

Synchronization in Generalized Erdős-Rényi Networks of Nonlinear Oscillators

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Abstract—In this paper, we study synchronization of complex random networks of nonlinear oscillators, with specifiable expected degree distribution. We review a sufficient condition for synchronization and a sufficient condition for desynchronization, expressed in terms of the eigenvalue distribution of the Laplacian of the graph and the coupling strength. We then provide a general way to approximate the Laplacian eigenvalue distribution for the case of large random graphs produced by a generalization, [2], of the Erdős-Rényi model. Our approach is based on approximating the moments of the eigenvalue density function. The analysis is illustrated by using a complex network of nonlinear oscillators, with a power-law degree distribution.

I. INTRODUCTION

The Erdős-Rényi (E-R) random graph model [3] is the oldest and one of the most studied techniques to generate complex networks. However, some interesting features of large-scale complex networks cannot be emulated by the E-R construction. One of its most important limitations is the infeasibility of generating arbitrary degree distributions. (The degree of a node in a graph is the number of edges connected to it; the degree distribution is then the monotonic nonincreasing sequence of the node degrees.). Specifically, while the original E-R models can only provide either binomial or Poisson degree distributions, many biological and technological networks might be described by other degree distributions [1],[4],[14],[16].

Our interest in this paper is the analytical study of the synchronization phenomenon in complex networks of nonlinear oscillators. In this direction, many efforts have been invested in determining conditions under which a set of dynamical nodes synchronize. Most of these studies mainly focus on analyzing the nonlinear dynamics of completely deterministic regular topologies [13], [17], excluding the complexity originated from stochastically-generated connectivity patterns.

In this paper, we merge the structural complexity of a complex network topology with the dynamical complexity of nonlinear oscillators. We review analytical conditions to achieve synchronization among coupled identical nonlinear systems connected in a graph. The mathematical structure of our problem is closely related to that of consensus problems

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in networks of agents [11], as well as the task of analyzing flocking/swarming conditions in a group of autonomous agents [5], [8].

The organization of the paper is as follows. In Section II, we review different approaches to constructing complex networks through a set of simple rules. The issue of synchronization in a network is discussed in Section III, giving conditions for synchronization in terms of the eigenvalue distribution of a matrix representing the topology of the network. The results are illustrated using a power-law interconnection of Rössler oscillators. Section IV presents a new procedure to approximate the lower-order moments of the eigenvalue density function of this type of random network in the case of a large number of nodes. We end this paper with some conclusions and comments for further research.

II. MODELING COMPLEX DYNAMICAL NETWORKS

The number of elements interacting in a complex large-scale network is such that answering even simple structural and dynamical questions is usually an infeasible task. Therefore, the development of models capable of matching empirically-observed behaviour and amenable to analysis is an important task.

A. Some Approaches to Generate a Complex Topology

A brief summary of the most popular ways to generate a complex network is presented in this section. The descriptions only try to give a flavor of the type of techniques one can use to stochastically generate a complex network.

1) *Barabási-Albert Model*: The following algorithm was suggested by Barabási and Albert [1] in order to grow a network presenting the so-called scale-free or power-law feature. In this recursive algorithm, we have a set of m_0 nodes at the origin of time. Then, at every discrete-time step, we add a new node with m edges that we link randomly to a set of m different nodes in the network. When randomly choosing the nodes to which to connect, assume node i is selected with probability μ_i linearly proportional to its degree. Growing a network with this algorithm gives rise to a power-law degree distribution in the limit of large time, i.e. $P(k) \sim 2m^2k^{-3}$.

2) *Watts-Strogatz-Newman Model*: In the original model by Watts and Strogatz [16], random graphs are obtained from regular lattices by rewiring edges or by making shortcuts between randomly chosen vertices. The resulting architecture is intermediate between regular lattices and classical random graphs. An exciting result observed in this model was the following: even for small probability of rewiring p , while

the local properties of the network are still nearly the same as for the original regular lattice and the clustering coefficient does not differ essentially from its initial value, the average shortest-path length is already of the order of that for the classical random graphs.

3) *Erdős-Rényi Random Network Model*: The basic E-R random network is defined as a random graph of N labeled nodes connected by l edges, which are chosen randomly from all the $N(N-1)/2$ possible edges. The network is generated with the following simple rule: Starting from N isolated nodes, add edges between every pair with a fixed probability p .

The main goal of the elegant theory developed for this type of network is to determine the value of p at which a particular property of the graph arises with high probability. In the case of large networks, this model renders a Poisson degree distribution; therefore, nodes in the network present a quite homogeneous degree. The fact that the degrees of the nodes are fairly evenly distributed makes this model inappropriate to match many real-world networks.

4) *Generalized Erdős-Rényi Model*: In this paper, we use a random graph model introduced in [2] as an attempt to generalize the Erdős-Rényi model to the case of a given degree distribution. Let us consider a given expected degree sequence $\mathbf{w} = (w_1, w_2, \dots, w_n)$. In this model, edges are independently assigned between each pair of nodes (i, j) with probability $p_{ij} = w_i w_j \rho$, where $\rho = \left(\sum_{i=1}^N w_i\right)^{-1}$. It is easy to verify that the expected degree of i is w_i . We can recover the original E-R model by using a uniform degree distribution, i.e., $w_i = w = Np$ for every node i in the graph.

At this point we assume that $\max_i w_i^2 < \sum_k w_k$ so that $p_{ij} \leq 1$ for all i and j . This assumption assures that the sequence w_i is 'graphical' (in the sense that it satisfies the necessary and sufficient condition for a sequence to be realized as a graph [3]).

Note that, contrary to the approaches presented above, this model does not attempt to explain the mechanisms under which a complex graph develops a given degree distribution. Rather, it takes the degree distribution as given from empirical data, and tries to derive the structures and properties (such as connected components, diameters, etc.).

III. SYNCHRONIZATION IN DYNAMICAL NETWORKS

Consider a dynamical network consisting of N linearly and diffusively coupled identical nodes, where each node is an n -dimensional nonlinear oscillator (possibly chaotic in isolation). The state equations of this network are

$$\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i) + \gamma \sum_{j=1, j \neq i}^N a_{ij} \Gamma (\mathbf{x}_j - \mathbf{x}_i), \quad i = 1, 2, \dots, N, \quad (1)$$

where \mathbf{x}_i represents the n -dimensional vector of state variables corresponding to node i , and $f(\cdot)$ is a nonlinear function describing the dynamics of the isolated nodes. We also have a positive scalar γ which can be interpreted as a

global coupling strength parameter, and an $n \times n$ matrix Γ representing how states in the neighboring oscillators linearly combine to affect the dynamics.

Let us denote by A the matrix whose elements $a_{ij} = a_{ji}$ are defined to be 1 when there is a connection between node i and j , 0 otherwise. This symmetric matrix is called the *adjacency matrix* of the undirected graph, and it comprises all the information regarding the topology of the network. We also define the degree k_i of node i as the number of edges connected to it. Using the degrees, we can obtain the so-called diagonal degree matrix $D \equiv \text{diag}(k_1, k_2, \dots, k_N)$, and the Laplacian matrix $L \equiv D - A = [l_{ij}]$. The dynamics (1) can then be written as

$$\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i) - \gamma \sum_{j=1}^N l_{ij} \Gamma \mathbf{x}_j, \quad i = 1, 2, \dots, N. \quad (2)$$

Assuming that L is a symmetric and irreducible matrix (meaning that the network presents a single connected component) it can be verified that its smallest eigenvalue λ_1 is zero with algebraic multiplicity one, and the other eigenvalues $\lambda_2 \leq \dots \leq \lambda_N$ are strictly positive.

Asymptotic synchronization in the coupled network (1) is said to be achieved if

$$\mathbf{x}_i(t) \rightarrow \mathbf{s}(t), \quad \text{for } i = 1, 2, \dots, N \text{ as } t \rightarrow \infty$$

where $\mathbf{s}(t)$ satisfies the isolated-node dynamics $\dot{\mathbf{s}}(t) = \mathbf{f}(\mathbf{s}(t))$. In the case of nonlinear systems, $\mathbf{s}(t)$ can be not only an equilibrium point, but also a periodic or even chaotic orbit, broadening the behavior of interest beyond the usual fixed-point stability.

In the following, two useful lemmas proven in [15] are presented.

Lemma 1: Consider the dynamical network (2). Let $0 = \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_N$ be the eigenvalues of the Laplacian matrix L , and let $Df(\mathbf{s})$ be the Jacobian of $\mathbf{f}(\cdot)$ at $\mathbf{s}(t)$. If the following $N-1$ linear time-varying systems are exponentially stable

$$\dot{\mathbf{w}} = [Df(\mathbf{s}(t)) - \gamma \lambda_k \Gamma] \mathbf{w}, \quad k = 2, \dots, N \quad (3)$$

then individual node state vectors synchronize exponentially fast.

Lemma 2: Consider the dynamical network (2). Suppose that there exists a diagonal matrix Λ and two constants $\bar{d} < 0$ and $\tau > 0$, such that

$$[Df(\mathbf{s}(t)) + \bar{d}\Gamma]^T \Lambda - \Lambda [Df(\mathbf{s}(t)) + \bar{d}\Gamma] \leq -\tau I_n \quad (4)$$

for all $d \leq \bar{d}$. If $\gamma \lambda_2 \geq \bar{d}$, the network synchronizes exponentially fast.

Therefore, Lemma 2 shows the existence of a threshold \bar{d}/λ_2 on the coupling strength γ for synchronization although it does not provide an explicit form for the value of this threshold.

For sake of clarity, we will restrict ourselves to the case of chaotic synchronization. Denote by $h_1 \geq h_2 \geq \dots \geq h_n$ the Lyapunov exponents of each individual dynamical node. A chaotic regime in the isolated dynamics of the nodes is

characterized by the maximum Lyapunov exponent h_1 being positive. In the following useful theorem, introduced in [13], we assume the isolated nodes are chaotic.

Theorem 3: Consider network (2) with identical nodes that are chaotic in isolation. If the network attains synchronization exponentially, then,

$$\gamma\lambda_N < h_1. \quad (5)$$

Therefore, this theorem indicates the counterintuitive idea that a sufficiently large coupling strength may lead to desynchronization [7] in the chaotic case. A similar phenomenon has been reported in the closely related problem of stability of multiagent systems with time-dependent communication links [10]. In that paper, interaction of dynamical systems is analyzed under the perspective of information flow. It is shown there that more information flow among the agents may lead to a loss of convergence of the individual agents' states.

The above analytical conditions have been illustrated for networks of deterministic topologies of relatively small size in [13]. In the following, we extend the illustration to the case of stochastically-generated large-sized networks. Consider a power-law network with 100 nodes generated by the generalized E-R method. We analyze the cases in which the exponent of the distribution is $\beta = 3.2$ and 2.4 . At each node, we locate a Rössler oscillator¹, allowing interaction through the edges of the power-law graph. The nonlinear dynamics of the coupled network of Rössler oscillators is given by

$$\begin{aligned} \dot{x}_i &= -(y_i + z_i) - 0.4 \sum_{j=1}^{100} l_{ij} x_j, \\ \dot{y}_i &= x_i + ay_i - 0.4 \sum_{j=1}^{100} l_{ij} y_j, \\ \dot{z}_i &= b + z_i(x_i - c) - 0.4 \sum_{j=1}^{100} l_{ij} z_j. \end{aligned} \quad (6)$$

In this paper we have chosen the parameters of each isolated oscillator in such a way that they present chaotic behaviour. In this way, we expect the synchronized state of the network to be chaotic as well. Under this assumption, and after numerically calculating the Lyapunov exponent of the oscillator, we determine the threshold in the second smallest eigenvalue λ_2 for the network to synchronize to be 26.4, for a coupling strength $\gamma = 0.4$.

In Fig. 2, we show the eigenvalue histograms for both $\beta = 3.2$ (above) and $\beta = 2.4$ (below). The threshold in the value of λ_2 to achieve synchronization is marked with a dashed line on the figures. We see that the values chosen must give rise to a synchronized network for $\beta = 3.2$, and could produce desynchronized behaviour for $\beta = 2.4$.

Fig. 3 represents the evolution of the state x_i for several oscillators in the network. We have chosen, in order of decreasing degree, the most connected node, the third, the

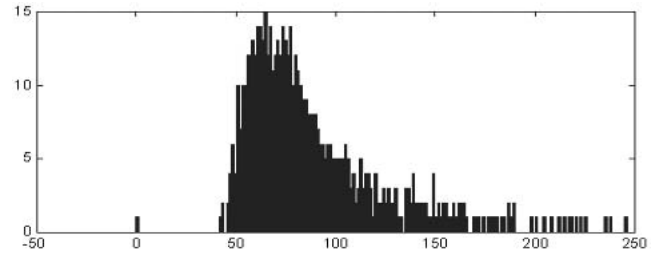


Fig. 3. Histogram of the eigenvalues of the Laplacian of a Power-Law graph with 600 nodes. Notice the trivial eigenvalue at the origin.

fifth, the thirtieth and the sixtieth as a representative set of the 100 nodes involved in our numerical simulations. In the upper part of the figure, we represent the case in which synchronization is achieved. We can observe how, even for initial conditions randomly chosen in quite a broad range (uniformly distributed between -10 and 10), the nodes tend to a synchronized state in a relatively short time. In the lower case, the synchronized state is not stable. Therefore, even for very closely chosen initial conditions, the values of x_i for the representative set of nodes diverge in time, as predicted.

IV. EIGENVALUE SPECTRA AND SYNCHRONIZATION IN RANDOM NETWORKS

From the theoretical results presented above, we conclude that the second smallest eigenvalue, λ_2 , of the network Laplacian L plays a central role in determining whether synchronization occurs. An example of the eigenvalue distribution of the Laplacian is in Fig. 1 for the case of a power-law graph with $N = 600$ nodes. Notice that the spectrum contains a single trivial eigenvalue at the origin, and a nontrivial spectrum beyond that. The cumulative distribution function (cdf) of the eigenvalues remains approximately invariant over different realizations of the 600-node power-law graph. Furthermore, the cdf of the eigenvalues of the size-normalized Laplacian $\frac{1}{N}L$ remains approximately invariant as N increases.

Therefore, our dynamical problem is reduced to the algebraic problem of determining the value λ_2 from the stochastic definition of the random graph (note that this eigenvalue determines the lower endpoint of the nontrivial support of the distribution). Then, combining this value with the numerically calculated largest Lyapunov exponent h_1 , we will be in a position to determine the range of values of the coupling strength γ to achieve synchronization in the network.

Before presenting our approach to estimating the eigenvalue spectrum of the Laplacian for the generalized E-R model, some notation is introduced.

Notation: We use the operator $tr_N \mathbf{E}$ to denote the averaged expected trace for an $N \times N$ random matrix M with entries m_{ij} :

$$tr_N E(M) := \frac{1}{N} \sum_{i=1}^N E[m_{ii}].$$

¹The Rössler oscillator was originally proposed to model the nonlinear dynamics of a chemical reaction. We have chosen this oscillator because it is one of the simplest models able to present a chaotic behavior [17].

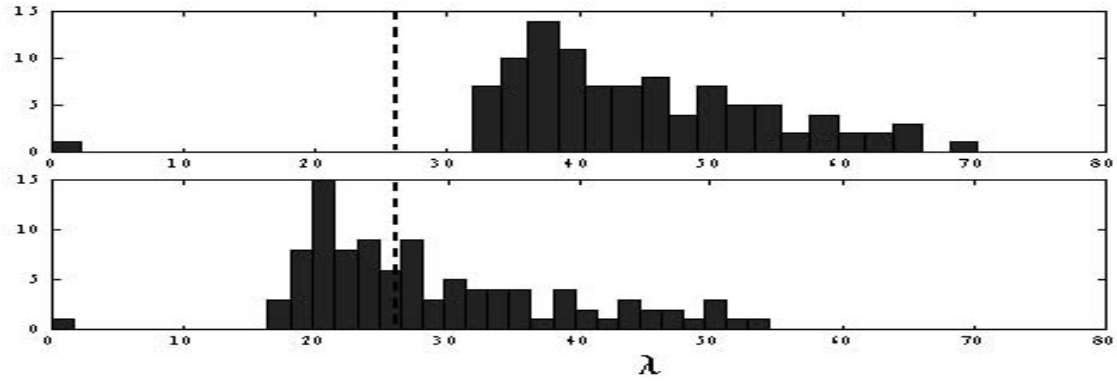


Fig. 1. Eigenvalue histograms for two power-law networks of 100 nodes. Synchronization is achieved for the case $\beta = 3.1$ (upper part of the figure), and for $\beta = 2.4$ we reach a desynchronized state (lower part).

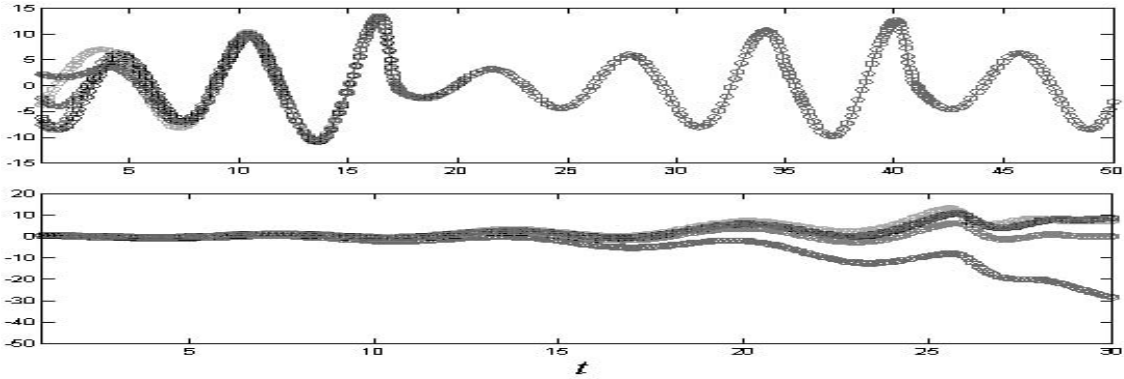


Fig. 2. Time evolution of the x state for the first, the third, the fifth, the thirtieth and the sixtieth most connected nodes. Synchronization can be observed in the upper case. A desynchronized behaviour is achieved in the lower case.

The expected n -th moment associated with the eigenvalue distribution of the size-normalized Laplacian $\frac{1}{N}L$ is then given by $\frac{1}{N^n}tr_N E(L^n)$. For well-behaved matrix ensembles, these moments are known to converge as $N \rightarrow \infty$ [9]. Let us denote by α_n the limit of the n -th moment,

$$\alpha_n := \lim_{N \rightarrow \infty} \frac{1}{N^n} tr_N E(L^n).$$

A. Moment-Based Reconstruction of the Eigenvalue Spectrum of the Laplacian

In this section, we present an approach to estimate the probability distribution of the eigenvalues of the size-normalized Laplacian matrix of a large network produced by the generalized E-R construction. Our method is based on approximating the moments of the distribution from knowledge of the expected degree distribution vector $\mathbf{w} = (w_1, w_2, \dots, w_N)$. Given all the moments, one can in principle recover the eigenvalue distribution; however, our focus is on the first few moments.

Determining the exact value of the moments of the Laplacian is an intricate combinatorial problem. Instead, we introduce a much simpler expression that is exact up to third-order moments and gives a good approximation for a few more moments beyond the third.

The proposed approximation for the n -th moment of the (non-normalized) Laplacian takes the form:

$$\begin{aligned} tr_N E[(D - A)^n] &\approx tr_N E \left[\sum_{k=0}^n \binom{n}{k} (-1)^k A^k D^{n-k} \right] \\ &= \sum_{k=0}^n \binom{n}{k} (-1)^k tr_N E[A^k D^{n-k}] \end{aligned} \quad (7)$$

The approximate expression on the right is what one obtains if D and A commute, which is not the case. Nevertheless, this approximation—which is considerably more tractable than the exact expression—yields quite accurate results.

The problem of approximating moments of the random matrix is now reduced to finding a closed form for the terms $tr_N E[A^i D^j]$, for all pairs of positive integers i, j . Based on the description in Section 2, the generalized E-R random graph can be generated by defining a matrix $P \equiv \rho \mathbf{w} \mathbf{w}^T$ obtained by collecting the terms $p_{ij} = \rho w_i w_j$ that define the probability of linking nodes i and j , $i \neq j$; since we don't allow self-loops, $p_{ii} = 0$. It is immediate to see that the expectation of the adjacency matrix of the graph is $E[A] = P = \rho (\mathbf{w}^T \mathbf{w})$.

In the following, we determine the expectation of the diagonal elements of the powers of the adjacency matrix A .

In the case of the second power,

$$\begin{aligned} (E[A^2])_{ii} &= E\left[\sum_k a_{ik}a_{ki}\right] = \sum_k E[a_{ik}^2] \\ &= \sum_k E[a_{ik}] = \rho w_i \sum_j w_j = w_i. \end{aligned} \quad (8)$$

In the cubic case,

$$\begin{aligned} (E[A^3])_{ii} &= E\left[\sum_r a_{ir} \sum_k a_{rk}a_{ki}\right] \\ &= \sum_r \left(E[a_{ir}^2 a_{ii}] + \sum_{k \neq i} E[a_{ir} a_{rk} a_{ki}] \right) \\ &= E[a_{ii}] + \sum_{k \neq i} E[a_{ii} a_{ik}] + \sum_{r \neq i} E[a_{ir} a_{ii}] \\ &\quad + \sum_{r \neq i} \sum_{k \neq i} E[a_{ir} a_{rk} a_{ki}] \\ &= p_{ii} + \sum_{k \neq i} p_{ii} p_{ik} + \sum_{r \neq i} p_{ir} p_{ii} \\ &\quad + \sum_{r \neq i} \sum_{k \neq i} p_{ir} p_{rk} p_{ki}. \end{aligned} \quad (9)$$

Since we don't allow self-loops in our network ($p_{ii} = 0$), only the last addend survives. Therefore, this term can be easily written in terms of w_i as

$$(E[A^3])_{i,i} = \rho^3 w_i^3 \sum_{r \neq i} \sum_{k \neq i} w_r^2 w_k^2 \quad (10)$$

In order to determine the expected value of the diagonal elements of A^n for general n , we will use a graph-theoretical interpretation. It is a well-known fact that the diagonal elements of the powers of the adjacency, i.e., $(A^k)_{ii}$ represents the number of walks of length k starting on i and ending in itself (allowing potential visits to node i in the middle of the path). This quantity can be computed as presented below.

First of all, it is useful to define a function called the *compression* of a set of elements, which will be denoted by $comp(s_{i_1, j_1}, s_{i_2, j_2}, \dots, s_{i_k, j_k})$, as the multiplication of the elements in the set after eliminating repetitive symbols. In doing that, we take into account that symbols with symmetric subindices are considered equivalent ($s_{i,j} = s_{j,i}$). For example, $comp(s_{1,2}, s_{1,2}, s_{3,4}, s_{2,1}, s_{4,3}) = s_{1,2} s_{3,4}$.

Let $q_i(n)$ be the expected number of walks of length n starting at i and ending at i for the *first time*. This quantity can be easily computed as

$$q_i(n) = \sum_{\substack{i_1, i_2, \dots, i_{n-1} \\ \{i_1, i_2, \dots, i_{n-1}\} \in \bar{N}^{(i)}}} comp(p_{i, i_1}, p_{i_1, i_2}, \dots, p_{i_{n-1}, i}),$$

where $\bar{N}^{(i)}$ is the set of integers $\{1, 2, \dots, N\}$ excluding i . Note that this quantity can be directly computed as a function of the expected degree sequence by direct substitution.

The total number of walks of length n can be computed as

$$\begin{aligned} E[(A^n)_{i,i}] &= q_i(n) + \sum q_i(n_1) q_i(n_2) \delta_{n_1+n_2, n} \\ &\quad + \sum q_i(n_1) q_i(n_2) q_i(n_3) \delta_{n_1+n_2+n_3, n} + \dots \end{aligned} \quad (11)$$

Therefore, we can compute the diagonal terms as a function of the given expected degree sequence. Based on these, we can also find an approximate expression for the terms $E[A^k D^{n-k}]$ involved in the decomposition of the spectral moments (7). By the strong law of large numbers, we can estimate the elements d_i of the degree matrix D in the case of large matrices as $d_i \simeq \sum_j E[a_{i,j}] = w_i$. Therefore,

$$(E[A^k D^{n-k}])_{i,i} \simeq w_i^{n-k} (E[A^k])_{i,i},$$

and we can use (8)-(10) to yield the required terms in the summation (7)

$$tr_N E[A^k D^{n-k}] \simeq \begin{cases} 0 & \text{for } k = 1, \\ \sum_i w_i^{n-1} & \text{for } k = 2, \\ \rho^3 \sum_i w_i^n \sum_{r \neq i} \sum_{k \neq i} w_r^2 w_k^2 & \text{for } k = 3. \end{cases}$$

For the case $k > 3$, the computation of $(E[A^k D^{n-k}])_{i,i}$ involves the solution of the expression (11). Therefore, we can estimate the moments of the Laplacian from knowledge of an expected degree sequence by using the above determined values of the terms in (7).

From the description of the distribution given by the moments, the eigenvalue spectrum of the Laplacian can be constructed in several ways, depending on the specific form of the expected degree distribution. Two alternatives are:

(i) Use the moments to determine the characteristic function by Taylor expansion

$$M_\alpha(jv) = \sum_{r=0}^{\infty} \alpha_r \frac{(jv)^r}{r!},$$

and obtain the spectrum by inverse Fourier transformation [12].

(ii) Use a multi-pole series expansion to determine the Stieltjes transform of the eigenvalue distribution,

$$S_a(z) = -\frac{1}{z} - \sum_{r=0}^{\infty} \frac{\alpha_r}{z^{r+1}},$$

and make use of the Stieltjes-Perron inversion formula

$$p_a(\lambda) = \frac{1}{\pi} \lim_{\xi \rightarrow 0} \text{Im } S_a(\lambda + j\xi).$$

where $p_a(\lambda)$ denotes the eigenvalue distribution.

B. Illustration: Synchronization in a Power-Law Network

In order to illustrate how the above results can be applied, we consider the case of a random graph with a power-law expected degree distribution. By definition, in a power-law network the probability that a node has degree k follows a power-law distribution, i.e. $P(k) \sim ak^{-\beta}$. To yield a power-law degree distribution from the generalized E-R model, we choose a degree sequence $\mathbf{w} = (w_1, w_2, \dots, w_N)$ satisfying

$w_i = ci^{-1/(\beta-1)} = ci^{-\gamma}$ for $i_0 \leq i \leq i_0 + N$ [2]. Here, c is determined by the average degree \bar{d} , and i_0 depends on the maximum degree m :

$$c = \frac{\beta - 2}{\beta - 1} \bar{d} N^{-1/(\beta-1)}, \quad (12)$$

$$i_0 = N \left[\frac{\bar{d}(\beta - 2)}{m(\beta - 1)} \right]^{\beta-1}.$$

As required for a power-law network, it can be verified that the number of vertices of degree k is proportional to $k^{-\beta}$.

Notice that many terms of the form $\sum_{i=1}^N w_i^s$, $s \in \mathbb{N}$ are involved in (12). In the case of large power-law networks, this summation has a closed form solution in terms of the powers of N [6]. By taking the term in the series with the highest power of N , we obtain

$$\sum_{i=1}^N w_i^s = \sum_{i=i_0}^{i_0+N} \left(ci^{-1/(\beta-1)} \right)^s$$

$$\simeq c^s \frac{(i_0 + N)^{1-s/(\beta-1)}}{1 - s/(\beta-1)}, \text{ for } N \gg i_0. \quad (13)$$

Therefore, by taking the leading term in the moments, we reach,

$$\alpha_n \simeq \frac{\rho^{2n} c^n (i_0 + N)^{1-n/(\beta-1)}}{N^{n+1} (1 - n/(\beta-1))}. \quad (14)$$

The existence of a closed-form expression of this sort is a consequence of the highly-structured expected degree distribution.

We have compared this analytical approximation with numerical values of the averaged spectra of 10 power-law networks with 600 nodes ($\beta = 3.2$). The following table contains a comparison of the first four moments:

α_n	Analytical moments	Numerical Values
1	0.1495	0.1491
2	2.575e-2	2.571e-2
3	5.21e-3	5.1972e-3
4	1.2534e-3	1.2187e-3

In this example, the relative errors between analytical and numerical results increase with the order of the moment; the maximum error is around 3% in the fourth moment.

V. CONCLUSIONS AND FUTURE RESEARCH

In this paper, we have reviewed several stochastic algorithms to construct complex networks by simple rules, and have presented explicit conditions to achieve synchronization of oscillators at the network nodes. We discussed the central role of the second smallest eigenvalue of the Laplacian in the stability of the synchronized state. A novel approach to approximate the moments of the spectrum of the Laplacian for stochastically generated large networks was presented.

Further analytical studies are needed, for example:

1) Improvement of our moment estimates by solving the exact combinatorial problem underlying the computation of the moments.

2) Extension of the approach to other types of stochastically generated networks, as well as switching topologies.

3) Determination of the robustness of the synchronized network under removal of edges or nodes.

4) Development of approaches to study synchronization of non-identical oscillators linked by heterogeneous connections.

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