -\varepsilon_3 p_m\,|x|^2\,.$$

Hence, the requirements of [1, Theorem 1, Remark 4] are satisfied, which establishes the UGAS of (21). In view of [3, Theorem 3.9], it is also UGES. ∎

We underline the fact that for an integrator of degree higher than 2, the first approach becomes heavier since it requires an iterative computation of the control law. As for the second approach, the main part of the previous reasoning remains valid, and we are left with condition (27), which does not depend on the dimension of the system. However, this bound may happen to be more and more conservative as the degree increases, leading to a more and more exciting signal. In addition, it may not be possible to explicitly give a condition as (19), *i.e.* based only on the signal $\alpha(t)$, due to the increasing difficulty of explicitly expressing p_m, p_M and λ_M.

III. CONCLUSIONS

We have presented a series of open questions in the stabilization of time varying systems, via persistency of excitation. These questions can be summarized as follows: can a system that has a non-invertible input term be stabilized by the control input that would be used in the case the input term were the identity (or invertible), multiplied by the input term, under a PE condition? We have presented several particular forms of this question and given examples of situations when they are significant.

We also presented two results that address the posed problem for the double integrator with a PE input term. The first is methodical and possibly sets a step forward towards the solution to Question 5. The second requires less regularity conditions on the input term and conserves, up to a normalization, better the ideal control that would be employed if the input term were constant.

REFERENCES

[1] D. Aeyels and J. Peuteman. A note on the liapunov approach to the study of asymptotic stability of differential equations. In *Proc. 4th. European Contr. Conf.*, 1997. paper no. 226.

[2] A. Astolfi and M. Lovera. Global spacecraft attitude control using magnetic actuators. In *Proc. American Control Conference*, pages 1331–1336, 2002.

[3] H. Khalil. *Nonlinear systems*. Macmillan Publishing Co., 2nd ed., New York, 1996.

[4] M. Krstić, I. Kanellakopoulos, and P. Kokotović. *Nonlinear and Adaptive control design*. John Wiley & Sons, Inc., New York, 1995.

[5] A. A. J. Lefeber. *Tracking control of nonlinear mechanical systems*. PhD thesis, University of Twente, Enschede, The Netherlands, 2000.

[6] A. Loría and E. Panteley. Uniform exponential stability of linear time-varying systems:revisited. *Syst. & Contr. Letters*, 47(1):13–24, 2002.

[7] A. Loría E. Panteley, and K. Melhem. UGAS of "skew-symmetric" time-varying systems: application to stabilization of chained form systems. *European J. of Contr.*, 8(1):33–43, 2002.

[8] A. Loría E. Panteley, D. Popović, and A. R. Teel. Matrosov's theorem using a family of auxiliary functions: an analysis tool to aid time-varying nonlinear control design. eprint no. arXiv:math.OC/0304255, April 2003. Available from `http://front.math.ucdavis.edu/math.OC/0304255`.

[9] A. Loría E. Panteley, D. Popović, and A. R. Teel. Persistency of excitation for uniform convergence in nonlinear control systems. eprint no. arXiv:math.OC/0301335, Jan 2003. Available from `http://front.math.ucdavis.edu/math.OC/0301335`.

[10] A. Loría E. Panteley, D. Popović, and A.R. Teel. A nested matrosov theorem and persistency of excitation for uniform convergence in stable nonautonomous systems. *IEEE Trans. on Automat. Contr.*, 2005.

[11] A. P. Morgan and K. S. Narendra. On the stability of nonautonomous differential equations $\dot{x} = [A + B(t)]x$ with skew-symmetric matrix $B(t)$. *SIAM J. on Contr. and Opt.*, 15(1):163–176, 1977.

[12] E. Panteley, A. Loría and A. Teel. Relaxed persistency of excitation for uniform asymptotic stability. *IEEE Trans. on Automat. Contr.*, 46(12):1874–1886, 2001.

[13] C. Samson. Control of chained system: Application to path following and time-varying point stabilization of mobile robots. *IEEE Trans. on Automat. Contr.*, 40(1):64–77, 1995.