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Model reduction via projection of generalized state space systems

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***Abstract*— We show that the projection of generalized state space models of SISO systems allows to construct arbitrary lower order models and that they can be obtained via the solution of particular generalized Sylvester equations. This generalizes the results already obtained for state space systems, where both the original models and low order models were constrained to be strictly proper. We also conjecture that for MIMO systems, this approach is as general as one can hope for.**

I. Introduction

It has been shown in the literature [3], [2] that most reduced order models $\hat{T}(s)$ of a given $p \times m$ transfer function $T(s) := C(sI_N - A)^{-1}B + D$ of Mc Millan degree N can be obtained via projection of the state vector of the system, i.e. $\hat{T}(s) := \hat{C}(sI_n - \hat{A})^{-1}\hat{B} + D$ where

$$sI - \hat{A} = Z^T(sI - A)V, \quad \hat{B} = Z^T B, \quad \hat{C} = CV, \quad \hat{D} = D.$$

This has been rigorously proven for SISO systems and shown to be true for almost all MIMO systems as well [4], [5]. It should be pointed out, however, that this is actually restrictive since both $\hat{T}(s)$ and $T(s)$ have to be equal at $s = \infty$. An equivalent statement is that it only holds for strictly proper systems, since then $D = \hat{D} = 0$.

That this is indeed restrictive follows from the following example, known as *singular perturbation approximation* [6]. Such a reduced order model for $T(s) := C(sI_N - A)^{-1}B + D$ is constructed as follows. Perform first a similarity transformation T such that

$$\left[\begin{array}{c|c} sI_N - T^{-1}AT & T^{-1}B \\ \hline -CT & D \end{array}\right] := \left[\begin{array}{cc|c} sI_{n_1} - A_{11} & -A_{12} & B_1 \\ -A_{21} & sI_{n_2} - A_{22} & B_2 \\ \hline -C_1 & -C_2 & D \end{array}\right]$$

then the reduced order model is defined as $\{A_r, B_r, C_r, D_r\}$, with

$$A_r = A_{11} + A_{12}(\gamma I - A_{22})^{-1}A_{21}, \; B_r = B_1 + A_{12}(\gamma I - A_{22})^{-1}B_2,$$

$$C_r = C_1 + C_2(\gamma I - A_{22})^{-1}A_{21}, \; D_r = D + C_2(\gamma I - A_{22})^{-1}B_2,$$

and where $\gamma = 0$ for a continuous-time system and $\gamma = 1$ for a discrete-time system.

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If we assume $m \geq p$, we can instead realize this same transfer function using

$$\left[\begin{array}{c|c} s\bar{E} - \bar{A} & \bar{B} \\ \hline -\bar{C} & 0 \end{array}\right] := \left[\begin{array}{ccc|c} sI_{n_1} - A_{11} & -A_{12} & 0 & B_1 \\ -A_{21} & sI_{n_2} - A_{22} & 0 & B_2 \\ -C_1 & -C_2 & I_p & D \\ \hline 0 & 0 & -I_p & 0 \end{array}\right], \quad (1)$$

which will typically be a minimal generalized state-space realization. For such a system, it suffices to choose the projection matrices

$$Z^T := \begin{bmatrix} I_{n_1} & Z_1^T & 0 \\ 0 & Z_2^T & I_p \end{bmatrix}, \quad V := \begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \\ 0 & I_p \end{bmatrix}, \quad (2)$$

where

$$Z_1^T := A_{12}(\gamma I - A_{22})^{-1}, \quad Z_2^T := C_2(\gamma I - A_{22})^{-1},$$

to obtain the realization $\{Z^T(sE - A)V, Z^T B, CV, 0\}$ in the form

$$\left[\begin{array}{cc|c} sI_{n_1} - A_r & 0 & B_r \\ -C_r & I_p & D_r \\ \hline 0 & -I_p & 0 \end{array}\right],$$

of the desired low order model. In this paper, we give a more general treatment of this problem. We give sufficient conditions for the existence of a projection of a generalized state space realization to yield a particular generalized state space realization of a given reduced order model.

II. Generalized state space realizations

Definition 2.1: Let $T(s)$ be an arbitrary rational matrix function. A quintuple (A, B, C, D, E) such that

$$T(s) = C(sE - A)^{-1}B + D$$

is called a *generalized* state space realization of $T(s)$. If $D = 0$, the quadruple (A, B, C, E) is also called a generalized state space realization of $T(s)$. The dimension of the square matrices A and E is called the order of the realization.

In this paper, we only consider generalized state space realizations with $D = 0$. As we will see, any (not necessarily strictly proper) rational function admits a generalized state space realization with $D = 0$. Since the inverse of $sE - A$ must exist, the pencil $sE - A$ must be nonsingular. This implies that the Kronecker form of $(sE - A)$ is [10]

$$\begin{bmatrix} sI - A_{fin} & 0 \\ 0 & I - sJ_{inf} \end{bmatrix},$$

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where J_{inf} is a block diagonal matrix where each diagonal block is a Jordan block, but with zero as only possible eigenvalue.

By rewriting

$$B = \left[\begin{array}{cc} B_{fin} & B_{inf} \end{array}\right], \quad C = \left[\begin{array}{c} C_{fin} \\ C_{inf} \end{array}\right]$$

with appropriate dimensions, one obtains

$$T(s) = C_{fin}(sI_{n_{fin}} - A_{fin})^{-1}B_{fin} + C_{inf}(I_{n_{inf}} - sJ_{inf})^{-1}B_{inf}.$$

¿From this, it is clear that $C_{fin}(sI_{n_{fin}} - A_{fin})^{-1}B_{fin}$ is strictly proper and that the polynomial part of $T(s)$ is

$$C_{inf}(I_{n_{inf}} - sJ_{inf})^{-1}B_{inf} = \sum_{k=0}^{n_{inf}} C_{inf}J_{inf}^{k}B_{inf}s^{k}.$$

The order of the state space realization is equal to $n_{fin} + n_{inf}$. This order is in general not equal to the Mc Millan degree of a rational function, defined as follows [10].

Definition 2.2: The Mc Millan degree of a rational matrix $T(s)$ is denoted by $\delta[T(s)]$, and given by

$$\delta[T(s)] = \nu[\bar{T}(s)] + \nu[D(s^{-1})],$$

where $\bar{T}(s)$, $D(s)$ are the strictly proper and polynomial parts of $T(s)$, respectively, and $\nu[\quad]$ denotes the regular order of an irreducible (regular) state-space realization of the associated strictly proper rational matrix.

The following definition is essential.

Definition 2.3: A quadruple (A,B,C,E) is called a *minimal* generalized state space realization of $T(s)$ when there exists no generalized state space realization $(\bar{A},\bar{B},\bar{C},\bar{E})$ of $T(s)$ of smaller order.

It is essential to keep in mind that we always consider here $D = 0$. Note that in general the order of a minimal state space realization is not equal to the Mc Millan degree of $T(s)$ (see [10], [8] for a discussion).

In the strictly proper case, it is well known that a state space realization (A,B,C,E) is minimal if and only if the matrix $\left[\begin{array}{cc} sE-A & B \end{array}\right]$ is of full row rank for any value of $s \in \mathbb{C}$ (the pair $(sE-A,B)$ is then called controllable) and the matrix $\left[\begin{array}{c} sE-A \\ C \end{array}\right]$ is of full column rank of any value of $s \in \mathbb{C}$ (the pair $(sE-A,C)$ is then called observable). Note that in the strictly proper case, the order of a minimal generalized state space realization is equal to the Mc Millan degree of the corresponding transfer function.

In the non proper case, the minimality (in the sense of 2.3) corresponds to the following conditions.

Theorem 2.1: Let (A,B,C,E) be a state space realization of $T(s)$. Decompose this into the strictly proper part and the polynomial part, i.e. choose invertible matrices M,N such that

$$\left[\begin{array}{c|c} M(sE-A)N & MB \\ \hline -CN & 0 \end{array}\right] = \left[\begin{array}{cc|c} sI_{n_{fin}} - A_{fin} & 0 & B_{fin} \\ 0 & I_{inf} - sJ_{inf} & B_{inf} \\ \hline -C_{fin} & -C_{inf} & 0 \end{array}\right].$$

The state space realization (A,B,C,E) is minimal if and only if

1) The realization $(A_{fin},B_{fin},C_{fin})$ of the strictly proper transfer function $T_{fin}(s) := C_{fin}(sI_{n_{fin}} - A_{fin})^{-1}B_{fin}$ is minimal.
2) The realization $(J_{inf},B_{inf},C_{inf})$ of the strictly proper transfer function $C_{inf}(sI_{n_{inf}} - J_{inf})^{-1}B_{inf}$ is minimal.

Proof: The minimality of $(A_{fin},B_{fin},C_{fin})$ is clearly necessary. Assume that the state space realization $(J_{inf},B_{inf},C_{inf})$ is non minimal. Then, there exists $(\hat{A},\hat{B},\hat{C})$ of smaller order $\hat{n} < n_{inf}$ such that $C_{inf}(sI_{n_{inf}} - J_{inf})^{-1}B_{inf} = \hat{C}(sI_{\hat{n}} - \hat{A})^{-1}\hat{B}$. This implies that $C_{inf}(I_{n_{inf}} - sJ_{inf})^{-1}B_{inf} = \hat{C}(I_{\hat{n}} - s\hat{A})^{-1}\hat{B}$, i.e. (A,B,C,E) is non minimal.

Let us assume that the minimality conditions are satisfied. If there exists another state space realization $(\bar{A},\bar{B},\bar{C},\bar{E})$ of smaller order for $T(s)$, this implies that either there exists a state space realization of $T_{fin}(s)$ of smaller order, or that there exists a state space realization of $C_{inf}(sI_{n_{inf}} - J_{inf})^{-1}B_{inf}$ of smaller order, i.e. one of the two minimality assumptions is not satisfied. ■

A proof of the following result can be found in [10].

Corollary 2.1: The Mc Millan degree of a transfer function realized by a minimal generalized state space realization (A,B,C,E) is equal to the rank of E.

Another corollary is the following.

Corollary 2.2: Every $p \times m$ transfer function $T(s)$ of Mc Millan degree n can be realized as a generalized state space model of the type $T(s) = C(sE-A)^{-1}B$, where the dimension of the pencil $sE-A$ is $N + min(m,p)$.

Proof: The normal rank of the polynomial part of $T(s)$ cannot be larger than $min(m,p)$. The proof is done by observing that the order of a minimal generalized state space realization of $T(s)$ (with $D = 0$) is equal to the Mc Millan degree of $T(s)$ plus the normal rank of the polynomial part of $T(s)$. See [10] for a discussion. ■

III. PRELIMINARY RESULTS

In the single input single output case, the following result has recently been proved [2].

Theorem 3.1: Let (A,B,C) and $(\hat{A},\hat{B},\hat{C})$ be arbitrary minimal state space realizations of SISO transfer functions of Mc Millan degree n and $k < n$, respectively. Then $(\hat{A},\hat{B},\hat{C})$ can be constructed via projection of (A,B,C).

In order to generalize this theorem to non proper transfer functions, we need some preliminary results.

Definition 3.1: Let (A,B,C,E) and $(\hat{A},\hat{B},\hat{C},\hat{E})$ be two generalized state space realizations of respective order n and $k < n$. The realization $(\hat{A},\hat{B},\hat{C},\hat{E})$ is *embedded* into (A,B,C,E) if there exists a pair of *projection* matrices $Z,V \in \mathbb{C}^{n\times k}$ such that

$$Z^T(sE-A)V = s\hat{E} - \hat{A}, \quad Z^TB = \hat{B}, \quad CV = \hat{C}. \qquad (3)$$

A straightforward consequence of this definition is the following

Proposition 3.1: If $(\hat{A},\hat{B},\hat{C},\hat{E})$ is embedded in (A,B,C,E), then for any invertible matrices $M,N,\hat{M},\hat{N}$ of appropriate dimension, $(\hat{M}\hat{A}\hat{N}, \hat{M}\hat{B}, \hat{C}\hat{N}, \hat{M}\hat{E}\hat{N})$ is embedded in (MAN, MB, CN, MEN).

Proof: Because $(\hat{A}, \hat{B}, \hat{C}, \hat{E})$ is embedded in (A, B, C, E), there exist Z, V such that (3) is satisfied. Straightforward computations show that $M^{-T}Z\hat{M}^T, N^{-1}V\hat{N}$ satisfies (3) for the transformed realizations. ∎

Proposition 3.2: Let (A, B, C, E) and $(\hat{A}, \hat{B}, \hat{C}, \hat{E})$ be two minimal state space realizations of the SISO strictly proper transfer functions $T(s)$ and $\hat{T}(s)$ of respective Mc Millan degree n and $k < n$. Then $(\hat{A}, \hat{B}, \hat{C}, \hat{E})$ is embedded in (A, B, C, E).

Proof: Because $T(s)$ and $\hat{T}(s)$ are strictly proper, the matrices E and $\hat{E}$ of the minimal state space realizations (A, B, C, E) and $(\hat{A}, \hat{B}, \hat{C}, \hat{E})$ are invertible. From Proposition 3.1, it is equivalent to prove that $(\hat{E}^{-1}\hat{A}, \hat{E}^{-1}\hat{B}, \hat{C}, I_k)$ is embedded into $(E^{-1}A, E^{-1}B, C, I_n)$. The proof follows from Theorem 3.1. ∎

It remains to consider the singular case. In order to handle this case, we will use the following result.

Proposition 3.3: Let (A, B, C, E) be a minimal state space realization of a possibly non proper SISO transfer function $T(s)$. For any value $\mu \in \mathbb{C}$ such that $E + \mu A$ is nonsingular, the generalized state space realization

$$(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{E}) := (A, B, C, E + \mu A)$$

is a minimal state space realization of the transfer function

$$\tilde{T}(s) := \tilde{C}(s\tilde{E} - \tilde{A})^{-1}\tilde{B},$$

and $\tilde{T}(s)$ is strictly proper.

Proof: First note that if $det(sE - A) \neq 0$ for some $s \in \mathbb{C}$, then $det\,(s(E + \mu A) - A)$ is not identically equal to zero, i.e. the inverse of $s\tilde{E} - \tilde{A}$ is well defined for every $\mu \in \mathbb{C}$. As μ is chosen such that $\tilde{E}$ is nonsingular, there is no Jordan block at infinity in the Kronecker form of $s\tilde{E} - \tilde{A}$. This implies that the transfer function $\tilde{T}(s)$ is strictly proper. Let us prove that $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{E})$ is a minimal generalized state space realization. As $\tilde{T}(s)$ is strictly proper, we only have to prove that the polynomial matrix $\begin{bmatrix} s\tilde{E} - \tilde{A} & \tilde{B} \end{bmatrix}$ has full row rank for any $s \in \mathbb{C}$ and that the polynomial matrix $\begin{bmatrix} (s\tilde{E} - \tilde{A})^T & \tilde{B}^T \end{bmatrix}^T$ has full column rank for any $s \in \mathbb{C}$. Let us consider the first matrix. Assume that there exists $\lambda \in \mathbb{C}$ such that there exists a nonzero vector y such that

$$y^T \begin{bmatrix} \lambda\tilde{E} - \tilde{A} & \tilde{B} \end{bmatrix} = 0. \quad (4)$$

Several cases are possible. If $\lambda = 0$, (4) implies that $y^T A = 0$ and $y^T B = 0$. This is impossible because A, B, C, E is a minimal realization.

Let us assume that $\lambda \neq 0$.

The equation (4) implies that

$$y^T (\lambda E - (1 - \lambda\mu)A) = 0.$$

If $\lambda\mu = 1$, (4) implies that $y^T E = 0$ and $y^T B = 0$. This is again impossible because (A, B, C, E) is minimal.

Finally, if $\lambda \neq 0$ and $\lambda\mu \neq 1$, (4) implies that $y^T \left(\frac{\lambda}{1-\lambda\mu}E - A\right) = 0$ and $y^T B = 0$, which is impossible because (A, B, C, E) is minimal. Similar arguments prove that the polynomial matrix $\begin{bmatrix} (s\tilde{E} - \tilde{A})^T & \tilde{B}^T \end{bmatrix}^T$ has full column rank for any $s \in \mathbb{C}$. This concludes the proof. ∎

IV. Main Results

Here is the main result of this paper.

Theorem 4.1: Let (A, B, C, E) be an arbitrary minimal state space realization of order n of an arbitrary, not necessarily proper, SISO transfer function $T(s)$. Let $(\hat{A}, \hat{B}, \hat{C}, \hat{E})$ be an arbitrary minimal state space realization of order $k < n$ of an arbitrary, not necessarily proper, SISO transfer function $\hat{T}(s)$, then $(\hat{A}, \hat{B}, \hat{C}, \hat{E})$ is embedded in (A, B, C, E).

Proof: Assume that $T(s)$ and/or $\hat{T}(s)$ has a polynomial part. This implies that either E or $\hat{E}$ or both are singular. Because the pencils $sE - A$ and $s\hat{E} - \hat{A}$ are regular, it is always possible to find $\mu \in \mathbb{C}$ such that the matrices $E + \mu A$ and $\hat{E} + \mu\hat{A}$ are invertible. ¿From Proposition 3.3, the state space realizations $(A, B, C, E + \mu A)$ and $(\hat{A}, \hat{B}, \hat{C}, \hat{E} + \mu\hat{A})$ are minimal state space realizations of strictly proper transfer functions. From Proposition 3.2, there exist Z, V such that

$$Z^T B = \hat{B}, \quad Z^T AV = \hat{A}, \quad CV = \hat{C} \quad (5)$$

and

$$Z^T (E + \mu A)V = \hat{E} + \mu\hat{A}. \quad (6)$$

Injecting (5) into (6) gives $Z^T EV = \hat{E}$. This concludes the proof. ∎

In [2], [9], it is shown that in the SISO strictly proper case, with standard state space realizations (A, B, C), the projecting matrices can always be obtained from Sylvester equations of the form

$$AVM_1 + VM_2 + BX = 0, \quad A^T ZN_1 + ZN_2 + C^T Y = 0.$$

As a consequence of the preceding results developed for generalized state space realizations of SISO rational matrices, the projection matrices can always been obtain from Sylvester equations of the form:

$$AVM_1 + EVM_2 + BX = 0, \quad A^T ZN_1 + E^T ZN_2 + C^T Y = 0.$$

The proof is omitted.

V. Special Case: Singular Perturbation

Proposition 5.1: Let (A, B, C, D) be a minimal standard state space realization of the $p \times m$ transfer function $T(s) := C(sI_n - A)^{-1}B + D$. Assume that $m \geq p$. If the rank of D is equal to p, then the generalized state space realization

$$\left[\begin{array}{cc|c} sI_n - A & 0 & B \\ -C & I_p & D \\ \hline 0 & -I_p & 0 \end{array}\right],$$

is minimal.

Proof: See [10], [8], [7]. ∎

The objective of the singular perturbation approximation is to maintain the DC gain between the original and the reduced-order transfer function. This corresponds to impose $\hat{T}(0) = T(0)$ in the continuous time case and $T(e^{j\omega}) = \hat{T}(e^{j\omega})$ in the discrete time case. Thanks to (2), an alternative way to verify that this interpolation condition is indeed satisfied consists in verifying that the generalized Krylov subspace $(\gamma\bar{E}-\bar{A})^{-1}\bar{B}$ is contained in one of the projecting matrices Z or V of (2) (see [1] for a discussion). This can be verified by straightforward computation.

VI. About non minimal realizations

If the state space realizations are allowed to be non minimal, everything is possible, as shown in the following proposition.

Proposition 6.1: Let $(T_1(s),T_2(s))$ be an arbitrary pair of $p\times m$ rational functions (not necessarily proper). Let (A_1,B_1,C_1,E_1) be an arbitrary state space realization of $T_1(s)$. There always exists a (possibly non minimal) generalized state space realization (A,B,C,E) of $T_2(s)$ such that (A_1,B_1,C_1,E_1) is embedded in it.

Proof: Let (A_2,B_2,C_2,E_2) be a generalized state space realization of $T_2(s)$. Define the state space realization

$$\left[\begin{array}{c|c} sE-A & B \\ \hline -C & 0 \end{array}\right] := \left[\begin{array}{cccc|c} sE_1-A_1 & 0 & 0 & 0 & 0 \\ 0 & sE_c-M_c & 0 & 0 & B_1 \\ 0 & 0 & sE_o-M_o & 0 & 0 \\ 0 & 0 & 0 & sE_2-A_2 & B_2 \\ \hline 0 & 0 & -C_1 & -C_2 & 0 \end{array}\right],$$

where the matrices M_c,E_c,E_o and M_o are arbitrary matrices of appropriate dimension. It is clear that (A,B,C,E) is a (non minimal) realization of $T_2(s)$. Moreover, one can obtain the realization (A_1,B_1,C_1,E_1) by projecting (A,B,C,E) with

$$Z := \begin{bmatrix} I \\ I \\ 0 \\ 0 \end{bmatrix}, \quad V := \begin{bmatrix} I \\ 0 \\ I \\ 0 \end{bmatrix}.$$

■

It should be pointed that the Mc Millan degree of $T_1(s)$ and $T_2(s)$ are also arbitrary. Moreover, this result is valid for any value of the natural numbers m and p. This is in contrast with minimal state space realizations. Let us be more precise.

Proposition 6.2: Let (A_1,B_1,C_1,E_1) be a minimal realization of order n_1 of the SISO rational function $T_1(s)$. Let (A_2,B_2,C_2,E_2) be a minimal realization of order n_2 of the SISO rational function $T_2(s)$.

1) If $n_1<n_2$, then (A_1,B_1,C_1,E_1) is embedded into (A_2,B_2,C_2,E_2).
2) If $n_1=n_2$, (A_1,B_1,C_1,E_1) is embedded into (A_2,B_2,C_2,E_2) if and only if $T_1(s)=T_2(s)$.
3) If $n_1>n_2$, (A_1,B_1,C_1,E_1) is not embedded into (A_2,B_2,C_2,E_2).

Proof: The first case is proven in Theorem 4.1. To prove the second case, assume that (A_1,B_1,C_1,E_1) is embedded into (A_2,B_2,C_2,E_2). This implies that there exist two square matrices Z,V such that $Z^T(sE_2-A_2)V=(sE_1-A_1)$. Because the pencil sE_1-A_1 is regular, the matrices Z and V must be invertible. This clearly implies that $T_1(s)=T_2(s)$. The last case is clear since $Z^T(sE_2-A_2)V$ cannot have a normal rank larger than $n_2<n_1$, contradicting the fact that sE_1-A_1 must be a regular pencil. ■

VII. Concluding Remarks

The fact that we impose the D matrix to be zero in the generalized state space realization permits to define correctly a concept of minimality that is preserved under allowed perturbations of the E matrix, i.e $\tilde{E}=E+\mu A$. In turns, this permit us to extent our results proven in the strictly proper case to the non proper case. By imposing $D=0$, we have thus been able to prove in the SISO case that any minimal state space realization can been obtained by any minimal state space realization of larger order. Thanks to the results of this paper, we have shown that all the existing model reduction techniques developed so far are particular cases of a projection-based model reduction framework. This permits to unify the theory. Moreover, this sheds new lights on the singular perturbation model reduction technique.

As shown in the preceding section, the minimality assumption of the state space realizations is also essential.

The problem of embedding of non minimal and MIMO state space realizations is certainly not a closed topic. Clearly, one can not hope to obtain much stronger results than what is available for the strictly proper case, and it was shown in [4], [5] that this problem is clearly more restrictive than the SISO case.

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