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Trajectory Tracking for a Threecycle Mobile Robot: the Vector Field Orientation Approach

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Abstract— **The paper presents a proposition of the trajectory tracking task solution for the car-like threecycle mobile vehicle from a theoretical point of view. Proposed concept comes from the Vector Field Orientation (VFO) approach introduced for example in [9] and [10]. The VFO methodology results from a simple geometrical interpretation of the possible time evolution of the considered kinematics in a response to specific control signals. The characteristic features of the obtained VFO controller are its simple and intuitive synthesis and *natural* transient states of the controlled vehicle in a task space. Control performances are illustrated by simulation results.**

I. INTRODUCTION

Control of nonholonomic vehicles is still a challenging problem for automatic community. In the literature there are many approaches, which allow to solve locally or globally the control tasks for such systems (for surveys see [20], [18], [6], [13]). The difficulty in the control design comes from the existence of nonintegrable velocity constraints, which restrict the instantaneous set of motion directions [4], [21]. Available propositions of solutions results from usage of different mathematical tools like static and dynamic linearization [23], [14], [22], homogeneous approximations [16], [15], Lyapunov stability theory [1], [12], [8], [7], model transformations and discontinuous approach [2], [5], [3] or Lie algebra and Lie groups techniques [19], [17]. Although the solutions are effective from a theoretical point of view, some aspects are often not considered deeply enough. Many proposed controllers involve state transformation to auxiliary spaces, so results are valid only locally. Such practical problem like simple controller synthesis ensuring good quality of transient states of controlled vehicles often is not satisfactory solved. Tuning of control parameters is usually really hard and non intuitive (design coefficients have got any physical interpretation). Moreover, shaping the desired transient states is very difficult to obtain and often depends on initial system's state. Due to these facts, alternative propositions of control solutions for nonholonomic systems can be useful to improve unsatisfactory performances.

In this paper the trajectory tracking problem will be considered for the threecycle mobile robot (TMR) with the vector field orientation (VFO) approach. This approach results from a simple and intuitive geometrical interpretation of a possible time evolution of the TMR kinematics in a response to specific controls. The VFO methodology is applicable to original kinematic model (no state transformation is needed) and can be treated as a unified approach to solve tracking or stabilizing tasks for different nonholonomic systems. It has been successfully applied to control several driftless models, among them unicycle vehicle [9] and nonholonomic 3-D manipulator [10]. Resulting VFO controllers give *natural* non-oscillatory transient responses of the controlled systems. Moreover, design parameters tuning is very simple, with clear influence on movement characteristics and can be compared with synthesis of simple linear controllers.

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II. KINEMATICS

We will consider the threecycle mobile robot with rear axle driven shown in Fig. 1. The kinematic model of the

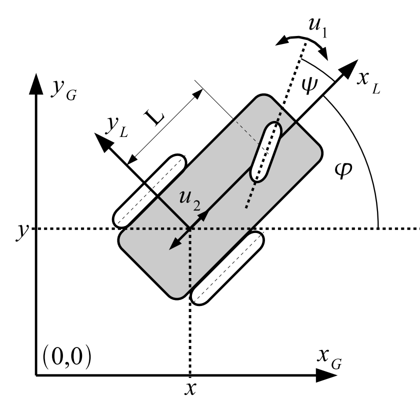

Fig. 1. Threecycle mobile robot in the global frame $\{G\}$.

robot for $L = 1$ can be represented as follows:

$$\begin{bmatrix} \dot\psi \\ \dot\varphi \\ \dot x \\ \dot y \end{bmatrix} \triangleq \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ \tan\psi \\ \cos\varphi \\ \sin\varphi \end{bmatrix} u_2, \tag{1}$$

where $\boldsymbol{q} \triangleq [\psi \ \varphi \ x \ y]^T \in Q \subset \mathbb{R}^4$ is a state vector: ψ – steering angle, φ – orientation angle, x, y – position coordinates in a global frame, u_1, u_2 are control signals and it is assumed, that

$$|\psi| \le \epsilon\frac{\pi}{2}, \quad \epsilon \in (0, 1) \tag{2}$$

to avoid the movement obstruction. Above model belongs to the class of driftless nonholonomic systems:

$$\dot{\boldsymbol{q}} = \boldsymbol{g}_1 u_1 + \boldsymbol{g}_2 u_2 = \boldsymbol{G}(\boldsymbol{q})\boldsymbol{u} \tag{3}$$

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with nonitegrable velocity constraints in the Pfaffian form:

$$\boldsymbol{A}(\boldsymbol{q})\dot{\boldsymbol{q}} = \boldsymbol{0}. \tag{4}$$

Considering absence of a lateral slippage of the rear robot's axle, the constraints matrix for the threecycle has the form:

$$\boldsymbol{A}(\boldsymbol{q}) = [0\ 0\ -\sin\varphi\ \cos\varphi]. \tag{5}$$

In this paper we will consider only admissible and persistently exciting reference trajectories $\boldsymbol{q}_t \triangleq [\psi_t\ \varphi_t\ x_t\ y_t]^T \in \mathbb{R}^4$ as a solution of the reference model

$$\begin{bmatrix} \dot{\psi}_t \\ \dot{\varphi}_t \\ \dot{x}_t \\ \dot{y}_t \end{bmatrix} \triangleq \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u_{1t} + \begin{bmatrix} 0 \\ \tan\psi_t \\ \cos\varphi_t \\ \sin\varphi_t \end{bmatrix} u_{2t}, \tag{6}$$

with reference controls $\boldsymbol{u}_t \triangleq [u_{1t}\ u_{2t}]^T$ and with assumptions

$$\forall_{\tau\geq 0} \quad u_{2t}(\tau) \neq 0 \quad \text{and} \quad |\psi_t| \leq \epsilon\frac{\pi}{2}, \quad \epsilon \in (0,1). \tag{7}$$

The tracking task is to find bounded controls $\boldsymbol{u} = [u_1\ u_2]^T$, to ensure the tracking errors

$$\boldsymbol{e} \triangleq \begin{bmatrix} e_\psi \\ e_\varphi \\ e_x \\ e_y \end{bmatrix} \triangleq \begin{bmatrix} \psi_t - \psi \\ \varphi_t - \varphi \\ x_t - x \\ y_t - y \end{bmatrix} \in \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{R}^2, \tag{8}$$

tend to zero for $\tau \to \infty$ (note: since $e_\varphi \in \mathbb{S}^1$, values $e_\varphi = \{0, \pm 2\pi, \pm 4\pi, \ldots\}$ will be treated equivalently).

III. VFO APPROACH

Vector field orientation approach comes from a simple geometrical interpretation of a possible time evolution of system (1). In the kinematics (1) the first vector field (vf) $\boldsymbol{g}_1$ is constant and oriented along the axis of the first component $\dot{\psi}$ of the tangent space. The second vf $\boldsymbol{g}_2(\boldsymbol{q})$ depends on two state variables: ψ and φ. Hence, instantaneous values of these two variables determine the instantaneous orientation (direction) of $\boldsymbol{g}_2$ in $\mathbb{R}^4$. Since the first component of $\boldsymbol{g}_2$ is zero, the orientation of vf $\boldsymbol{g}_2^*(\boldsymbol{q}) \triangleq [\tan\psi\ \cos\varphi\ \sin\varphi]^T \in \mathbb{R}^3$ is only non-trivially defined. We can call ψ and φ the *orienting variables*, although only the first orienting variable ψ can be controlled in the direct manner through the signal u_1. This variable will be called the *orienting variable directly controlled*, and the control u_1 – the *directly orienting control*. On the other hand, the instantaneous value of the second orienting variable φ depends on the *orienting variable directly controlled* ψ and on the second control signal u_2. The second input signal u_2 can be interpreted as a control, which pushes the 3-D state vector $\boldsymbol{q}^* \triangleq [\varphi\ x\ y]^T$ along the current direction of $\boldsymbol{g}_2^*(\boldsymbol{q})$ vf. One can call u_2 as the *pushing control*. In this way, we have decomposed the control process for (1) into two subprocesses: orienting and pushing [3]. Let us now introduce a convergence vector field:

$$\boldsymbol{h} \triangleq \begin{bmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \end{bmatrix} \triangleq \begin{bmatrix} h_1 \\ h_2 \\ \boldsymbol{h}^{**} \end{bmatrix} \triangleq \begin{bmatrix} h_1 \\ \boldsymbol{h}^* \end{bmatrix} \in \mathbb{R}^4, \tag{9}$$

which defines a convergence direction to the reference posture $\boldsymbol{q}_t \triangleq [\psi_t\ \varphi_t\ x_t\ y_t]^T$ during a tracking task. Taking classically $\boldsymbol{h} \triangleq \boldsymbol{e}$, where $\boldsymbol{e}$ is defined in (8), the convergence vector becomes the posture error vector. But taking another definition of $\boldsymbol{h}$, it is easier to shape the transient behavior of the controlled vehicle and to ensure convergence of all tracking errors asymptotically to zero [9], [10]. Let us assume, that the convergence vector $\boldsymbol{h}$ is given. Recalling previous geometrical interpretations of the time evolution of (1), the control strategy can be divided into two subtasks: 1) orienting control subtask, in which u_1 puts the direction of $\boldsymbol{g}_2^*(\boldsymbol{q})$ on the current direction defined by $\boldsymbol{h}^* \in \mathbb{R}^3$, 2) pushing control subtask, in which u_2 pushes the subsystem

$$\dot{\boldsymbol{q}}^* \triangleq \boldsymbol{g}_2^*(\boldsymbol{q})u_2, \tag{10}$$

along the current direction of $\boldsymbol{g}_2^*(\boldsymbol{q})$, where for TMR:

$$\dot{\boldsymbol{q}}^* \overset{(1)}{=} \begin{bmatrix} \dot{\varphi} \\ \dot{x} \\ \dot{y} \end{bmatrix}, \qquad \boldsymbol{g}_2^*(\boldsymbol{q}) \overset{(1)}{=} \begin{bmatrix} \tan\psi \\ \cos\varphi \\ \sin\varphi \end{bmatrix}. \tag{11}$$

Since the orienting process is not simultaneous, it seems to be reasonable to push (10) only proportionally to the orthogonal projection of $\boldsymbol{h}^*$ on the current direction of $\boldsymbol{g}_2^*(\boldsymbol{q})$. Concluding the whole above interpretation, design methodology for the orienting control u_1 can be described by the following relation:

$$\text{find } u_1 : \left\{ \lim_{\tau\to\infty} \ (\boldsymbol{g}_2^*(\boldsymbol{q})\, k = \boldsymbol{h}^*) \right\}, \tag{12}$$

where $k = k(.)$ is some nonzero scalar function and methodology for the pushing control u_2 can be written as follows:

$$\text{find } u_2 : \{ \ \dot{\boldsymbol{q}}^* \propto \| \boldsymbol{h}^* \| \cos\alpha, \quad \alpha\angle(\boldsymbol{g}_2^*, \boldsymbol{h}^*) \}. \tag{13}$$

The above VFO strategy should assure convergence of the vehicle position $\boldsymbol{q}^{**} \triangleq [x\ y]^T$ to the reference one $\boldsymbol{q}_t^{**} \triangleq [x_t\ y_t]^T$ (see the next section). Additionally to solve the tracking task, a definition of the vf $\boldsymbol{h}$ has to ensure convergence of the orienting variables ψ and φ to the reference ones ψ_t, φ_t near the reference position $\boldsymbol{q}_t^* = [x_t\ y_t]^T$. This imposes some limitations on the $\boldsymbol{h}$ vf design. In the subsequent section a proposition for $\boldsymbol{h}$ vf will be given.

IV. VFO TRACKING CONTROLLER

Now we derive the proposition for orienting control u_1. Using definitions (11) and (9) one can rewrite condition (12) in the following form:

$$\text{find } u_1 : \lim_{\tau\to\infty} \left\{ \begin{bmatrix} \tan\psi \\ \cos\varphi \\ \sin\varphi \end{bmatrix} k = \begin{bmatrix} h_2 \\ h_3 \\ h_4 \end{bmatrix} \right\}. \tag{14}$$

Hence to put the direction of $\boldsymbol{g}_2^*(\boldsymbol{q})$ on the direction of $\boldsymbol{h}^*$, it suffices to meet (at least at the limit) three above right hand side equations. Combining two last equations we can rewrite (14) as follows:

$$\text{find } u_1 : \lim_{\tau\to\infty} \left\{ \begin{array}{c} \psi \to \arctan\left(\frac{h_2}{k}\right) \\ \varphi \to \text{Atan2}\,(sgn(k)h_4, sgn(k)h_3) \end{array} \right\}, \tag{15}$$

where Atan2 (., .) denotes the four-quadrant inverse tangent function and

$$sgn(z) \triangleq \begin{cases} 1, & \text{for} \quad z \geq 0, \\ -1, & \text{for} \quad z < 0. \end{cases} \tag{16}$$

From the last relation of (15) it results, that the sign of function k determines the movement strategy for the TMR (*forward/backward movement*). To make the strategy compatible with the movement of the reference system also during transient states (to ensure smooth entering on the reference trajectory) let us define:

$$k(u_2, u_{2t}) \triangleq |u_2|\, sgn(u_{2t}), \tag{17}$$

where u_{2t} is the reference control along the reference trajectory and $sgn(u_{2t})$ does not change in time according to the first of assumptions (7). To fulfill requirements (15) let us introduce the following auxiliary variables: *auxiliary steering angle*:

$$\psi_d \triangleq \arctan\left(\frac{h_2}{|u_2|\, sgn(u_{2t})}\right) \in \left[-\epsilon\frac{\pi}{2}, \epsilon\frac{\pi}{2}\right], \tag{18}$$

where ϵ comes from (2), and the *auxiliary direction angle*:

$$\varphi_d \triangleq \text{Atan2}\left(sgn(u_{2t})h_4, sgn(u_{2t})h_3\right). \tag{19}$$

According to (15), to ensure putting direction of $\boldsymbol{g}_2^*$ on the direction of $\boldsymbol{h}^*$, one should make variables ψ and φ tend to ψ_d and φ_d, respectively. Since u_1 directly influences ψ variable and h_1 constitutes the first component of the convergence vector $\boldsymbol{h}$ we propose to choose:

$$h_1 \triangleq k_1 e_{\psi d} + \dot{\psi}_d, \qquad e_{\psi d} \triangleq \psi_d - \psi \tag{20}$$

and

$$u_1 \triangleq h_1 \overset{(20)}{=} k_1(\psi_d - \psi) + \dot{\psi}_d, \tag{21}$$

where $k_1 > 0$ is a design coefficient. Moreover we propose to take:

$$h_4 \triangleq k_p e_y + \dot{y}_t \tag{22}$$

$$h_3 \triangleq k_p e_x + \dot{x}_t \tag{23}$$

$$h_2 \triangleq k_p e_{\varphi d} + \dot{\varphi}_d, \tag{24}$$

where $k_p > 0$ is a design coefficient and

$$e_{\varphi d} \triangleq \varphi_d - \varphi, \tag{25}$$

where φ_d is defined in (19). Introduced errors $e_{\psi d}$, $e_{\varphi d}$ are called *auxiliary steering error* and *auxiliary direction error*, respectively. Velocity forms of $\dot{\psi}_d$ and $\dot{\varphi}_d$ are described in Appendix. Taking into account the methodology for design of *pushing control* u_2 we propose to choose:

$$u_2 \triangleq k_2 \|\boldsymbol{h}^*\| \cos\alpha, \quad k_2 \triangleq \frac{1}{\|\boldsymbol{g}_2^*\|} \overset{(1)}{=} \frac{1}{\sqrt{1+\tan^2\psi}}, \tag{26}$$

where $\boldsymbol{h}^* \overset{(9)}{=} [h_2\ h_3\ h_4]^T$, $\alpha\angle(\boldsymbol{g}_2^*, \boldsymbol{h}^*)$ and hence

$$\cos\alpha \triangleq \frac{\boldsymbol{g}_2^{*T}\boldsymbol{h}^*}{\|\boldsymbol{g}_2^*\|\,\|\boldsymbol{h}^*\|}, \quad \alpha \in [-\pi, \pi). \tag{27}$$

Substituting (27) into (26) allows to rewrite the u_2 definition in the simpler form well determined also for $\|\boldsymbol{h}^*\| = 0$:

$$u_2 \triangleq \frac{1}{\|\boldsymbol{g}_2^*\|^2}\boldsymbol{g}_2^{*T}\boldsymbol{h}^* \overset{(1)}{=} \frac{h_2\tan\psi + h_3\cos\varphi + h_4\sin\varphi}{1+\tan^2\psi}. \tag{28}$$

Now we can formulate the following theorem.

Theorem 1: Assuming:
A1. persistence movement of (6): $\forall_{\tau\geq 0}\ u_{2t} \neq 0$,
A2. $x_t, y_t \in C^3$, $\quad u_{1t}, u_{2t} \in \mathcal{L}_\infty$,
A3. $\forall_{\tau\geq 0}\, h_3^2 + h_4^2 \neq 0$ and $\forall_{\tau\geq 0}\, h_2^2 + u_2^2 \neq 0$,

the VFO control law (21) and (28) applied to system (1) ensures global asymptotic convergence of tracking errors (8) to zero at the limit $\tau \to \infty$.

Proof: First we consider the behavior of two auxiliary errors: $e_{\psi d}$ and $e_{\varphi d}$. Substituting (21) into the first equation in model (1) one gets:

$$\dot{e}_{\psi d} + k_1 e_{\psi d} = 0 \quad \Rightarrow \quad \lim_{\tau\to\infty} \psi = \psi_d. \tag{29}$$

From definition (18) $|u_2|\, sgn(u_{2t})\tan\psi_d = h_2$. Combining this formula with definition (24) gives:

$$|u_2|\, sgn(u_{2t})\tan\psi_d = k_p e_{\varphi d} + \dot{\varphi}_d. \tag{30}$$

It can be shown (see Appendix), that

$$\exists_{\hat{\tau}<\infty} : \forall_{\tau>\hat{\tau}}\ sgn(u_2) = sgn(u_{2t}). \tag{31}$$

Taking this into account and putting ψ instead of ψ_d in (30) (at the limit $\tau \to \infty$), one yields due to (29) and from the second equation in (1):

$$\lim_{\psi\to\psi_d} \dot{e}_{\varphi d} + k_p e_{\varphi d} = 0 \quad \Rightarrow \quad \lim_{\tau\to\infty} \varphi = \varphi_d. \tag{32}$$

Asymptotic convergence of $e_{\psi d}$ and $e_{\varphi d}$ to zero allows us to conclude, that the aim denoted by (12) is met at the limit $\tau \to \infty$. Therefore since $\alpha\angle(\boldsymbol{g}_2^*, \boldsymbol{h}^*)$ it is clear, that the following relation must hold:

$$\lim_{\psi\to\psi_d, \varphi\to\varphi_d}(1-\cos^2\alpha) = 0 \overset{(29),(32)}{\Longrightarrow} \lim_{\tau\to\infty}(1-\cos^2\alpha) = 0. \tag{33}$$

Now let us consider the convergence of position errors e_x, e_y. To do this we introduce the following error vector and velocity vector:

$$\boldsymbol{e}_d^* \triangleq [e_{\varphi d}\ e_x\ e_y]^T = \boldsymbol{q}_d^* - \boldsymbol{q}^*, \qquad \dot{\boldsymbol{q}}_d^* \triangleq [\dot{\varphi}_d\ \dot{x}_t\ \dot{y}_t]^T. \tag{34}$$

Recalling definitions (22)-(24) we can write

$$\boldsymbol{h}^* \triangleq k_p\boldsymbol{e}_d^* + \dot{\boldsymbol{q}}_d^*, \quad \dot{\boldsymbol{e}}_d^* = \dot{\boldsymbol{q}}_d^* - \dot{\boldsymbol{q}}^* \tag{35}$$

Combining two above formulas gives the following equation

$$\dot{\boldsymbol{e}}_d^* + k_p\boldsymbol{e}_d^* = \boldsymbol{r}, \qquad \boldsymbol{r} = \boldsymbol{h}^* - \dot{\boldsymbol{q}}^*. \tag{36}$$

It can be shown (see Appendix), that the following relation is true:

$$\|\boldsymbol{r}\|^2 = \|\boldsymbol{h}^*\|^2\gamma^2, \quad \text{where} \quad \gamma \triangleq \sqrt{1-\cos^2\alpha}. \tag{37}$$

One proposes the positive definite Lapunov function:

$$V(\boldsymbol{e}_d^*) \triangleq \frac{1}{2}\boldsymbol{e}_d^{*T}\boldsymbol{e}_d^*. \tag{38}$$

The time derivative of above function along trajectories of (36) can be estimated as follows:

$$\begin{aligned}
\dot{V} &= \boldsymbol{e}_d^{*T}\dot{\boldsymbol{e}}_d^* \overset{(36)}{=} -k_p \,\|\, \boldsymbol{e}_d^*\|^2 + \boldsymbol{e}_d^{*T}\boldsymbol{r} \leq \\
&\leq -k_p \,\|\, \boldsymbol{e}_d^*\|^2 + \|\, \boldsymbol{e}_d^*\| \,\|\, \boldsymbol{r}\| \overset{(37)}{=} \\
&\overset{(37)}{=} -k_p \,\|\, \boldsymbol{e}_d^*\|^2 + \|\, \boldsymbol{e}_d^*\| \,\|\, \boldsymbol{h}^*\| \,\gamma \overset{(35)}{=} \\
&\overset{(35)}{=} -k_p \,\|\, \boldsymbol{e}_d^*\|^2 + \|\, \boldsymbol{e}_d^*\| \,\|\, k_p\boldsymbol{e}_d^* + \dot{\boldsymbol{q}}_d^*\| \,\gamma \leq \\
&\leq -k_p \,\|\, \boldsymbol{e}_d^*\|^2 + \|\, \boldsymbol{e}_d^*\| \,(k_p \,\|\, \boldsymbol{e}_d^*\| + \|\, \dot{\boldsymbol{q}}_d^*\|)\gamma = \\
&= -k_p(1-\gamma) \,\|\, \boldsymbol{e}_d^*\|^2 + \|\, \boldsymbol{e}_d^*\| \,\|\, \dot{\boldsymbol{q}}_d^*\| \,\gamma \,.
\end{aligned}$$

Hence,

$$\dot{V} \leq 0 \quad \Leftrightarrow \quad \|\, \boldsymbol{e}_d^*\| \,\|\, \dot{\boldsymbol{q}}_d^*\| \,\gamma \leq k_p(1-\gamma) \,\|\, \boldsymbol{e}_d^*\|^2 \tag{39}$$

and finally the convergence condition can be written as follows:

$$\dot{V} \leq 0 \quad \Leftrightarrow \quad \|\, \boldsymbol{e}_d^*\| \geq \frac{\|\, \dot{\boldsymbol{q}}_d^*\| \,\gamma}{k_p(1-\gamma)}. \tag{40}$$

Boundedness of $\|\, \dot{\boldsymbol{q}}_d^*\|$ results from (34), assumptions **A2**, **A3** and definition (48) (see Appendix). Since $\|\, \dot{\boldsymbol{q}}_d^*\| \in \mathcal{L}_\infty$ and $\gamma \overset{(33)}{\to} 0$ one can conclude, that:

$$\lim_{\tau\to\infty} e_{\varphi d} = 0 \;\wedge\; \lim_{\tau\to\infty} e_x = 0 \;\wedge\; \lim_{\tau\to\infty} e_y = 0. \tag{41}$$

Recalling definitions (19), (22) and (23) we can write:

$$\lim_{e_x,e_y\to 0} \varphi_d = \varphi_t \mod 2\pi \quad \overset{(32)}{\Rightarrow} \quad \lim_{\tau\to\infty} e_\varphi = 2\pi n, \tag{42}$$

where $n = 0, \pm 1, \pm 2, \ldots$ (see the note after eq.(8)). It can be shown (see Appendix), that

$$\lim_{e_\varphi,e_x,e_y\to 0} \left\{ u_2 = u_{2t}\frac{\sqrt{1+\tan^2\psi_t}}{\sqrt{1+\tan^2\psi}} \right\}. \tag{43}$$

Regarding this fact and since $\lim_{e_x,e_y\to 0}\varphi_d = \varphi_t \;\wedge\; \lim_{\tau\to\infty} e_{\varphi d} = 0 \;\Rightarrow\; \lim_{\tau\to\infty} h_2 \overset{(24)}{\to} \dot{\varphi}_t$, one may rewrite (18) as follows:

$$\lim_{\tau\to\infty} \; u_{2t}\frac{\sqrt{1+\tan^2\psi_t}}{\sqrt{1+\tan^2\psi}}\tan\psi_d = \dot{\varphi}_t \tag{44}$$

Using (29) we get

$$\begin{gathered}
\lim_{\tau\to\infty} \; u_{2t}\frac{\sqrt{1+\tan^2\psi_t}}{\sqrt{1+\tan^2\psi_d}}\tan\psi_d = \dot{\varphi}_t \;\Rightarrow \\
\lim_{\tau\to\infty} \; u_{2t}^2(1+\tan^2\psi_t)\tan^2\psi_d = \dot{\varphi}_t^2(1+\tan^2\psi_d) \;\Rightarrow \\
\lim_{\tau\to\infty} \; u_{2t}^2\tan^2\psi_d + (u_{2t}^2\tan^2\psi_t - \dot{\varphi}_t^2)\tan^2\psi_d = \dot{\varphi}_t^2 \overset{(6)}{\Rightarrow} \\
\lim_{\tau\to\infty} \; u_{2t}^2\tan^2\psi_d = \dot{\varphi}_t^2 \;\Rightarrow\; \lim_{\tau\to\infty} \; \psi_d = \psi_t
\end{gathered}$$

and combining above result with (29) gives:

$$\lim_{\tau\to\infty} \psi = \psi_t \quad \Rightarrow \quad \lim_{\tau\to\infty} e_\psi = 0. \tag{45}$$

Since $e_{\psi d}, \dot{\psi}_d, \|\, \boldsymbol{e}_d^*\|, \|\, \dot{\boldsymbol{q}}_d^*\| \in \mathcal{L}_\infty$, signals $u_1, u_2 \in \mathcal{L}_\infty$. ■

Remark 1: The auxiliary signals $\psi_d,\ \varphi_d$ can not be determined by (18) and (19) at time instants $\overline{\tau}$, in which respectively $h_2(\overline{\tau}) = u_2(\overline{\tau}) = 0$ and $h_4(\overline{\tau}) = h_3(\overline{\tau}) = 0$ (also the time derivatives $\dot{\psi}_d,\ \dot{\varphi}_d$ can not be directly defined from (49), (48)). Although such conditions theoretically can occur (only during transient stage), in practice (in simulations) it is very rare situation[1]. However, in such a case we propose to choose, following the work [22], the auxiliary signals as follows:

$$\psi_d(\overline{\tau}) = \psi_d(\overline{\tau}_-), \qquad \varphi_d(\overline{\tau}) = \varphi_d(\overline{\tau}_-). \tag{46}$$

The feed-forward velocities $\dot{\psi}_d(\overline{\tau}),\ \dot{\varphi}_d(\overline{\tau})$ can be computed using the de L'Hospital analysis to (49) and (48) (see Appendix) [22] or as $\dot{\psi}_d(\overline{\tau}) = \dot{\psi}_d(\overline{\tau}_-)$ and $\dot{\varphi}_d(\overline{\tau}) = \dot{\varphi}_d(\overline{\tau}_-)$. As a consequence, generally only a piecewise continuity for control inputs u_1, u_2 can be guaranteed.

V. SIMULATION RESULTS

Performances of the VFO controller defined by (21) and (26) have been examined by simulations. During conducted simulations the reference trajectory has been computed by the numerical integration[2] of the model (6) with fixed time step $T_p = 0.005[s]$ for the time horizon $\tau_h = 10[s]$ and for the following reference controls: $u_{1t} = 0.1[rad/s]$, $u_{2t} = 1[m/s]$ and initial conditions: $\psi_{t0} = \pi/8$, $\varphi_{t0} = 0[rad]$ and $x_{t0} = y_{t0} = 2[m]$. Controller parameters have been chosen as follows: $k_p = 5$, $k_1 = 5$. Initial conditions for the controlled vehicle (1) have been assumed to be: $\psi_0 = 2\pi/5$, $\varphi_0 = -\pi/2[rad]$, $x_0 = y_0 = 0[m]$. In both kinematics (the reference and controlled ones) the steering angle saturation coefficient has been set as $\epsilon = 0.8$ (see (2) and (7)). During simulations the velocity $\dot{\psi}_d$ included in (21) has been computed numerically using *Euler backward* discretization algorithm ($n \in \mathbb{N}$):

$$\frac{d\psi_d}{d\tau} \quad \to \quad \frac{\Delta\psi_d}{T_p} \triangleq \frac{\psi_d(nT_p) - \psi_d(nT_p - 1)}{T_p}. \tag{47}$$

Since the continuous variable $\varphi(\tau) \in \mathbb{R}$ has not been limited to the range $[-\pi, \pi)$, to avoid discontinuity (due to the definition (19)) during computing the error (25), the continuous method for determining $\varphi_d(\tau) \in \mathbb{R}$ from (19) has been applied. In that way the term $e_{\varphi d} \in \mathbb{R}$ becomes continuous.

Figures 2-5 show obtained simulation results. From Fig. 2 it results, that the VFO controller gives *natural*, non-oscillatory movement of the controlled vehicle during transient stage, which has a practical significance and ensures relative low control cost (compare Fig. 5). It is worth to note, that the orienting process (putting the direction of $\boldsymbol{g}_2^*$ on the direction of $\boldsymbol{h}^*$) is effective and fast (see Fig. 4).

[1] Vehicle trajectory does dot get stuck in such discontinuity points. Namely, the set of discontinuity is non-attractive (chattering phenomenon does not occur).

[2] The *Euler backward* integration algorithm with the fixed time step $T_p = 0.005[s]$ has been used.

This fact can justify the geometrical interpretation and the whole VFO methodology given in the paper. Obtained rate of convergence for tracking errors also seems to be satisfactory fast. Notice, that controller synthesis is very simple and clear concerning the influence of both design coefficients k_p and k_1 on the control performance.

Finally, it is worth to note, that introduction of the auxiliary variables ψ_d and φ_d causes some kind of division of tracking process onto two subtasks (although smoothly and implicitly connected): 1) decreasing the position errors e_x, e_y when the robot is far from the reference trajectory and 2) eliminating the orientation and steering errors e_φ, e_ψ, but just near the reference trajectory. This manner of control is rather intuitive for a common car driver and gives natural vehicle behavior.

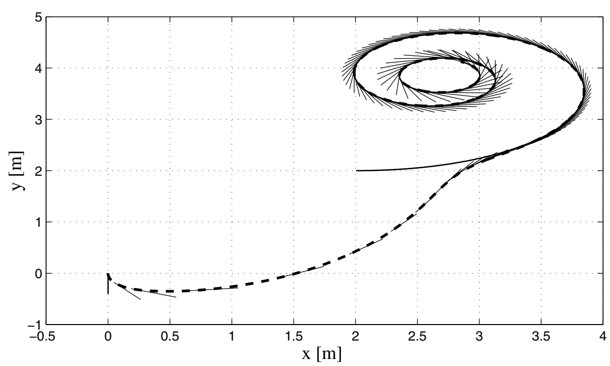

Fig. 2. The reference (–) and the robot's (- -) paths in the task space. Short straight lines denote the instantaneous orientation of the robot.

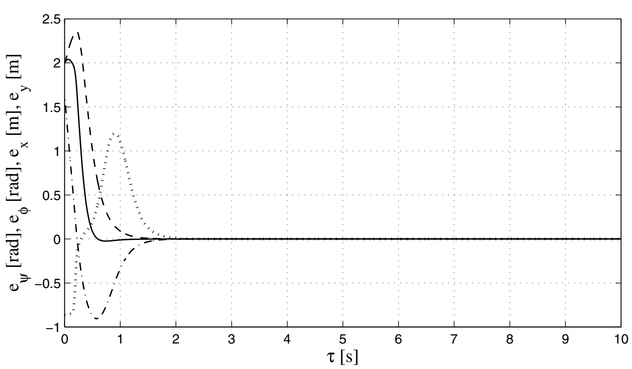

Fig. 3. Time plots of tracking errors: e_ψ (...), e_φ (-·-) – denoted as e_ϕ, e_x (–), e_y(- -).

VI. CONCLUSIONS AND FUTURE WORKS

In the paper the vector field orientation (VFO) approach to solve the tracking task for the threecycle mobile robot has been presented. Derived VFO controller results directly from the simple geometrical interpretation of a possible time evolution of the robot kinematics as a response to specific control signals applied. Presented VFO methodology is a continuation of the VFO approach development introduced in [9], [10] and [11] for 3-D state nonholonomic systems. Obtained results for the VFO controller seem to justify the VFO methodology and show general features of VFO control laws:

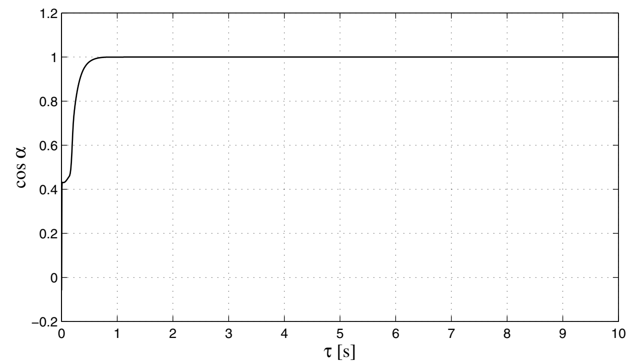

Fig. 4. Time plot of $\cos\alpha$, where $\alpha\angle(\boldsymbol{g}_2^*, \boldsymbol{h}^*)$.

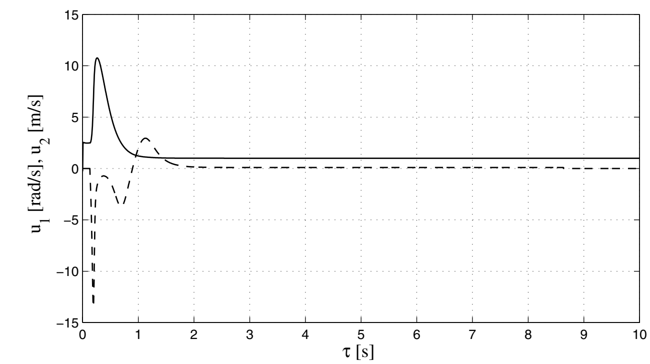

Fig. 5. Time plots of orienting u_1 (- -) and pushing u_2 (–) controls.

asymptotic tracking, *natural* and non-oscillatory movement of the controlled system (hence relative low control cost) and very simple and clear controller synthesis (comparable to those from the linear systems theory).

Future works will be focused on extending the VFO concept to a more difficult point stabilization task for 3-D and 4-D nonholonomic systems. Some preliminary works have been carried out and it seems to be possible to apply the VFO methodology to this control problem [11]. Moreover, an experimental validation of the VFO tracking controller with an universal mobile robot system is planned.

APPENDIX

Derivation of the formula (37).

$$
\begin{aligned}
\boldsymbol{r} &\stackrel{\Delta}{=} \boldsymbol{h}^* - \dot{\boldsymbol{q}}^* = \begin{bmatrix} h_2 \\ h_3 \\ h_4 \end{bmatrix} - \begin{bmatrix} \tan\psi\, u_2 \\ \cos\varphi\, u_2 \\ \sin\varphi\, u_2 \end{bmatrix} = \\
&= \|\boldsymbol{h}^*\| \begin{bmatrix} \frac{h_2}{\|\boldsymbol{h}^*\|} - \tan\psi\, k_2\cos\alpha \\ \frac{h_3}{\|\boldsymbol{h}^*\|} - \cos\varphi\, k_2\cos\alpha \\ \frac{h_4}{\|\boldsymbol{h}^*\|} - \sin\varphi\, k_2\cos\alpha \end{bmatrix} = \|\boldsymbol{h}^*\| \begin{bmatrix} r_{h1} \\ r_{h2} \\ r_{h3} \end{bmatrix}
\end{aligned}
$$

Now one can compute:

$$
\begin{aligned}
\|\boldsymbol{r}\|^2 &= \|\boldsymbol{h}^*\|^2 \left(r_{h1}^2 + r_{h2}^2 + r_{h3}^2\right) = \\
&= \|\boldsymbol{h}^*\|^2 \left[1 + (1+\tan^2\psi)k_2^2\cos^2\alpha\, - \right. \\
&- \left. 2k_2\cos\alpha \frac{h_2\tan\psi + h_3\cos\varphi + h_4\sin\varphi}{\|\boldsymbol{h}^*\|}\right] \stackrel{(26),(27)}{=} \\
&= \|\boldsymbol{h}^*\|^2 \left[1 + \cos^2\alpha\, - 2k_2\cos^2\alpha\,\|\boldsymbol{g}_2^*\|\right]
\end{aligned}
$$

Hence from $k_2 \triangleq \frac{1}{\| \boldsymbol{g}_2^* \|}$ one gets:

$$\begin{aligned} \| \boldsymbol{r} \|^2 &= \| \boldsymbol{h}^* \|^2 \left(1 + \cos^2 \alpha - 2\cos^2 \alpha\right) = \\ &= \| \boldsymbol{h}^* \|^2 \left(1 - \cos^2 \alpha\right). \end{aligned}$$

Below the exact formulas for velocity terms in definitions (21) and (24) are presented. Using assumption **A3** one gets:

$$\dot{\varphi}_d \overset{(19)}{=} \frac{\dot{h}_4 h_3 - h_4 \dot{h}_3}{h_3^2 + h_4^2}, \tag{48}$$

where

$$\dot{h}_3 \overset{(23)}{=} k_p(\dot{x}_t - \dot{x}) + \ddot{x}_t, \qquad \dot{h}_4 \overset{(22)}{=} k_p(\dot{y}_t - \dot{y}) + \ddot{y}_t$$

and

$$\dot{\psi}_d \overset{(18)}{=} \frac{\dot{h}_2 \, |u_2| \, sgn(u_{2t}) - h_2 \dot{u}_2 sgn(u_{2t}) sgn(u_2)}{h_2^2 + u_2^2}, \tag{49}$$

where

$$\dot{h}_2 \overset{(24)}{=} k_p(\dot{\varphi}_d - \dot{\varphi}) + \ddot{\varphi}_d$$

$$\begin{aligned} \ddot{\varphi}_d \overset{(19)}{=}& \frac{1}{(h_3^2 + h_4^2)^2} \Big[(\ddot{h}_4 h_3 - h_4 \ddot{h}_3)(h_3^2 + h_4^2) + \\ -& 2\dot{h}_3 \dot{h}_4 (h_3^2 - h_4^2) + 2 h_3 h_4 (\dot{h}_3^2 - \dot{h}_4^2) \Big], \end{aligned}$$

$$\ddot{h}_3 \overset{(23)}{=} k_p(\ddot{x}_t - \ddot{x}) + \dddot{x}_t, \qquad \ddot{h}_4 \overset{(22)}{=} k_p(\ddot{y}_t - \ddot{y}) + \dddot{y}_t$$

and $\dot{u}_2$ can be directly computed using (28).

Let us show the relation (31). Directly from (14) and (15) we can write:

$$\lim_{\psi \to \psi_d, \; \varphi \to \varphi_d, h_1 \to 0} \{ k(.) \boldsymbol{g}_2^* = \boldsymbol{h}^* \}. \tag{50}$$

Now using (29) and (32) we can write:

$$\lim_{\tau \to \infty} \{ k(.) \boldsymbol{g}_2^* = \boldsymbol{h}^* \} \;\Rightarrow\; \lim_{\tau \to \infty} \left\{ \begin{array}{l} k(.) \tan \psi = h_2, \\ k(.) \cos \varphi = h_3, \\ k(.) \sin \varphi = h_4. \end{array} \right. \tag{51}$$

Substituting above relations into the definition of $\cos \alpha$ (see (27)) we get

$$\begin{aligned} \lim_{\tau \to \infty} \cos \alpha &= \frac{1}{k(.)} \frac{(h_2^2 + h_3^2 + h_4^2)}{\| \boldsymbol{g}_2^* \| \, \| \boldsymbol{h}^* \|} = \frac{1}{k(.)} \frac{\| \boldsymbol{h}^* \|}{\| \boldsymbol{g}_2^* \|} \overset{(51)}{=} \\ &\overset{(51)}{=} \frac{|k(.)| \, \| \boldsymbol{g}_2^* \|}{k(.) \, \| \boldsymbol{g}_2^* \|} = sgn(k(.)) \overset{(17)}{=} sgn(u_{2t}). \end{aligned}$$

By continuity of α one obtains:

$$\begin{array}{lll} \exists_{\hat{\tau} < \infty} : \; \forall_{\tau > \hat{\tau}} & sgn(\cos \alpha) = sgn(u_{2t}) & \overset{(26)}{\Rightarrow} \\ \exists_{\hat{\tau} < \infty} : \; \forall_{\tau > \hat{\tau}} & sgn(u_2) = sgn(u_{2t}) & . \end{array} \tag{52}$$

The limit in (43) results from the following simple calculations:

$$\lim_{e_x, e_x, e_\varphi \to 0} \boldsymbol{h}^* \overset{(22)-(24),(42)}{\longrightarrow} \dot{\boldsymbol{q}}_t^*, \qquad \dot{\boldsymbol{q}}_t^* \triangleq [\dot{\varphi}_t \; \dot{x}_t \; \dot{y}_t]^T. \tag{53}$$

From (26), (52) and using (42) one obtains:

$$\lim_{e_x, e_x, e_\varphi \to 0} \left\{ \begin{aligned} u_2 &= \frac{\| \dot{\boldsymbol{q}}_t^* \|}{\| \boldsymbol{g}_2^* \|} sgn(u_{2t}) \overset{(6)}{=} \\ &\overset{(6)}{=} |u_{2t}| \frac{\| \boldsymbol{g}_{2t}^* \|}{\| \boldsymbol{g}_2^* \|} sgn(u_{2t}) \overset{(6),(1)}{=} \\ &\overset{(6),(1)}{=} u_{2t} \frac{\sqrt{1 + \tan^2 \psi_t}}{\sqrt{1 + \tan^2 \psi}} \end{aligned} \right\}.$$

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