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The Stock Markets as an Ineffective Sampler

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Abstract— **The ability to forecast the trend of the stock markets is of obvious importance. A lot of effort has been devoted at modelling the stock markets as deterministic or stochastic dynamic systems. In this paper the stock markets are view as a sampler for the continuous-time price dynamics and its effectiveness will be discussed. It will be shown that data extrapolated from stock markets are corrupted by noise and, in most cases, it appears that no useful information can be extracted from such data.**

I. Introduction

The possibility to forecast market trends stimulates a lot of effort of the researchers with different backgrounds: economists, psychologists, mathematicians tried to find a technique, or a model, to understand the stocks' price variations. Unfortunately, while most of the methods worked *a-posteriori*, there is no evidence of a successful *a-priori* approach. On the other side, sometimes researchers search for models that can *reproduce* the price behavior, even if this model can not be used for forecast purposes.

The first idea of Efficient Market is given in the Master Thesis of Louis Bachelier, a Poincaré's student, in 1900. At the time, this theory was quite ignored; it was then re-proposed, among the others, by Fama in [3] and, in 1973, by the popular work of Malkiel [11], further presented with the name of Random Walk Theory, strictly related to the Efficient Market Theory that models the price variations as an in-correlated random variable and the price, thus, as a random walk model, or a Brownian model. With a concise expression those theories state that *markets discount everything*. Recently, Lo and Mac Kinlay [10], propose a book where the existence of correlation among data is deeply studied. It is worth noticing that few researchers now claim the existence of a pure random walk model [11]. A divulgative point of view of a mathematician is given in [17]. The work by Mandelbrot [12], [13] tries to implement the fractal mathematics to stock markets. The classical financial models, among them the portfolio theory, are critiqued since they do not represent the real markets behavior. It is shown that multifractals can better reproduce the behavior of stocks' prices. It is worth noticing that this approach is structurally discrete, in fact, the Author does not give so much importance in the sample time selection: *"The intervals themselves are chosen arbitrarily; they may represent a second, an hour, a day or a year"* [12]. It is worth noticing that this approach tries to find a model that reproduces the stock markets behavior rather than to predict their evolution.

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In the book written by Murphy [15] the fundament for the Technical Analysis is illustrated; among the most popular chart-based techniques, the Japanese Candlestick, known from some century, applied also to the stock markets data. It is worth noticing that, in general, the Technical Analysis theory does not accept the central role of randomness, in fact, quoting from [15] *"The illusion of randomness gradually disappears as the skill in chart reading improves"*.

The possibility to use stock markets information on a fixed, e.g., daily, basis in used by [2], under the framework of the Information Theory, in order to efficiently exploit eventual investment information. The work [20] uses the concept of non uniform information availability in order to proposes a continuous-time model. Reference [18] uses the ARFIMA model of difference equation to extract information about the Japanese stock market.

The case of *computational finance models*, that, as pointed out by [6], are different from stock markets models will not be considered in this paper. Those are mathematical tools useful to, e.g., determine the price of a financial derivative.

Citing from the work of Lo and Mac Kinlay [10], *"…the overwhelming rejections of the Random Walk Hypothesis that we obtained for weekly US stock returns from 1962 to 1985 implied that a more powerful test was not needed– the random walk could have been rejected on the basis of the simple first-order autocorrelation coefficient, which we estimated to be 30 % for the equal-weighted weekly returns index!…At first, we attributed this to our using weekly returns–priori studies used either daily or monthly. We chose a weekly sampling interval to balance the desire for a large sample size against the problems associated with high-frequency financial data, e.g., non-synchronous prices, bid/ask bounce, etc."*. In this paper a tentative answer to the problems raised by these Authors will be given from a different perspective. The stock markets are considered simply as a sampler of the continuous-time dynamics of the stocks' price and the consequence of this approach will be discussed. It will be shown that the obtained data are structurally corrupted by different sources of noise.

II. Modelling

A. Continuous-time or discrete-time?

A wide literature has modelled the stock trend by resorting to discrete-time dynamic systems, both deterministic or stochastic ([8], [10], [11], [14], [18]). The reason for this choice is obvious since the data are discrete. However, why the dynamics of the stock markets should be discrete-time-based is not clear; in fact, evidence shows an opposite behavior. Just to give few examples, most of the *AdCom* meetings

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are settled during the week-ends, i.e., intentionally when the markets are closed; moreover, globalization causes events to happen in another part of the world when, again, the markets are closed, and a consequence to the markets is expected.

Moreover, discrete-time modelling is implicit in a long series of methods to forecast stock values; examples are the Japanese Candlestick, and all the chart-based methods, i.e., the methods that imply the graphical representation of the data [15], the study of the complexity [17], the Chaos [12], etc.

Another possible approach is to model the stock price in the continuous time; some well known models have been proposed such as the one of Bachelier, dated 1900, Samuelson [21] or, recently, Shepp [20]. In Bachelier's model the price $p(t)$ varies following

$$p(t) = p(0) + \mu t + \sigma w(t) \qquad t \geq 0, \tag{1}$$

where $p(t)$ is the price, $p(0)$ its initial value, $w(t)$ is a Brownian motion process and μ and σ represent the drift and the volatility, respectively. Samuelson, instead, proposes the following

$$p(t) = p(0) e^{\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma w(t)} \qquad t \geq 0. \tag{2}$$

In [20] the Samuelson's model is further elaborated by including also the existence of increased or nonuniform information by considering $\mu = \mu(\zeta(t))$ and $\sigma = \sigma(\zeta(t))$, in which $\zeta(t)$ is an hidden Markov model.

In the following only two assumptions are made:

- the stock price is an unknown continuous-time process;
- the markets are an observation of this process.

Using a systems' language, the markets are a sampler of the continuous time dynamics of a stock's price. It will be shown that the information eventually contained in the price time series is significantly corrupted.

B. Time invariance

In many studies of the stock market trends it is possible to observe very long time series, often several years or decades of data. It must be pointed out, however, that markets are obviously a time-varying process. Major and minor changes arise continuously in every company; a model that does not take into account these changes, thus, suffers from major drawbacks and can not produce reliable results. The *rate* of changes is obviously unknown and it is composed by abrupt changes, in response to a major change in the company/society, or slow changes, to adapt to slow modification of the environment; most models simply ignore this characteristic of the real world.

On the other side, the problem of time variance of the stock markets is well known from the researchers that, however, tried to find out some heuristic evidence for its influence on the overall model [3]. It seems very unlikely to consider the complex issues that arise with the time variance property of a model by simply making some heuristic comments. The stock markets are affected by several sources of non-modelled effects; time variance is just one of them. Since it is not possible to isolate these effects as in a laboratory-scale experiment, the only reasonable conclusion is that time variance strongly influences the stock markets but in an unknown way.

Finally, it is worth noticing that the time behavior of the systems is differently considered in two different domains. In the engineering community the data coming out from a time variant model are often weighted so that the recent data are taken in more consideration than the past [7]. The chartists, on the opposite, assume that *the past behavior of a security's price is rich in information concerning its future behavior* [3].

III. Data observation

Figure 1 represents a block diagram of the model of the price formation process. The price is considered as the output of an unknown continuous time and time varying model; its behavior is influenced by some unknown inputs and by the investors' behavior. Firstly, the signal is subject to the bid/ask gap, i.e., a quantization; then, a non uniform sampling is performed on the signal itself. This is the signal available to the intra-day investors that feed it back by their actions; this causes the quantization to be no longer a simple white noise additive signal and certainly not a simple transaction cost. When the trading day is over, the closing, or reference price is determined by resorting to known algorithms that vary from one market to another. Those data are stored as daily data and used by the investors to determine their decision thus feeding it back again. All these steps cannot be avoided and have as a consequence the *corruption* of the information eventually contained in the price behavior $p(t)$. In the following the single steps will be analyzed in detail.

A. Bid-Ask Gap/Quantization

The presence of the bid-ask gap causes the price signal to suffer from a quantization effect (see, e.g., [5] for a detailed discussion on the quantization). In the case of the stock's price, the quantization is present since the value can assume only finite-length levels; this is known as the spread, i.e., the difference between the bid and ask price in any stock's book. The spread is commonly considered as some kind of price to pay to play the game [11]; its effect, however, is deeper than a simple transaction cost.

The quantization can be modelled by resorting to different methods: among them, the stochastic approach. Given the value of the function as $p(t)$ it holds:

$$p_q(t) = p(t) + \varepsilon(t) \tag{3}$$

where $p_q(t)$ is the quantized value and $\varepsilon(t)$ is the quantization error. By preliminarily making the assumption that the quantization error is white, i.e., its samples are uncorrelated, it is possible to compute the mean value μ_ε and the standard

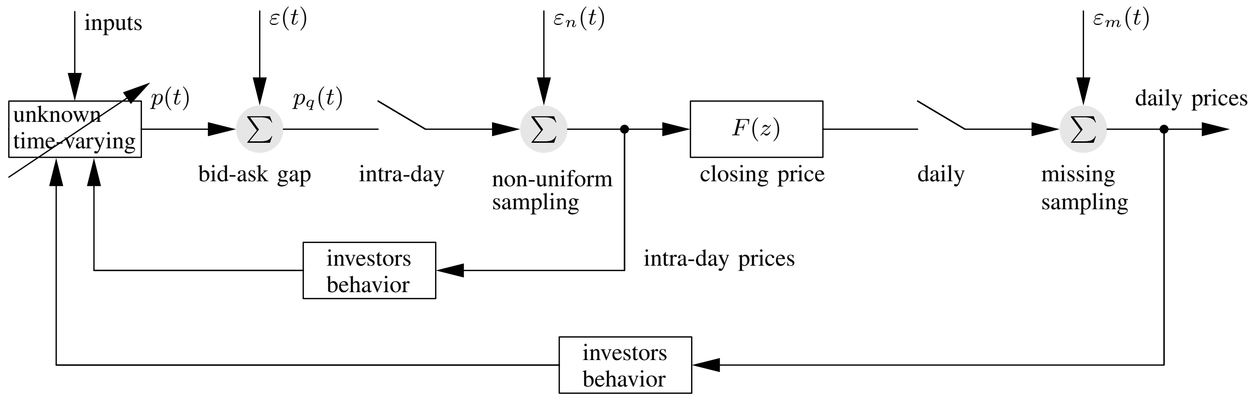

Fig. 1. Sketch of the effects of the observation of the prices. Corruption of the unknown, time-varying dynamics $p(t)$ is *structural* of the data acquisition process. Is the price an useful signal?

deviation σ_ε:

$$\begin{aligned}\mu_\varepsilon &= 0\\ \sigma_\varepsilon &= \frac{q}{2\sqrt{3}}.\end{aligned}$$

To have an idea of this effect let us consider the Fiat title belonging to the Milan Stock Exchange. In January, 2005 its price was ≈ 6 €, while the spread, or quantization, was 0.01 € ($\approx 0.16\,\%$). Considering the stochastic model, thus, a noise with standard deviation $\sigma_\varepsilon \approx 0.003$ € needs to be considered, i.e., a standard deviation of $0.05\,\%$ of the stock's price. As a consequence, since variations in a range of $\pm 3\sigma_\varepsilon$ can be considered due to the presence of the sole bid-ask gap, it means that variations of $\pm 0.15\,\%$ are information void.

In case of the warrants this effect is quite different from the simple stock; in fact, the strike price changes a lot during the option's life and can eventually decrease to zero. When approaching small values, thus, the effect of the quantization variance can reach very large percentage values. Moreover, the options experience the presence of the *market maker* that has to maintain constantly a certain bid/ask level; however, it is well known that, in case of absence of investors, the market maker enlarges the spread. In practice, the noise given by the spread in the options' markets can be very large and very difficult to predict. It is very common, in fact, to observe spreads raising from $3\,\%$ to $100\,\%$ of the warrant's value leading to extraordinary standard deviations.

While this analysis has been made with the hypotheses of the whiteness of the variable $\varepsilon(t)$, it is reasonable to assume that $\varepsilon(t)$ is not completely white. The variation of the price, and thus of $\varepsilon(t)$, in fact, are *feed back* by the investors that pretend to extrapolate information by looking at the price variations (see Figure 1).

The overall effect of the presence of a bid-ask gap, thus, is some, probably colored, noise added to the price, whose eventual stochastic properties can not be easily identified.

It is worth noticing that [20], looking at the presence of a bid-ask gap, finds one of the motivations for modelling the price with a continuous-time model. At the best of the author's knowledge, the bid/ask gap is considered in the literature only as a kind of transaction's cost [11] giving what appears a misleading interpretation of one of the price characteristics.

B. Intra-day Ideal Sampling

The word *ideal* is used in this Subsection for a sampling achieved with a fixed rate, i.e., at a constant sample time, 1 minute, 1 day or 1 month denoted in the following as T. However, even if ideal, the sampling of an analog signal is not risk-free and may cause loss of the information content of the signal due to the results of the well known Shannon's theorem [9]. In short, let consider as $r^\star(t)$ the sampled signal

$$r^\star(t) = \sum_{k=-\infty}^{\infty} r(t)\delta(t-kT) \tag{4}$$

where $\delta(t-kT)$ is the Dirac's impulse function, in $t = kT$. The Laplace transformation is given by infinite replicas of the un-sampled signal, i.e.,

$$R^\star(s) = \frac{1}{T}\sum_{k=-\infty}^{\infty} R(s-jk\omega_s). \tag{5}$$

In many works ([3], [10]) it is recognized that the choice of the sampling interval has an importance in the results, usually supported by heuristic considerations. In the view of the Shannon's theorem, however, it is clear that the sampling time is of crucial importance and it is also likely that, for daily or longer sampling time, the Theorem is not satisfied, i.e., the eventual information in the data is lost. Also, an arbitrary choice of the intervals as proposed by Mandelbrot [12] might be ineffective.

C. Non-uniform sampling

Real sampling is not uniform, i.e., the rate at which the signal is sampled is not constant but it is function of the communication channel. Currently, the orders on the markets travel mostly via internet, i.e., mainly using the TCP/IP protocol; the communication channel, thus, is stochastic, function of dozen of different variables, leading as a consequence a stochastic non-uniform sampling.

In case of non-uniform, also denoted as irregular, sampling, it is well known that an average sampling frequency higher than the Nyquist frequency is required in order to avoid alias. In [19], the non-uniformity is intentionally used in order to sample an analogical signal, with an average sampling frequency higher than the Nyquist frequency. For band-limited signals, it is possible to correct the non-uniform sampling if the sampling frequency is high enough. In particular, non-uniform sampling has been used in signal processing in order to optimize some criteria where it is possible to select the sampling criterion; in this case, however, it is worth noticing that is not possible to control the irregularity of the sampling, moreover, it is also not-possible to practically reconstruct the irregularity even working off-line.

In detail, let suppose that only one sample is subject to a delay. The sample expected for $k=\overline{k}$ arrives at $k=\overline{k}+\kappa$, the sampled signal is given by:

$$\begin{aligned} r^\star(t) &= \sum_{k=-\infty}^{\overline{k}-1} r(t)\delta(t-kT) + \\ &+ r((\overline{k}+\kappa)T)\delta(t-(\overline{k}+\kappa)T) + \sum_{k=\overline{k}+1}^{\infty} r(t)\delta(t-kT) \end{aligned}$$

that, by adding $\pm r(\overline{k}T)\delta(t-\overline{k}T)$ (see Figure 2), can be rearranged as

$$\begin{aligned} r^\star(t) &= \sum_{k=-\infty}^{\infty} r(t)\delta(t-kT) + \\ &+ r((\overline{k}+\kappa)T)\delta(t-(\overline{k}+\kappa)T) - r(\overline{k}T)\delta(t-\overline{k}T) \end{aligned}$$

whose Laplace transformation is given by:

$$R^\star(s) = \frac{1}{T}\sum_{k=-\infty}^{\infty} R(s-jk\omega_s) + r((\overline{k}+\kappa)T) - r(\overline{k}T).$$

Non-uniformity of the sampling adds a constant, but unknown, value to the signal spectrum, i.e., a white noise over the time series.

Non-uniform sampling affects quite all the samples; its effect, however, is larger for larger values of κ and for high frequency signals. While non-uniform sampling is absent in daily stock prices, it is present in intra-day trading. Moreover, it is rather reasonable to consider that its effect changed a lot during the last decades due to the different technologies used in the world's markets. Comparing, e.g., an intra-day price of a company in the beginning of the 1990 (when internet was not diffused) with the same kind of data in the beginning of the 2000 (when private investors commonly started using internet) means compare data affected by strongly different non-uniformity in the sampling.

In [10], the problem of "non-synchronicity", or "non-trading" problem, of the prices is apparently solved by resorting to a stochastic model. The emphasis is given in the autocorrelation and cross-correlation properties that might arise because of the non-synchronicity of sampled data.

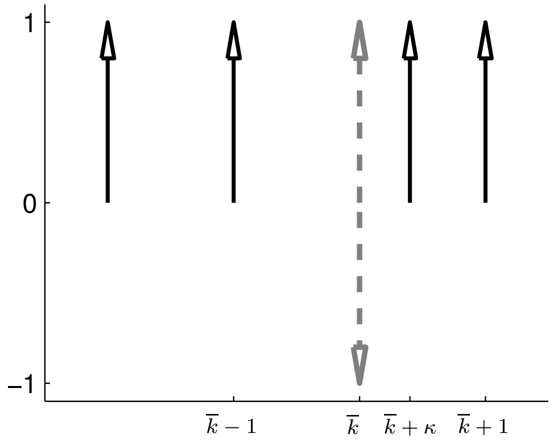

Fig. 2. Non-uniform sampling arises when the samples are not at a fixed sample T (dark arrows); in this case, it is possible to add and subtract a virtual sample at the desired instant time (dashed arrows).

D. Missing samples

Every week the markets are closed during the week-ends. In view of the markets as samplers of the price dynamics it means simply that the investor is missing same samples. For band-limited, time-invariant signals, it is possible to reconstruct the missing samples if the sampling frequency is high enough [4], [16]. In particular this is possible if the samples are missed with a certain, known, probability [1]. Concerning the financial community, the importance of missing samples is commonly recognized even if their effect is approached only on an heuristic base [3].

The effect of missing the samples of a signal can be easily analyzed in the framework of the Laplace transform and it can be observed that, in the frequency domain, missing some samples is equivalent to adding a constant, but unknown, value at the spectrum. In the time domain this is equivalent to adding some white noise to the signal itself. Let suppose that one single sample is missed for $k=\overline{k}$, the sampled signal is thus given by:

$$r^\star(t) = \sum_{k=-\infty}^{\overline{k}-1} r(t)\delta(t-kT) + \sum_{k=\overline{k}+1}^{\infty} r(t)\delta(t-kT)$$

that, by adding $\pm r(\overline{k}T)\delta(t-\overline{k}T)$, can be rearranged as

$$r^\star(t) = \sum_{k=-\infty}^{\infty} r(t)\delta(t-kT) - r(\overline{k}T)\delta(t-\overline{k}T)$$

whose Laplace transformation is given by:

$$R^\star(s) = \frac{1}{T}\sum_{k=-\infty}^{\infty} R(s-jk\omega_s) - r(\overline{k}T).$$

In case of several missed samples the last term needs to be substituted by the summation of all of them. The *significance* of the corruption, thus, is related to the number and to the amplitude of missed samples.

For daily stock prices the missing samples phenomena arises for all the festivities when, i.e., the markets are closed. In one year, approximatively the 30 % of the samples is missed.

E. Heuristic estimation of the raise time

Evidence that the Shannon's theorem is not satisfied can be observed easily. As a consequence, it is not possible to reconstruct the original signal and it is not possible to fill the data with the missing values.

Classical identification theory says that, for linear systems and in in case of a step input, one needs to store from 4 to 9 samples of the consequent step response in order to have proper data to work with [7]. Let analyzes the consequences in case of stock markets with an example, let us imagine that that AdCom takes a major decision during Sunday, this can be considered as a step input to the system. Ignoring the pre-opening trading, it is evident that the investors will trade the proper price of the stock in the very first minutes of the Monday. The closing Monday's price, thus, already discounts the Sunday's changes while one might need a transient time of at least 4 days in order to *absorbe* the Sunday's decision. Let see it from another point of view, the investors might need 4 days in order to understand the correct stock's price after the Sunday's decision; it would be really underestimating the intelligence of the investors.

F. Correlation given by finite-length data

Numerical representations of theoretical properties of numbers often do not show the property itself; in other words, apparently, even for computer-generated numbers, the theoretical property seems to be un-verified.

For the reasons above, correlation in stock markets data is often only apparent and given by the finite-length data. As it is well known in the mathematic community and pointed out e.g., in [11], [17], one can found practical correlation even between the price of a stock and the temperature on the moon at midnight but it would be difficult to use it to select among investments.

Let consider for seek of example the auto-correlation of a zero-mean Gaussian signal; its theoretical auto-correlation is impulsive, i.e., the only value different from zero is the origin. The only method to have numerical impulsive auto-correlation is to use an *infinite* number of data, i.e., a long enough number of data. In case of stock markets, however, *infinite* number of data can not be used because of the lack of the time-invariant property, as it will be shown in next subsection. Figure 3 reports the normalized numerical autocorrelation function for a sequence of 10^5 samples (dot) and for its first 100 samples (square); it can be observed that an apparent correlation arises when working with only the first samples. Notice that, in case of daily data, 100 samples correspond to ≈ 5 months.

It might be possible to use different statistical indexes in order to prove that the series shown in the example above is white and, for example, that stock prices are not white, but what it is important to point out is that the correlation

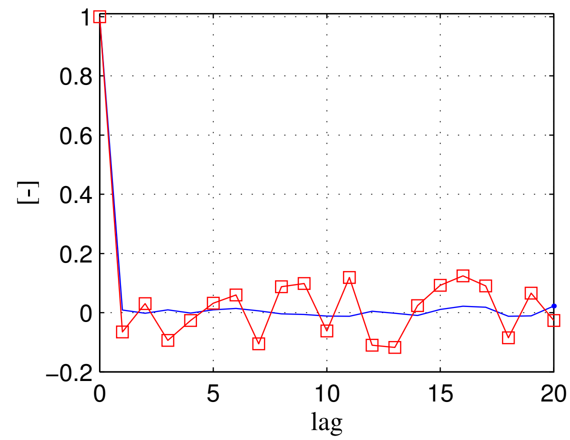

Fig. 3. Example of the normalized autocorrelation (lag of 20 samples) of a finite-length zero-mean Gaussian signal numerically generated: dot, 10^5 samples, square, first 100 samples.

properties depend on the data length and on the specific realization.

G. Correlation given by time-varying dynamics

The stock markets face two different needs: the need to work with long time series in order to magnify properties of infinite-length data and the need to work with short time series in order to avoid working with data corresponding to different models.

The markets environment changes continuously, if there is a mathematical model that effectively represents the price's variations it is obviously changing adaptively with the environment. To go further, one does not know neither the structure (or the model) neither the parameters' value. Moreover, some changes are abrupt and other are slowly varying, for this reason it seems very difficult to find correct the use of long time series of data such as, e.g., data over several years.

To understand it let just ask to an investor what was its idea of the telecommunication market in the second half of the 90 th and its idea now; probably he/she would describe the situation, and the stocks with their expectations, with totally different words and would probably avoid to make comparison of two different stocks in those two different time instant. Why should be different with the numerical data? Why should we be able to extract any information from yearly data?

H. Opening, closing or reference prices

It is worth noticing that only the intra-day prices are exactly the price of the last transactions. Several algorithms are used, in the different markets, to determine the opening, closing or reference prices. While it is difficult to examine the specific algorithms in detail they share one aspect: the price is some kind of average of the last transactions feeding some correlation on the price.

In detail, the dynamic equation representing this average is given by

$$p(k) = \frac{1}{N}\sum_{i=1}^{N} a_i p(k-i) \qquad (6)$$

where N is the number of past transactions considered and a_i is a coefficient that represents a weight. Notice that the last N transactions generally occur at a non-uniform sampling; the coefficient a_i, moreover, are often related to the volume of the transactions and are, thus, non-linearly related to one of the inputs.

I. Other sources of noise

When working with such a complex system as the stock markets are, it is very difficult to retrieve homogeneous data to work with. This is specially true when the time series is long. One possible source of *disturbance* is given by the presence of the dividends; usually the researchers are used to consider the dividends as part of the price for the ex-dividends days even if they recognize that it is a debatable approach [3]. Other aspects that are difficult to take into account for are the taxation and the transactions costs; those are obviously changing with time, with the country where the data are observed and with the operator the investor selects for its trading purposes.

The simplest way to take into account for these aspects is to consider that they add some *noise*, in a wide sense, to the data.

IV. CONCLUSIONS

In this paper the utility of the time series of the stock prices has been investigated. The stock markets can be considered as a strange kind of ineffective samplers of the prices dynamics. In the literature it is often recognized that the choice of the sampling time is critic; this, however, is not further related to the goodness of the data itself. Figure 1 reports all the modifications at which the price is subject before that its value can be stored for further analysis: let consider first a mechanism that generates a continuous time price (here we do not care about its characteristics); due to the presence of the bid-ask gap a first noise signal is added to the price; the price is then sampled with an apparently small sampling time such as, e.g., 1 minute; this may cause alias to corrupt the signal spectrum. Since the sampling is not uniform one needs to take into account another additive noise to the signal. The closing price is obtained by considering some kind of averaging operation causing the price to be filtered by a known function $F(z)$. The resulting value is used for daily stock data; again, alias might eventually corrupt this data. The sampling is not ideal: $\approx 30\,\%$ of the data are missed. This implies that another white noise needs to be added to the price value. Those effects are structural in the price definition and can not be avoided. Depending on the environmental condition, however, some effects might be mitigated and some other increased.

The analysis conducted has shown that:

- weekly or monthly data are mainly affected by alias, finite length data and time variance;
- daily data are mainly affected by alias, finite length data, time variance and missing samples;
- intra-day data are mainly affected by the quantization effect, unknown, non-uniform sampling.

As significant consequence, it seems to be possible to extrapolate that all the data-based analysis, from the chart-based technique of the Technical Analysis to the stochastic-dynamic-systems-based analysis, ignore significant sources of noises. This might be the reason why in the literature still is possible to find contradictory experimental data about price variation characteristics.

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