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Approximate Trajectories and Sampling Methods For Impulsive Systems

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Abstract— **This paper introduces sampling techniques for impulsive systems analogous to the classical Euler one-step method. Moreover a method of approximating measures with absolutely continuous controls is also offered. The paper concludes with an invariance result that is an application of the sampling technique.**

I. INTRODUCTION

Impulsive systems arise when state variable is allowed to move at different time scales. The usual time progression is regarded as the "slow" movememnt and the "fast" movement happens over a small time interval that resembles the effect of a point-mass measure. We adopt the mathematical formalism introduced in [8], [9], [10], in which the controlled dynamic inclusion is the sum of a slow time velocity belonging to a set $F(x)$ and a fast time contribution coming from another set $G(x)d\mu$, where μ is a vector valued measure.

Throughout the paper, the following data with accompanying assumptions are given:

(H1) A closed convex cone $K \subseteq \mathbb{R}^m$;

(H2) A multifunction $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ with closed graph and convex values, and satisfying

$$f \in F(x) \implies |f| \leq c(1 + |x|) \quad \forall x \in \mathbb{R}^n.$$

(where $c > 0$ is a given constant);

(H3) A multifunction $G : \mathbb{R}^n \rightrightarrows \mathcal{M}_{n\times m}$ (where $\mathcal{M}_{n\times m}$ denotes the $n \times m$ dimensional matrices with real entries) with closed graph and closed convex values, and satisfying

$$g \in G(x) \implies \|g\| \leq c(1 + |x|) \quad \forall x \in \mathbb{R}^n.$$

The set of vector-valued Borel measures defined on the interval $[0, T] \subset \mathbb{R}$ with values in K is denoted by $\mathcal{B}_K([0, T])$. Suppose $\mu \in \mathcal{B}_K([0, T])$ is given. The impulsive system considered in this paper is described by a differential inclusion of the form

$$\left\{ \begin{array}{rcl} dx & \in & F\big(x(t)\big)\, dt + G\big(x(t)\big)\, d\mu(dt) \\ x(0-) & = & x_0. \end{array} \right. \tag{1}$$

The trajectory $x(\cdot)$ is a function of bounded variation, however additional information must be available in order to give a definition of solution with desirable properties. In particular, we seek that a solution concept to (1) for which solutions can be obtained as a limit of (1) time-discretized Euler-type arcs, and (2) solutions to a classical system in which the measure μ is approximated by an absolutely continuous measure.

Even in the case when the multifunctions $F(\cdot)$ and $G(\cdot)$ are singleton-valued (that is, they are just regular functions), these properties are not available without additional considerations. A simple example in [1] illustrates the dilemma, and thereby shows that naive time-discretization schemes can lead to wildly different solutions. In this example, the approximate controls converge to two different graph completions (which are defined in the next paragraph).

The main goals of this paper are (1) to develop a time discretization method analogous to the classical Euler one-step method (see Section III) and show that they appropriately limit to solutions of (1), and (2) to introduce a concept of graph convergence of measures for which solutions of the approximate control problems appropriately limit to solutions of (1). In the final section of this paper we consider applications to flow invariance.

II. THE SOLUTION CONCEPT

Suppose $u(\cdot) : [0, T] \to \mathbb{R}^m$ is the distribution function of μ given by $u(t) = \mu([0, t])$. A *graph completion* of $u(\cdot)$ is a Lipschitz map $(\phi_0, \phi) : [0, S] \to [0, T] \times \mathbb{R}^m$ satisfying

(g1) $\phi_0(\cdot)$ is non-decreasing with $0 \leq \dot{\phi}_0(s) \leq 1$,

(g2) for every $t \in [0, T]$ there exists $s \in [0, S]$ so that $\big(\phi_0(s), \phi(s)\big) = \big(t, u(t)\big)$ and

(g3) for almost all $s \in [0, S]$, $\dot{\phi}(s) = (1 - \dot{\phi}_0)k(s)$ where $k(\cdot) : [0, S] \to K$ is measurable and satisfies $|k(s)| \leq 1$ almost everywhere.

This definition is called a *normalized graph completion* in [12], however, as explained there, it is equivalent to more general definitions but avoids further technical complications. Note that $|\dot{\phi}(s)| \leq 1$ for almost all $s \in [0, S]$.

We now discuss a definition of solution to (1) that is a modification of the definition given in [1], [2], [3]. Suppose the measure $\mu \in \mathcal{B}_K([0, T])$ is given. Consider a three-tuple

$$X_\mu := \big(x(\cdot), \phi(\cdot), \{y_i(\cdot)\}_{i\in\mathcal{I}}\big) \tag{2}$$

with the following constituents: $x(\cdot) : [0, T] \to \mathbb{R}^n$ is of bounded variation with its points of discontinuity equal to the set $\mathcal{T}$ of μ's atoms, $\phi(\cdot) : [0, S] \to \mathbb{R}^m$ is a graph completion of μ's distribution function $u(\cdot)$, and $\{y_i(\cdot)\}_{i\in\mathcal{I}}$ is a collection of Lipschitz functions, each defined on the nondegenerate interval $I_i := [s_i^-, s_i^+] := \phi_0^{-1}(t_i)$ and satisfying $y_i(s_i^{\pm}) = x(t_i\pm)$.

Definition 2.1 (Bressan-Rampazzo (B-R)): Consider a three-tuple X_μ as in (2), and let

$$y(s) = \left\{ \begin{array}{ll} x(t) & \text{if } s \notin \cup_{i\in\mathcal{I}} I_i, \quad t = \phi_0(s) \\ y_i(s) & \text{if } s \in I_i. \end{array} \right. \tag{3}$$

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Then X_μ is a *Bressan-Rampazzo (B-R) solution* of (1) provided $y(\cdot)$ is Lipschitz on $[0,S]$ and satisfies

$$\left\{\begin{array}{lcl} \dot{y}(s) & \in & F\big(y(s)\big)\dot{\phi}_0(s) + G\big(y(s)\big)\dot{\phi}(s) \\ y(0) & = & x_0. \end{array}\right. \tag{4}$$

An alternate definition of solution is developed in [11] that is framed in the original t-domain. It is shown there that the two solution concepts coincide.

III. A Sampling Method With Prescribed Measure

An Euler-type discretization procedure is introduced in this section that produces approximate discrete solutions (called *sampled trajectories*) when the measure μ and a graph completion are given. The limit of a subsequence of approximations will be shown to graph-converge in the Hausdorff metric to some solution X_μ of (1). With X_μ as in (2), its graph is defined as the set

$$\text{gr } X_\mu = \{(t, x(t)) : t \in [0,T]\} \cup \{(t_i, y_i(s)) : s \in I_i, i \in \mathcal{I}\}.$$

The idea is to discretize the ordinary trajectory $y(\cdot)$ that is defined in (3), where the "compactness of trajectories" is known to hold, and to project it down into t-space.

Let N be a positive integer, and let $h := \frac{S}{N}$ be the step-size parameter. Let $s_0 = 0 = t_0$, and for each $j = 1, \ldots, N$, let $s_j = jh$, $t_j = \phi_0(s_j)$, and $\lambda_j = t_j - t_{j-1}$. Sampled points $\{x_j\}_{j=1}^N$ are defined and "velocity" data are selected as follows (the parameter N is suppressed in this notation):

$$\begin{array}{lll} x_0 = x_0 & f_0 \in F(x_0) & g_0 \in G(x_0) \\ \multicolumn{3}{l}{x_1 = x_0 + \lambda_1 f_0 + g_0\big(\phi(s_1) - \phi(s_0)\big)} \\ & f_1 \in F(x_1) & g_1 \in G(x_1) \\ \vdots & \vdots & \vdots \\ \multicolumn{3}{l}{x_{j+1} = x_j + \lambda_j f_j + g_j\big(\phi(s_j) - \phi(s_{j-1})\big)} \\ & f_{j+1} \in F(x_{j+1}) & g_{j+1} \in G(x_{j+1}) \\ \vdots & \vdots & \vdots \\ \multicolumn{3}{l}{x_N = x_{N-1} + \lambda_N f_{N-1} + g_{N-1}\big(\phi(s_N) - \phi(s_{N-1})\big)} \end{array}$$

We denote by Ω^N the graph of a sampled trajectory:

$$\Omega^N := \big\{(t_j, x_j) : j = 0, \ldots, N\big\}. \tag{5}$$

The Hausdorff distance between two compact subsets A and B of $I\!\!R^n$ is denoted by $\text{dist}_{\text{H}}(A, B)$. A multifunction $\Gamma : I\!\!R^n \rightrightarrows I\!\!R^n$ with compact values is locally Lipschitz if for all compact $C \subset I\!\!R^n$, there exists a constant c so that

$$\text{dist}_{\text{H}}\big(\Gamma(x), \Gamma(y)\big) \le c\|x - y\| \quad \forall x,\, y \in C.$$

The main result of this section follows.

Theorem 1: Suppose $\mu \in \mathcal{B}_K([0,T])$ and a graph completion $\phi(\cdot)$ are given.

(a) For every sequence $\{\Omega^N\}_N$ of graphs of sampled trajectories, there is a solution X_μ of (1) and a subsequence $\{\Omega^{N_k}\}_k$ of $\{\Omega^N\}_N$ so that $\text{dist}_{\text{H}}\big(\Omega^{N_k},\ \text{gr } X_\mu\big) \to 0$ as $k \to \infty$.

(b) Assume F and G are locally Lipschitz. For every solution X_μ of (1), there exists a sequence $\{\Omega^N\}_N$ of graphs of sampled trajectories so that

$$\text{dist}_{\text{H}}\big(\Omega^N,\ \text{gr } X_\mu\big) \to 0 \quad \text{as} \quad N \to \infty.$$

Proof: Suppose the sequences $\{f_j\}$, $\{g_j\}$, $\{x_j\}$ are constructed by the sampling method described above. We first show there exists a constant c_1 independent of N so that

$$\max_j\big\{|x_j|,\, |f_j|,\, \|g_j\|\big\} \le c_1 \tag{6}$$

for all j and $N \in I\!\!N$. Indeed, since 1 is the Lipschitz constant of $\phi(\cdot)$ and with c as in (H2) and (H3), we have

$$\begin{array}{rcl} |x_{j+1}| & \le & |x_j| + h|f_j| + \|g_j\|h \\ & \le & |x_j| + \big[c(1 + |x_j|) + c(1 + |x_j|)\big]h \\ & = & h\alpha + \big[1 + h\alpha\big]|x_j|, \end{array}$$

where $\alpha := 2c$. It follows from the discrete Gronwall inequality that $|x_j| \le e^{\alpha S}(1 + |x_0|) - 1$, and that then (6) holds by (H2) and (H3) with $c_1 := c[e^{\alpha S}(1 + |x_0|)]$.

Define the multifunction $M : [0,S] \times I\!\!R^n \rightrightarrows I\!\!R^n$ by

$$M(s, y) = F(y)\dot{\phi}_0(s) + G(y)\dot{\phi}(s), \tag{7}$$

which is $\mathcal{L} \times \mathcal{B}$ measurable, has nonempty compact convex values, and has linear growth. Moreover, $M(s, \cdot)$ has closed graph for almost all $s \in [0,S]$. For each $N \in \mathbb{N}$, let $\tilde{\Omega}^N$ be the sampled trajectory in s-time:

$$\tilde{\Omega}^N := \big\{(s_j, x_j) : j = 0, \ldots, N\big\}. \tag{8}$$

Also consider its related polygonal arc $y^N(\cdot)$ defined on $[0,S]$ given as a linear interpolation of points in Ω^N. Note for later use that

$$\text{dist}_{\text{H}}\big(\tilde{\Omega}^N,\ \text{gr } y^N(\cdot)\big) \le \max\big\{h, 2c_1 h\big\}. \tag{9}$$

We claim there exist sequences of positive numbers δ_N and r_N so that $\delta_N \to 0$ and $r_N \to 0$ and a sequence of measurable sets $A_N \subseteq [0,S]$ so that $m(A_N) \to 0$, where all limits are as $N \to \infty$, and that these sequences satisfy

$$\inf\big\{|\dot{y}^N(s) - v| :\ v \in M\big(s, y^N(s) + \delta_N \overline{I\!\!B}\big)\big\} \ \le\ r_N, \tag{10}$$

almost everywhere on $[0,S] \backslash A_N$. To see this, let $\delta_N = 2c_1 S/N$ where c_1 is as in (6). Note for each $j = 1, 2, \ldots, N-1$ and $s \in [s_{j-1}, s_j]$ that

$$\begin{array}{rcl} \big|y^N(s) - x_j\big| & \le & \big|x_{j+1} - x_j\big| \\ & = & \big|\lambda_{j+1} f_j + g_j\big(\phi(s_{j+1}) - \phi(s_j)\big)\big| \\ & \le & h\big[|f_j| + \|g_j\|\big] \\ & \le & \delta_N. \end{array}$$

Next, for $s \in [0, S-h]$, define

$$\begin{array}{rcl} \Phi_0^N(s) & := & \frac{1}{h}\int_s^{s+h} \dot{\phi}_0(s')ds' \quad \text{and} \\ \Phi^N(s) & := & \frac{1}{h}\int_s^{s+h} \dot{\phi}(s')ds', \end{array}$$

and recall that $\Phi_0^N(s) \to \dot{\phi}_0(s)$ and $\Phi_0^N(s) \to \dot{\phi}(s)$ for almost all $s \in [0,S]$ as $N \to \infty$. By Egoroff's Theorem [7], there exist measurable sets $A_N \subseteq [0,S]$ with $m(A_N) \to 0$ (and for notational simplicity, we may assume $[S-h, S] \subseteq A_N$) and satisfying $r_N \to 0$ as $N \to \infty$, where r_N was taken to be

$$r_N = c_1 \max_{s \in [0,S] \backslash A_N} \Big\{\big|\Phi_0^N(s) - \dot{\phi}_0(s)\big|,\ \big|\Phi^N(s) - \dot{\phi}(s)\big|\Big\}$$

Now let $v^N(s) := f_j\dot{\phi}_0(s) + g_j\dot{\phi}(s)$ on $[s_j, s_{j+1}]$ and note that $v^N(s) \in M(s, x_j)$ for almost all $s \in [s_j, s_{j+1}]$. Recall that $\dot{y}^N(s) = \Phi_0^N(s_j)f_j + g_j\Phi^N(s_j)$, and thus

$$\max_{s\in[0,S]\setminus A_N} \left|\dot{y}^N(s) - v^N(s)\right| \le r_N.$$

We have shown that (10) holds.

From the compactness of trajectories theorem [4, Theorem 4.1.11], there exists a trajectory $y(\cdot)$ of M and a subsequence (we label as $\{y^{N_k}(\cdot)\}_k$) of $\{y^N(\cdot)\}_N$ so that $y^{N_k}(\cdot) \to y(\cdot)$ uniformly on $[0, S]$. One sees easily that this means

$$\mathrm{dist}_{\mathbf{H}}\big(\mathrm{gr}\; y^{N_k}(\cdot),\ \mathrm{gr}\; y(\cdot)\big) \to 0 \tag{11}$$

as $k \to \infty$. We define the components of a solution X_μ to (1) as follows. Let $x(\cdot) : [0, T] \to I\!R^n$ be given by $x(t) = y\big(\eta(t)\big)$, and define the functions $y_i(\cdot)$ (for each $i \in \mathcal{I}$) as the restriction of $y(\cdot)$ to I_i.

Now recall Ω^N as in (5) and $\tilde{\Omega}^N$ as in (8), and observe the second coordinates are the same for each $j = 1, \ldots, N$. Similarly, the second coordinates of gr X_μ and gr $y(\cdot) := \big\{\big(s, y(s)\big)\ :\ s \in [0, S]\big\}$ are the same for each $t \notin \mathcal{T}$, $t = \phi_0(s)$; and when $t \in \mathcal{T}$, the set of projections onto the second coordinate are the same. Thus the difference between the Hausdorff distances of Ω^N and gr X_μ on the one hand, and $\tilde{\Omega}^N$ and gr $y(\cdot)$ on the other is affected by only the first coordinate. It follows that

$$\mathrm{dist}_{\mathbf{H}}\big(\Omega^N,\ \mathrm{gr}\; X_\mu\big) \le \mathrm{dist}_{\mathbf{H}}\big(\tilde{\Omega}^N,\ \mathrm{gr}\; y(\cdot)\big), \tag{12}$$

where the righthand is at most h larger than the left side. By the triangle inequality, one has

$$\begin{aligned}&\mathrm{dist}_{\mathbf{H}}\big(\tilde{\Omega}^N,\ \mathrm{gr}\; y(\cdot)\big)\\ &\qquad \le \mathrm{dist}_{\mathbf{H}}\big(\tilde{\Omega}^N,\ \mathrm{gr}\; y^N(\cdot)\big) + \mathrm{dist}_{\mathbf{H}}\big(\mathrm{gr}\; y^N(\cdot),\ \mathrm{gr}\; y(\cdot)\big)\end{aligned}$$

Finally, passing to the subsequence $\{N_k\}$ and starting from (12), it follows from the previous inequality, (9), and (11) that $\mathrm{dist}_{\mathbf{H}}\big(\Omega^{N_k},\ \mathrm{gr}\; X_\mu\big) \to 0$, which finishes the proof of part (a).

To prove part (b), assume now that F and G are locally Lipschitz, and X_μ is as in (2) and is a solution of (1). Let $y(\cdot)$ be defined as in (3), and so there exist measurable selections $f(\cdot)$ and $g(\cdot)$ of $F\big(y(\cdot)\big)$ and $G\big(y(\cdot)\big)$ respectively so that $\dot{y}(s) = f(s)\dot{\phi}_0(s) + g(s)\dot{\phi}(s)$ a.e. $s \in [0, S]$. In a manner similar to proving the discrete bound (6), one can show there exists a constant c_2 so that $|y(s)| \le c_2$. Observe that for $0 \le \bar{s} < \hat{s} \le S$, and $c_3 := 2(1 + c_2)$, one has

$$|y(\hat{s}) - y(\bar{s})| \le \textstyle\int_{\bar{s}}^{\hat{s}} |\dot{y}(s)|\, ds \le c_3(\hat{s} - \bar{s}). \tag{13}$$

Let $L > 0$ be the Lipschitz constant for F and G on $c_2\overline{I\!B}$, and denote by $\mathrm{proj}_{F(y)}(f)$ the projection of f into $F(y)$ (which is unique since $F(y)$ is convex). If $|y_j| \le c_2$ ($j = 1, 2$) and $f \in F(y_1)$, then $|f - \mathrm{proj}_{F(y_2)}(f)| \le L|y_1 - y_2|$. Similar considerations hold with F replaced by G.

We use the notation of the sampling method, and will show there exists a sequence $\{\Omega^N\}$ that graph converges to gr X_μ.

Now, define $f_0 := \frac{1}{h}\int_0^{s_1} \mathrm{proj}_{F(x_0)}\big(f(s)\big)\, ds$, $g_0 := \frac{1}{h}\int_0^{s_1} \mathrm{proj}_{G(x_0)}\big(g(s)\big)\, ds$, and x_1 as described in our sampling method. We observe

$$\begin{aligned}
&x_1 - y(s_1) = I + II + III + IV, \quad \text{where}\\
&I := \tfrac{\phi_0(s_1)-\phi_0(0)}{h} \textstyle\int_0^{s_1} \big[\mathrm{proj}_{F(x_0)}\big(f(s)\big) - f(s)\big]\, ds,\\
&II := \textstyle\int_0^{s_1} \big[\mathrm{proj}_{G(x_0)}\big(g(s)\big) - g(s)\big] \left(\frac{\phi(s_1)-\phi(0)}{h}\right) ds,\\
&III := \textstyle\int_0^{s_1} \left(\frac{\phi_0(s_1)-\phi_0(0)}{h} - \dot{\phi}_0(s)\right) f(s)\, ds,\\
&IV := \textstyle\int_0^{s_1} g(s) \left(\frac{\phi(s_1)-\phi(0)}{h} - \dot{\phi}(s)\right) ds.
\end{aligned}$$

Recall $\phi_0(\cdot)$ is Lipschitz of rank 1, and so by the Lipschitz property of F, we have

$$|I| \le L \textstyle\int_0^{s_1} \big|y(s) - x_0\big|\, ds \le Lc_3 \int_0^{s_1} s\, ds = \frac{Lc_3}{2}h^2,$$

where the second inequality follows from (13). In the same way, one can show $|II| \le \frac{Lc_3}{2}h^2$ since $\phi(\cdot)$ is Lipschitz of rank r. To estimate III and IV, we re-use earlier notation to redefine $\Phi^N(\cdot)$ on $[0, S]$ by setting $\Phi^N(s) := \max\left\{\left|\frac{\phi_0(s_{j+1})-\phi_0(s_j)}{h} - \dot{\phi}_0(s)\right|, \left|\frac{\phi(s_{j+1})-\phi(s_j)}{h} - \dot{\phi}(s)\right|\right\}$ whenever $s \in [s_j, s_{j+1}]$. Then it follows that both $|III|$ and $|IV|$ are bounded above by $c(1 + c_2)\int_0^{s_1} \Phi^N(s)\, ds$. Putting all this together, we have

$$|x_1 - y(s_1)| \le Lc_3h^2 + 2c(1 + c_2) \textstyle\int_0^{s_1} \Phi^N(s)\, ds.$$

Proceed inductively: $f_j := \frac{1}{h}\int_{s_j}^{s_{j+1}} \mathrm{proj}_{F(x_j)}\, f(s)\, ds$ and $g_j := \frac{1}{h}\int_{s_j}^{s_{j+1}} \mathrm{proj}_{G(x_j)}\, g(s)\, ds$, and let x_{j+1} be as in the sampling method construction. The same argument used above can operate at each iteration, and inductively, one has the following estimate:

$$|x_j - y(s_j)| \le Lc_3jh^2 + 2c(1 + c_2) \textstyle\int_0^{s_j} \Phi^N(s)\, ds.$$

Since $\Phi^N(s)$ is bounded above and converges to 0 almost everywhere, it follows that $\tilde{\Omega}^N := \{(s_j, x_j) : j = 1, \ldots, N\}$ satisfies $\mathrm{dist}_{\mathbf{H}}\big(\tilde{\Omega}^N, \mathrm{gr}\; y(\cdot)\big) \to 0$ as $N \to \infty$. The bound in (12) is still valid here, and the conclusion of (b) readily follows. ∎

IV. Approximate controls

The original and perhaps most natural approach to defining solutions to the impulsive inclusion (1) is to consider limits of a sequence of solutions $x^N(\cdot)$ of an *approximate* control problem of the form

$$\dot{x}^N(t) \in F\big(x(t)\big)\dot{\phi}_0(t) + G\big(x(t)\big)\dot{u}^N(t), \tag{14}$$

where $d\mu^N = \dot{u}^N(\cdot)dt$ are absolutely continuous measures that approximate μ in some sense. See, for example, the discussion in [1]. We introduce in this section a concept of "graph convergence" of measures that is appropriate to carry out such an analysis. Graph convergence as defined below seems to be a stronger condition than expected, however even if the solutions of (14) are unique (which happens, for example, in the singleton case $F(x) = \{f(x)\}$ and $G(x) = \{g(x)\}$ with $f(\cdot)$ and $g(\cdot)$ Lipschitz functions), the limit arc may in general not be unique if the measures converge in a weaker sense.

Suppose we are given the following: a measure $\mu \in \mathcal{B}_K([0,T])$, an associated graph completion $(\phi_0, \phi) : [0,S] \to I\!R \times I\!R^n$ and a sequence $\{\mu^N\}$ of absolutely continuous measures belonging to $\mathcal{B}_K([0,T])$ whose associated distribution functions $u^N(t) := \mu^N\big([0,t]\big)$ are Lipschitz.

Definition 4.1: The sequence $\{\mu^N\}_N$ of absolutely continuous measures *graph-converges* to (μ, ϕ) provided

(i) there exist numbers $S^N > 0$ such that $S^N \to S$;

(ii) for each N, there exists a strictly increasing function $\phi_0^N(\cdot) : [0, S^N] \to [0,T]$ that is onto and Lipschitz of rank at most one, and such that

$$\int_0^{\min\{S,S^N\}} \big|\dot\phi_0^N(s) - \dot\phi_0(s)\big|\, ds \to 0 \quad \text{as} \quad N \to \infty$$

(iii) for each N, the sequence of functions defined by $\phi^N(s) := \big(u^N \circ \phi_0^N\big)(s)$ are Lipschitz with $\limsup_{N\to\infty} \|\dot\phi^N(\cdot)\|_\infty \le 1$, and satisfy

$$\int_0^{\min\{S,S^N\}} \big|\dot\phi^N(s) - \dot\phi(s)\big|\, ds \to 0 \quad \text{as} \quad N \to \infty.$$

The main result in this section follows.

Theorem 2: Suppose the measure $\mu \in \mathcal{B}_K([0,T])$ and an associated graph completion $\phi(\cdot) : [0,S] \to I\!R^n$ are given.

(a) Suppose $\{\mu^N\}$ is a sequence of absolutely continuous measures that graph-converges to $\big(\mu, \phi(\cdot)\big)$, and $\{x^N(\cdot)\}$ is a sequence of absolutely continuous arcs satisfying

$$\dot x^N(t) \in F\big(x^N(t)\big) + G\big(x^N(t)\big)\dot u^N(t). \tag{15}$$

Then there exists a solution X_μ of (1) and a subsequence $\{x^{N_k}(\cdot)\}$ of $\{x^N(\cdot)\}$ such that

$$\mathrm{dist}_{\mathrm{H}}\big(\mathrm{gr}\, x^{N_k}(\cdot), \mathrm{gr}\, X_\mu\big) \to 0 \quad \text{as} \quad k \to \infty.$$

(b) Conversely, suppose F and G are locally Lipschitz multifunction and $X_\mu := \big(x(\cdot), \phi(\cdot), \{y_i(\cdot)\}_{i\in\mathcal{I}}\big)$ is a solution of (1). Then there is a sequence $\{\mu^N\}$ of absolutely continuous measures that graph converge to $\big(\mu, \phi(\cdot)\big)$, and a sequence $x^N(\cdot)$ of solutions to (15) so that $\mathrm{dist}_{\mathrm{H}}\big(\mathrm{gr}\, x^N(\cdot), \mathrm{gr}\, X_\mu\big) \to 0 \quad$ as $\quad N \to \infty$.

Proof: Suppose we are given the measures $d\mu^N = \dot u^N(t)dt$, the functions $\phi_0^N(\cdot)$ and $\phi^N(\cdot)$ satisfying Definition 4.1, and solutions $x^N(\cdot)$ of (15). Set $\bar S^N := \min\{S, S^N\}$. Let $y^N(s) = (x^N \circ \phi_0^N)(s)$. Note, for almost all $s \in [0, \bar S^N]$, $\dot y^N(s) \in F\big(y^N(s)\big)\dot\phi_0^N(s) + G\big(y^N(s)\big)\dot\phi^N(s)$.

It follows that there exist measurable selections $f^N(s) \in F\big(y^N(s)\big)$ and $g^N(s) \in G\big(y^N(s)\big)$ so that $\dot y^N(s) = f^N(s)\dot\phi_0^N(s) + g^N(s)\dot\phi^N(s)$. Recall Definition 4.1 imposes a priori bounds on the Lipschitz rank of $\phi_0^N(\cdot)$ and $\phi^N(\cdot)$, and that $F(\cdot)$ and $G(\cdot)$ satisfy linear growth assumptions. A standard argument involving Gronwall's inequality implies there exists a constant c_4 independent of N that is an upper bound of both $\|f^N(\cdot)\|_\infty$ and $\|g^N(\cdot)\|_\infty$.

Let $M : [0,S] \times I\!R^n \rightrightarrows I\!R^n$ be defined as in (7), and define $\dot z^N(\cdot) : [0, \bar S^N] \to I\!R^n$ by $\dot z^N(s) := f^N(s)\dot\phi_0(s) + g^N(s)\dot\phi(s)$, and define $z^N(\cdot) : [0, \bar S^N] \to I\!R^n$ by $z^N(s) := x_0 + \int_0^s \dot z^N(s')\, ds'$. It is clear from the definitions that

$$\dot z^N(s) \in M\big(s, y^N(s)\big) \quad \text{a.e.} s \in [0, \bar S^N]. \tag{16}$$

Furthermore, it is readily seen that

$$\sup_{s\in[0,\bar S^N]} \big|z^N(s) - y^N(s)\big| \le c_4\big\{\|\dot\phi_0^N - \dot\phi_0\|_1 + \|\dot\phi^N - \dot\phi\|_1\big\}$$

which implies via the assumption of the graph convergence of the measures that $y^N - z^N$ approaches zero uniformly. In view of (16) and the compactness of trajectories theorem [4, Theorem 4.1.11], there exists $y(\cdot) : [0,S] \to I\!R^n$ that is a trajectory of M and to which a subsequence of $\{z^N(\cdot)\}$, and hence also of $\{y^N(\cdot)\}$, converges uniformly. That is, there exists a subsequence N_k for which

$$\mathrm{dist}_{\mathrm{H}}\big(\mathrm{gr}\, y^{N_k}(\cdot), \mathrm{gr}\, y(\cdot)\big) \to 0 \quad \text{as } k \to \infty. \tag{17}$$

We now define X_μ as before - see the paragraph containing (11) in the previous section. Similar reasoning as employed there shows also that $\mathrm{dist}_{\mathrm{H}}\big(\mathrm{gr}\, x^{N_k}(\cdot), \mathrm{gr}\, X_\mu\big)$ is bounded above by

$$\mathrm{dist}_{\mathrm{H}}\big(\mathrm{gr}\, y^{N_k}(\cdot), \mathrm{gr}\, y(\cdot)\big) + \sup_{s\in[0,\bar S^{N_k}]} |\phi_0^{N_k}(s) - \phi_0(s)|$$

which goes to zero as $k \to \infty$ by (17) and the assumption contained in Definition 4.1(ii). This finishes the proof of part (a).

We turn to part (b). Suppose F and G are now locally Lipschitz and X_μ is a solution to (1). For $N = 1, \ldots$, we proceed to construct the absolutely continuous measures μ^N and solutions $x^N(\cdot)$ of (15) that will converge in graph to X_μ. Fix $N > 0$ and set $h = \frac{S}{N}$, and for $j = 1, \ldots, N$, set $s_j = jh$ and $t_j = \phi_0(s_j)$. We will first introduce a new partition $\{\bar t_j\}$ of $[0,T]$ consisting of N distinct points that resembles the partition $\{t_j\}$ but has repeated nodes "pulled apart" and indexed accordingly, so that

$$|\bar t_j - t_j| \le h^2 \quad \text{for all } j. \tag{18}$$

Next, we define $\phi_0^N(\cdot) : [0,S] \to [0,T]$ as a linear interpolation of points $\{(s_j, \bar t_j\}$, which is onto and Lipschitz of rank at most 1. We claim that $\dot\phi_0^N(\cdot)$ converges to $\dot\phi_0(\cdot)$ in $L^1[0,S]$. Indeed, let $\tilde\phi_0^N(\cdot) : [0,S] \to [0,T]$ be a linear interpolation of points $\{s_j, t_j\}$. The difference between the linear interpolations $\phi_0^N(\cdot)$ and $\tilde\phi_0^N(\cdot)$ is that $\phi_0^N(\cdot)$ maps s_j to $\bar t_j$, whereas $\tilde\phi_0^N(\cdot)$ maps s_j to t_j. For $s \in [s_j, s_{j+1}]$, we have

$$\big|\dot\phi_0^N(s) - \dot{\tilde\phi}^N{}_0(s)\big| = \frac{1}{h}|\bar t_{j+1} - \bar t_j - t_{j+1} + t_j| \le 2h, \tag{19}$$

where the inequality is justified by (18). The Lebesgue differentiation Theorem says that $\dot{\tilde\phi}^N{}_0(s) \to \dot\phi_0(s)$ as $N \to \infty$ for almost all $s \in [0,S]$, and since these functions are bounded above by 1, the Dominated Convergence Theorem implies that $\dot{\tilde\phi}^N{}_0(\cdot) \to \dot\phi_0(\cdot)$ in $L^1[0,S]$. It follows from this and (19) that $\dot\phi_0^N(\cdot) \to \dot\phi_0(\cdot)$ in $L^1[0,S]$ as claimed.

Now define $u^N(\cdot) : [0,T] \to I\!R^n$ as the piecewise linear interpolation satisfying $u^N(\bar t_j) = \phi(s_j)$. Let $\phi^N(\cdot) := (u^N \circ \phi_0^N)(\cdot)$, and note $\phi^N(s_j) = \phi(s_j)$ for all j and for $s \in [s_j, s_{j+1}]$ that

$$\begin{aligned}\dot\phi^N(s) &= \dot u^N\big(\phi_0^N(s)\big)\dot\phi_0^N(s)\\ &= \tfrac{\phi(s_{j+1})-\phi(s_j)}{\bar t_{j+1}-\bar t_j}\ \tfrac{\bar t_{j+1}-\bar t_j}{h} = \tfrac{\phi(s_{j+1})-\phi(s_j)}{h}.\end{aligned}$$

Since $\phi(\cdot)$ is Lipschitz of rank 1, it follows that each of $\phi^N(\cdot)$ are also of rank at most 1. Completely analogous to the proof above showing $\dot{\phi}_0^N(\cdot) \to \dot{\phi}_0(\cdot)$ in $L^1[0,S]$ as $N \to \infty$, one has that $\dot{\phi}^N(\cdot) \to \dot{\phi}(\cdot)$ in $L^1[0,S]$ as $N \to \infty$. Therefore, with μ^N the absolutely continuous measure satisfying $d\mu^N = \dot{u}^N(t)dt$, we have shown that μ^N graph converges to $(\mu, \phi(\cdot))$ as $N \to \infty$ (where $S^N = S$ for all N in Definition 4.1).

We now turn to approximating a given a solution X_μ by a solution of (15). By Theorem 1(b), there exists a sequence of sampled trajectories whose graphs converge to gr X_μ. Denote these graphs as we did in (5), with $\{x_j\}$, $\{f_j\}$, and $\{g_j\}$ are constructed by a sampling scheme described in Section III. A new sampled set of points $\{\bar{x}_j\}$ is defined by replacing the partition $\{t_j\}$ by $\{\bar{t}_j\}$ and "tracking" the given sampled data. This is done as follows. Let $\bar{f}_0 = f_0$ and $\bar{g}_0 = g_0$ and define

$$\bar{x}_1 = \bar{x}_0 + (\bar{t}_1 - \bar{t}_0)\bar{f}_0 + \big(\bar{g}_0\big)\big(\phi(s_1) - \phi(s_0)\big)$$

Having chosen the data at stage j, inductively let $\bar{f}_j \in F(\bar{x}_j)$ and $\bar{g}_j \in G(\bar{x}_j)$ be the projections of f_j and g_j onto $F(\bar{x}_j)$ and $G(\bar{x}_j)$, respectively. Define the next node by

$$\bar{x}_{j+1} = \bar{x}_j + (\bar{t}_{j+1} - \bar{t}_j)\bar{f}_j + \big(\bar{g}_j\big)\big(\phi(s_{j+1}) - \phi(s_j)\big).$$

The linear growth assumptions on F and G guarantee that all of the sampled data remains in a bounded set, and let c_1 be as in (6) but which also bounds the newly sampled data. With L a Lipschitz constant for both F and G on $c_1\overline{I\!\!B}$, one has

$$|\bar{f}_j - f_j| \le L|\bar{x}_j - x_j|, \text{ and } \|\bar{g}_j - g_j\| \le L|\bar{x}_j - x_j|. \quad (20)$$

The estimate between the nodes x_j and $\bar{x}_j$ is calculated by

$$\begin{aligned}\big|\bar{x}_{j+1} - x_{j+1}\big| &\le \big|\bar{x}_j - x_j\big| + |t_{j+1} - t_j - \bar{t}_{j+1} + \bar{t}_j|\,|f_j| \\ &\quad + |\bar{t}_{j+1} - \bar{t}_j|\,|\bar{f}_j - f_j| \\ &\quad + \|\bar{g}_j - g_j\|\,|\phi(s_{j+1}) - \phi(s_j)| \\ &\le |\bar{x}_j - x_j| + 2h^2c_1 + 2hL|\bar{x}_j - x_j| \\ &= 2h^2c_1 + (1+2hL)|\bar{x}_j - x_j|,\end{aligned}$$

where (18), (20), and that $\phi(\cdot)$ is Lipschitz of rank 1 were invoked to deduce the second inequality. Gronwall's inequality implies

$$|\bar{x}_j - x_j| \le 2hc_1\frac{e^{2LS}-1}{2L}$$

for each $j = 0, 1, \ldots, N$, and in particular implies that

$$\text{dist}_{\text{H}}(\Omega^N, \bar{\Omega}^N) \to 0, \text{ as } N \to \infty \quad (21)$$

where $\bar{\Omega}^N$ is the newly sampled graph: $\bar{\Omega}^N := \{(\bar{t}_j, \bar{x}_j)\,|\, j = 1, ..., N\}$. Next, let $\bar{x}^N(\cdot)$ be the piecewise linear arc interpolating the points in $\bar{\Omega}^N(\cdot)$, which implies

$$\dot{\bar{x}}^N(t) = \bar{f}_j + \bar{g}_j\dot{u}^N(t) \in F(\bar{x}_j) + G(\bar{x}_j)\dot{u}^N(t) \quad (22)$$

whenever $t \in (\bar{t}_j, \bar{t}_{j+1})$. Let $\Gamma^N(\cdot) : [0,T] \times I\!\!R^n \rightrightarrows I\!\!R^n$ be given by $\Gamma^N(t,x) := F(x) + G(x)\dot{u}^N(t)$, which is the multifunction appearing in (15). It has convex compact values, is measurably Lipschitz (see [6]), and has linear growth in x. We will find a trajectory $x^N(\cdot)$ of Γ^N that is close to $\bar{x}^N(\cdot)$. Following the notation in [6], we have

$$\begin{aligned}\rho_\Gamma\big(\bar{x}^N(\cdot)\big) &:= \textstyle\sum_{j=0}^{N-1}\int_{t_j}^{t_{j+1}} \text{dist}\big(\dot{\bar{x}}^N(t), \Gamma^N\big(t, \bar{x}^N(t)\big)\big)\,dt \\ &\le \textstyle\sum_{j=0}^{N-1}\int_{t_j}^{t_{j+1}} \text{dist}_{\text{H}}\big(\Gamma^N\big(t, \bar{x}_j\big), \Gamma^N\big(t, \bar{x}^N(t)\big)\big)\,dt \\ &\le L\textstyle\sum_{j=0}^{N-1}\int_{t_j}^{t_{j+1}}\big(1 + |\dot{u}^N(t)|\big)\big|\bar{x}^N(t) - \bar{x}_j\big|\,dt,\end{aligned} \quad (23)$$

where (22) was used in the first inequality, and the Lipschitz property of F and G in the second. For $t \in [\bar{t}_j, \bar{t}_{j+1}]$, one has

$$\begin{aligned}\big|\dot{\bar{x}}^N(t) - \bar{x}_j\big| &\le \tfrac{t-\bar{t}_j}{\bar{t}_{j+1}-\bar{t}_j}\big|\bar{x}_{j+1} - \bar{x}_j\big| \\ &\le \tfrac{t-\bar{t}_j}{\bar{t}_{j+1}-\bar{t}_j}\big[(\bar{t}_{j+1}-\bar{t}_j)|\bar{f}_j| + \|\bar{g}_j\|\,|\phi(s_{j+1}) - \phi(s_j)| \\ &\le c_1\left[1 + \tfrac{h}{\bar{t}_{j+1}-\bar{t}_j}\right](t - \bar{t}_j) \text{ and}\end{aligned}$$

$$|\dot{u}^N(t)| = \left|\tfrac{\phi(s_{j+1})-\phi(s_j)}{\bar{t}_{j+1}-\bar{t}_j}\right| \le \tfrac{h}{\bar{t}_{j+1}-\bar{t}_j}.$$

We thus have

$$\begin{aligned}&\textstyle\int_{t_j}^{t_{j+1}}\big(1 + |\dot{u}^N(t)|\big)\big|\bar{x}^N(t) - \bar{x}_j\big|\,dt \\ &\quad\le [1 + \tfrac{h}{\bar{t}_{j+1}-\bar{t}_j}]c_1[1 + \tfrac{h}{\bar{t}_{j+1}-\bar{t}_j}]\textstyle\int_{t_j}^{t_{j+1}}(t - \bar{t}_j)\,dt \\ &\quad= c_1[1 + \tfrac{h}{\bar{t}_{j+1}-\bar{t}_j}]^2\tfrac{(\bar{t}_{j+1}-\bar{t}_j)^2}{2} \le c_6h^2\end{aligned}$$

for some constant c_6. Combining with (23), this estimate yields that $\rho_\Gamma\big(\bar{x}^N(\cdot)\big) \le LSc_6h$, and so by Filippov's Theorem (see [6, Theorem 3.1.6. page 115]), for each N there exists a trajectory $x^N(\cdot)$ of Γ^N such that $x^N(0) = x_0$ and for which

$$\text{dist}_{\text{H}}\big(\text{gr } x^N(\cdot), \text{gr } \bar{x}^N(\cdot)\big) \to 0 \quad (24)$$

as $N \to \infty$. Finally, we have by the triangular inequality

$$\begin{aligned}\text{dist}_{\text{H}}\big(\text{gr } x^N(\cdot), \text{gr } X_\mu\big) &\le \text{dist}_{\text{H}}\big(\text{gr } x^N(\cdot), \text{gr } \bar{x}^N(\cdot)\big) + \\ &+ \text{dist}_{\text{H}}\big(\text{gr } \bar{x}^N(\cdot), \bar{\Omega}^N\big) + \text{dist}_{\text{H}}\big(\bar{\Omega}^N, \Omega^N\big) \\ &+ \text{dist}_{\text{H}}\big(\Omega^N, \text{gr } X_\mu\big)\end{aligned}$$

which approaches 0 as $N \to \infty$ by (24), (21), and Theorem 1. ∎

V. Constructing Measure Via Sampling

One of the drawbacks of the sampling method given in Section III is that the measure μ must be specified in order to sample. This problem arises particularly when a measure μ has to be chosen so that a certain state constraint is satisfied. For example, invariance requirement when trajectory stays within a closed set. In this section, measure μ featured in (1) will not be specified directly, it will rather be constructed along with trajectory using another discretization scheme, given only a positive number S, multifunctions $F(\cdot)$ and $G(\cdot)$ satisfying the standing hypotheses, and a closed cone $K \subset I\!\!R^m$. At the end of this section we will see an example on how the new sampling technique is used for the invariance result. Due to the limited number of pages assigned to the paper, we offer only sketches of the proofs in this section. Details will be published elsewhere.

Let $N > 0$ be an integer and let $h := \frac{S}{N}$ be the step size. Let $s_0 =$ and for each $j = 1, ..., N$, let $s_j = jh$. Let us now define the sampled points $\{y_j\}_{j=1}^N$:

$$\lambda_0 \in [0,1] \qquad k_0 \in K_1 \qquad f_0 \in F(x_0) \qquad g_0 \in G(x_0)$$

$$
\begin{array}{llll}
x_1 := x_0 + \lambda_0 h f_0 + (1-\lambda_0) h g_0 k_0 & & & \\
\lambda_1 \in [0,1] & k_1 \in K_1 & f_1 \in F(x_1) & g_1 \in G(x_1) \\
\vdots & \vdots & \vdots & \vdots \\
x_{j+1} := x_j + \lambda_j h f_j + (1-\lambda_j) h g_j k_j & & & \\
\lambda_{j+1} \in [0,1] & k_{j+1} \in K_1 & f_{j+1} \in F(x_{j+1}) & g_{j+1} \in G(x_{j+1}) \\
\vdots & \vdots & \vdots & \vdots \\
x_N := x_{N-1} + \lambda_{N-1} h f_{N-1} + (1-\lambda_{N-1}) h g_{N-1} k_{N-1} & & &
\end{array}
$$

Here, $K_1 = K \cap S_1$, where S_1 is a unit sphere in $\mathbb{R}^m$.

Let polygonal arc $y^N(\cdot)$ be a linear interpolation of points $\{s_j, x_j\}$. Let us reuse notation of Ω^N, and let it this time be

$$\Omega^N := \big\{(s_j, x_j) : j = 0, \ldots, N\big\}.$$

The following theorem holds.

Theorem 3: Suppose that $S > 0$ is given. For every sequence $\{\Omega^N\}_N$ of graphs of sampled trajectories, there is a measure $\mu \in \mathcal{B}([0,T]$, normalized graph completion $(\phi_0, \phi)(\cdot)$, solution X_μ of (1) and a subsequence $\{\Omega^{N_k}\}_k$ of $\{\Omega^N\}_N$ such that $\mathrm{dist}_{\mathrm{H}}\big(\Omega^{N_k},\ \mathrm{gr}\, y(\cdot)\big) \to 0$ as $k \to \infty$.

Proof: Define $\lambda^N(\cdot)$ and $k^N(\cdot)$ on $[0,S]$ so that $\lambda^N(s) := \lambda_j$ and $k^N(s) := k_j$ on $[s_j, s_{j+1}]$. Let $\lambda(\cdot)$ be any uniform limit of $\{\lambda^N(\cdot)\}_N$ and let $k(\cdot)$ be any uniform limit of $\{k^N(\cdot)\}_N$. These limits exist because sequences $\{\lambda^N(\cdot)\}_N$ and $\{|k^N(\cdot)|\}_N$ are uniformly bounded by 1. We proceed by utilizing the compactness of trajectory result in a similar way as it was done in the proof of Theorem 1. There exists a bound c_1 independent of N for sequences $\{|x_j|\}$, $\{|f_j|\}$ and $\{\|g_j\|\}$ like in equation (6). Multifunction $M : [0,S] \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is here defined as $M(s,y) := F(y)\lambda(s) + \big(1-\lambda(s)\big)G(y)k(s)$, number δ_N is taken to be the same value as in the proof of Theorem 1 and number r_N is defined as $r_N := c_1 \max_{s\in[0,S]}\big\{\big|\lambda^N(s) - \lambda(s)\big|, \big|\big(1-\lambda^N(s)\big)k^N(s) - \big(1-\lambda(s)\big)k(s)\big|\big\}$. On each interval $[s_j, s_{j+1}]$, arc $v^N(s) \in M(s, x_j)$ was taken to be $v^N(s) := \lambda(s) f_j + \big(1-\lambda(s)\big) g_j k(s)$. Now, one shows that $\inf\{|\dot{y}^N(s) - v| \ : \ v \in M(s, y^N(s) + \delta_N \overline{\mathbb{B}})\} \le r_N$, on $[0,S]$, which assures existence of a trajectory $y(\cdot)$ of M and a subsequence $\{y^{N^k}(\cdot)\}_{N_k}$ of $\{y^N(\cdot)\}_N$ so that $\{y^{N^k}(\cdot)\}_{N_k}$ converges uniformly to $y(\cdot)$ on $[0,S]$. Since $\dot{y}(s) \in M(s, y(s))$, there are selections $f(s) \in F(y(s))$ and $g(s) \in G(y(s))$ so that $\dot{y}(s) = f(s)\lambda(s) + (1 - \lambda(s))g(s)k(s)$. The pair $(\phi_0, \phi) : [0,S] \mapsto [0,T] \times \mathbb{R}^m$, where $T := \phi_0(S)$,

$$\phi_0(s) := \textstyle\int_0^s \lambda(s')ds', \quad \phi(s) := \int_0^s (1-\lambda(s'))k(s')ds'$$

generates functions $\eta : [0,T] \to [0,S]$ and $u : [0,T] \to \mathbb{R}^m$:

$$\eta(t) := \phi_0^{-1}(t+), \qquad u(t) := \phi(\eta(t)).$$

Evidently, the pair $(\phi_0, \phi)(\cdot)$ is a normalized graph completion to the measure $\mu \in \mathcal{B}_K[0, \phi_0(S)]$ generated by distribution $u(\cdot)$.

We define other components of a solution X_μ to (1), as follows. Let $x(\cdot) : [0,T] \to \mathbb{R}^n$ be given by $x(t) = y\big(\eta(t)\big)$, and define the functions $y_i(\cdot)$ (for each $i \in \mathcal{I}$) as the restriction of $y(\cdot)$ to I_i. Similar procedure as in Theorem 2 shows that there is a subsequence $\{\Omega^{N_k}\}_k$ of $\{\Omega^N\}_N$ such that $\mathrm{dist}_{\mathrm{H}}\big(\Omega^{N_k},\ \mathrm{gr}\, y(\cdot)\big) \to 0$ as $k \to \infty$. ■

Example 5.1: In this example we briefly describe how the sampling method described in this section can be used in construction of measure μ which forces trajectory to remain in a closed set. Suppose that C is a closed set and assume that the system (1) is such that for each $x \in C$ and $\zeta \in N_C^P(x)$(:= proximal normal cone to set C at x, [4]) there exist $\lambda \in [0,1]$ and $k \in K_1$, $f \in F(x)$ and $g \in G(x)$ so that $\langle \lambda f + (1-\lambda) g k, \zeta \rangle \le 0$. Then one can show that $y(s) \in C$ for all $s \in [0,S]$, where $y(\cdot)$ is as in Definition 2.1. Indeed, for all N, one constructs $\{x_j\}$ in the sampling method described in this section, by choosing $\{\lambda_j\}$,$\{k_j\}$, $\{f_j\}$ and $\{g_j\}$ so that

$$\langle \lambda_j h f_j + (1-\lambda_j) h g_j k_j, x_j - c(x_j) \rangle \le 0$$

for a $c(x_j) \in \mathrm{proj}_C(x_j)$. One follows the proof of non-impulsive weak invariance characterization [5, Theorem 2.1.] to see that $d_C(x_j)^2 \le 4c_1^2 h$ for all $j = 1 \ldots N$. In other words, the nodes $\{x_j\}$ converge to the points in C as $N \to \infty$, and so does their uniform limit $y(\cdot)$. Given a closed set C, we say that the impulsive system (1) is *weakly invariant* in C, if for every $x_0 \in C$ and $S > 0$, there exists a time $0 \le T \le S$, a measure $\mu \in \mathcal{B}_K[0,T]$, and a solution X_μ of (1) so that $y(s) \in C$ for all $s \in [0,S]$,

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