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Identifiability and estimation of aircraft parameters and delays by using optimal input design

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Abstract— **This paper considers test of identifiability, optimal input design and estimation problem given by aerospace domain describing aircraft nonlinear dynamics with time delays. The original idea is to use an approximation well in line with the given system and an algebraic approach to analyze identifiability. Then the approximate model is considered to obtain the optimal input and to estimate the parameters and delays of the original model. Numerical results are given and compared.**

I. INTRODUCTION

System identification based on physical laws often involves parameter estimation. Before performing estimation problem, it is necessary to investigate its identifiability. Even if parameters are theoretically identifiable they may be poorly estimable for a given experiment. Thus a significant increase in accuracy of parameter estimation may be obtained by a suitable choice of experimental conditions. This paper considers a model given by aerospace domain describing aircraft dynamics. This model is derived from the general equations of motion:

$$\begin{cases} m\dfrac{dV}{dt} = & \sum F_e, \\ \dfrac{d\mathcal{C}}{dt} = & \sum M_e, \end{cases} \tag{1}$$

where $\frac{dV}{dt}$ denotes the acceleration of the gravity center, $\mathcal{C}$ the kinetic moment, $\sum F_e$ the sum of the forces acting on the aircraft and $\sum M_e$ the sum of the moments of these forces. In order to improve the model accuracy, the input which consists here in the turbulence, is supposed to act at three different points of the fuselage. Let σ_i denote the shift operator associated to $\tau_i > 0$ and defined for some function $v(t)$ by:

$$(\sigma_i v)(t) := v(t - \tau_i), \quad i = 1, .., 3, \tag{2}$$

and let also $\tau_3 = \tau_2 - \tau_1$. The projection of equations (1) on the aerodynamic reference frame of the aircraft yields the following nonlinear and retarded equations [5]:

$$\sum\nolimits^p \begin{cases} \dot{V} = & k_0 \sin(\theta - \alpha) + V^2(k_1 + k_2\alpha), \\ V(\dot{\theta} - \dot{\alpha}) = & k_0 \cos(\theta - \alpha) + k_3\, q\, V + V^2\, [k_4 + \\ & p_1\,(\alpha + u) + p_2\,(\alpha + \sigma_1 u) + \\ & p_3(\alpha + \sigma_2 u) + p_3\, p_4(\sigma_3\alpha + \sigma_2 u)], \\ \dot{q} = & k_5\, q\, V + V^2\, [k_6 + p_5\,(\alpha + u) + \\ & p_6(\alpha + \sigma_1 u) + p_7(\alpha + \sigma_2 u) + \\ & p_4\, p_7\,(\sigma_3\alpha + \sigma_2 u)], \\ \dot{\theta} = & q, \end{cases} \tag{3}$$

where V denotes the speed of the aircraft, α the angle of attack, θ the pitch angle, q the pitch rate, and u the input. The set of parameters k_i, $i = 0, ..., 6$ is assumed a priori known (here for instance $k_0 = g$, the gravity coefficient) and doesn't need to be estimated. The state vector is defined by:

$$x := (V, \alpha, q, \theta) \tag{4}$$

and is assumed available from measurements $y = x$. The unknown parameter vector:

$$p = (\underline{p}, \tau_1, \tau_2) \quad \text{with } \underline{\mathrm{p}} = (\mathrm{p}_1, ..., \mathrm{p}_7) \tag{5}$$

is assumed to be in some subset Ω of R^9, and the main problem comes from the introduction of the delays in the set of unknown parameters.

The identifiability of linear delay-differential systems has been analyzed by [2], [15], [16] and [17]. To our knowledge, there is no existing method for solving the identifiability problem for general nonlinear and retarded systems with a priori unknown delays.

Like the free delay case where the properties one can obtain from the linearized system still hold for the original plant in the vicinity of the linearization point, the extension of such result to the retarded case has been done with some specific assumptions [7]. The identifiability of the model $\sum^p$ with full state observation has been proved in [7] as an application of this result.

The aim of this paper is to show how a particular approximation derived from the original nonlinear and retarded system can be used to get an optimal input and to estimate the parameters and delays of the model. The presentation is organized as follows. An algebraic approach to test identifiability of the approximated model is developed in Section 2, where the nonlinearities are maintained and a particular class of inputs (a practicable input) is used with a Padé approximation on the state retarded terms. In Section 3, a classical experiment design approach based on the information matrix is given and the optimal input is computed by using the approximated model. Then, in Section 4, the parameters and delays of the model are estimated using the optimal input obtained in Section 3 and the approximated model. Section 5 presents some numerical results. Most of the computations are implemented in Maple for the aircraft models identifiability, and in Matlab for optimal design and estimation procedures.

II. ALGEBRAIC APPROACH

A. Input-output approach for testing identifiability

The input-output approach can be used for testing the identifiability of some nonlinear systems. This method is based on differential algebra [11] and consists in rewriting, when it is possible, the nonlinear system as a differential polynomial system that will be completed with $\dot{p} = 0$ where p is the parameter vector of the system. The resulting system Γ^p can be described by the following polynomial system:

$$\Gamma^p \begin{cases} R(\dot{x}, x, u, p) = 0, \\ S(x, y, p) = 0, \\ \dot{p} = 0. \end{cases} \tag{6}$$

The notion of identifiability is strongly connected to observability. In the 90s Fliess and Diop have proposed a new approach of nonlinear observability and identifiability based on differential algebra [8]. The initial conditions are ignored as they are in the model Γ^p. Thus, the solution of (6) may not be unique and some solutions may be degenerate. Therefore the set of non-degenerate solutions $\bar{x}(p, u)$, corresponding to every possible initial condition, have to be involved in the identifiability definition. Here we adopt the definition introduced in [12].

Definition 2.1: The model Γ^p is globally identifiable with respect to Ω at p if for any $p^* \in \Omega$, $p^* \neq p$ there exists a control u, such that $\bar{y}(p, u) \neq \emptyset$ and $\bar{y}(p, u) \cap \bar{y}(\tilde{p}, u) = \emptyset$.

where the set $\bar{y}(p, u)$ is the set of the outputs corresponding to the non-degenerate solutions.

The previous definition is well in line with the usual formulation [18] which considers initial conditions.

In most models there exist atypical points in Ω where the model is unidentifiable. Therefore, the previous definition can also be generically extended so that:

Definition 2.2: Γ^p is said to be globally structurally identifiable if it is globally identifiable at all $p \in \Omega$ except at the points of a subset of zero measure in Ω.

In this approach I is the radical of the differential ideal generated by the equations of Γ^p and the ranking:

$$[p] \prec [y, u] \prec [x] \tag{7}$$

is chosen in order to eliminate the state variables. In general, I is rewritten as an intersection of regular differential ideals whose each one admits a characteristic presentation. A characteristic presentation [4] is a set of polynomials which is a canonical representant of the ideal and leads to the exhaustive summary of the system in order to obtain identifiability results.

The computations are achieved with an algorithm implemented in Maple [6]. It is based on the *Rosenfeld-Groebner* algorithm which simplifies differential systems and has been realized and implemented by F. Boulier in the package Diffalg [4]. More recently the exhaustive summary is analyzed with the *Triade* algorithm which simplifies algebraic systems by using a triangular decomposition [14].

But if the delays are unknown, this approach cannot be applied except with some modifications. In the following an approximation of the retarded state variable is done and a specific class of inputs is used in order to obtain a suitable input and to estimate the parameters and delays of the model with a good accuracy (3).

B. Identifiability analysis

The delay expression in frequency domain is given by an exponential function which is approached with Padé approximants at first order [1]. In time domain, it yields the differential equation:

$$z_3(t) + \frac{\tau_3}{2}\dot{z}_3(t) = \alpha(t) - \frac{\tau_3}{2}\dot{\alpha}(t). \tag{8}$$

where $z_3 = \sigma_3\alpha$. The input u is given by:

$$u(t) = ae^{kt}\frac{l_1 - l_0}{(e^{kt} + l_0)(e^{kt} + l_1)}, \tag{9}$$

and a, l_0, l_1, k are known constants. Let us define $z(t) = e^{kt}$ ($\dot{z}(t) = kz(t)$) and $z_1(t)$, $z_2(t)$ such that:

$$\begin{cases} z_1(t) = \cos(\theta(t) - \alpha(t)), \\ z_2(t) = \sin(\theta(t) - \alpha(t)). \end{cases} \tag{10}$$

Therefore, $u(t)$ verifies:

$$\begin{cases} u(t) = & \dfrac{a(l_1 - l_0)z(t)}{(z(t) + l_0)(z(t) + l_1)}, \\ \sigma_1 u(t) = & \dfrac{aa_1(l_1 - l_0)z(t)}{(z(t) + a_1 l_0)(z(t) + a_1 l_1)}, \\ \sigma_2 u(t) = & \dfrac{aa_2(l_1 - l_0)z(t)}{(z(t) + a_2 l_0)(z(t) + a_2 l_1)}, \end{cases} \tag{11}$$

where $a_i = e^{k\tau_i}, i = 1, 2$ are the parameters to be identified (instead of $\tau_i, i = 1, 2$).

Finally the original problem is rewritten in the following rational form:

$$D^p \left\{ \begin{array}{rl} \dot{V} = & k_0 z_2 + V^2(k_1 + k_2\alpha), \\ V(\dot{\theta} - \dot{\alpha}) = & k_0 z_1 + k_3\, q\, V + V^2\, [k_4 + \\ & p_1\,(\alpha + u) + p_2\,(\alpha + \sigma_1 u) + \\ & p_3(\alpha + \sigma_2 u) + p_3\, p_4(z_3 + \sigma_2 u)], \\ \dot{q} = & k_5\, q\, V + V^2\, [k_6 + p_5\,(\alpha + u) + \\ & p_6(\alpha + \sigma_1 u) + p_7(\alpha + \sigma_2 u) + \\ & p_4\, p_7\,(z_3 + \sigma_2 u)], \\ \dot{\theta} = & q \\ z_1^2 + z_2^2 = & 1, \\ \dot{z_1} = & -(\dot{\theta} - \dot{\alpha}) z_2, \\ \dot{z_2} = & (\dot{\theta} - \dot{\alpha}) z_1, \\ \frac{\tau_3}{2}\dot{z_3} = & \alpha - z_3 - \frac{\tau_3}{2}\dot{\alpha}, \\ u = & \frac{a(l_1 - l_0) z}{(z + l_0)(z + l_1)}, \\ \sigma_i u = & \frac{a a_i (l_1 - l_0) z}{(z + a_i l_0)(z + a_i l_1)},\ i = 1, 2, \\ \dot{z} = & kz, \\ y = & (V, \alpha, q, \theta, z, z_1, z_2). \end{array} \right.$$

where the parameter vector is $p = (\underline{p}, a_1, a_2, \tau_3)$. The input-output approach described above gives the general characteristic presentation which contains two input-output polynomials which are too large to be written here. The resulting exhaustive summary is analyzed, which leads to the structural global identifiability of D^p. Obviously the model D^p is structurally globally identifiable with $p = (\underline{p}, \tau_1, \tau_2, \tau_3)$

III. EXPERIMENT DESIGN

A. Introduction

In order to obtain a suitable input and consequently obtain more accurate estimates of model parameters, the information content in the system response during the test must be maximized. The information contained in the response is embodied in the information matrix, whose elements are combinations of partial derivatives of the system response variables with respect to the model parameters. The experimental conditions and admissible experimental conditions are respectively denoted by $\mathcal{E}$ (a set) and Ξ (a vector). Thus the set $\mathcal{E}$ gives an admissible input set. In our case Ξ represents parameters which characterize the input. The dispersion matrix $D(p, \Xi)$ corresponds to the inverse of the well known Fisher information matrix denoted here $F(p, \Xi)$ [9].

B. Numerical results

The following cost function has been chosen:

$$j(\Xi) = \mathrm{tr}(\mathrm{D}(\mathrm{p}, \Xi) = \sum_{\mathrm{j}=1}^{\mathrm{l}} \mathrm{d}_{\mathrm{jj}}, \qquad (12)$$

where $d_{jj},\ j = 1, ..., l$, represent the diagonal terms of $D(p, \Xi)$. The matrix $F(p, \Xi)$ is computed from the measurement noise covariance matrix R given by:

$$R = \begin{bmatrix} 25.10^{-4} & 0 & 0 & 0 \\ 0 & 0.04 & 0 & 0 \\ 0 & 0 & 0.04 & 0 \\ 0 & 0 & 0 & 0.04 \end{bmatrix}.$$

Experimental conditions are given by:

$$\Xi = (a, t_0, t_1, k)\ , \qquad (13)$$

$$\mathcal{E} = \{\Xi \in \mathbb{R}^4 |\ |a| \le 10,\ 0 \le t_0 < t_1 \le 1, 100 \le k \le 1000\}, \qquad (14)$$

with $exp(kt_1) = l_0$ and $exp(kt_2) = l_1$.

Since the cost function is regular, a Quasi-Newton method can be applied for minimizing it with an initial guess: $a = 10,\ t_1 = 0.2,\ t_2 = 0.8,\ k = 1000$ (these values are obtained by using a method based on dynamic programming given in [9]) By this way, an optimal solution u_{opt} ($a = 8.8,\ t_1 = 0.11693,\ t_2 = 0.8374,\ k = 1000$) is computed. The input trajectory corresponding to u_{opt} is given on Fig 1.

The table 1 gives the criterion values computed with u_{opt} and another non optimal input u_{nopt}($a = 10,\ t_1 = 0.3, t_2 = 0.6,\ k = 1000$) the results show that the computation using u_{opt} improves the criterion significantly.

TABLE I

CRITERION VALUES

u_{opt}	non optimal input
0.0632	0.2618

IV. PARAMETERS AND DELAYS ESTIMATION

A. Introduction

The aim of this section is to estimate parameters and delays of the model (3) by using a classical least-squares objective function given by:

$$J(p) = \sum_{i=1}^{N} (y_d(t_i) - y(t_i, p))^\top R^{-1} (y_d(t_i) - y(t_i, p)), \qquad (15)$$

where $p = (\underline{p}, \tau_1, \tau_2)$, $\tau_3 = \tau_2 - \tau_1$, and R is the measurement noise covariance matrix. The vector y_d represents the data. The measurement noise is assumed to be white Gaussian with zero mean. The initial vector $p_0 = (\underline{p_0}, \tau_{1_0}, \tau_{2_0})$ is assumed to be known.

The cost function is minimized with respect to the unknown parameters and delays. This problem is solved by a Quasi-Newton method which is implemented in the toolbox optimization of MATLAB 5.3. It leads to an estimation $\hat{p}_{ls}$.

B. Estimation results

The following study was conducted in simulation. The columns of the Table II and III successively give the true parameter vector $\bar{p}$, the estimated parameter vector $\hat{p}^{ls}_{u_{nopt}}$ and $\hat{p}^{ls}_{u_{opt}}$ and the corresponding relative errors as indicated respectively obtained with the non optimal input u_{nopt} and with u_{opt}.

TABLE II

ESTIMATION RESULTS WITH u_{nopt}

Parameter	$\bar{p}$	$\hat{p}^{ls}_{u_{nopt}}$	$\frac{\lvert\hat{p}^{ls}_{u_{nopt}}-\bar{p}\rvert}{\lvert\bar{p}\rvert}$
p_1	0.2440	0.4009	0.6430
p_2	4.6260	5.0489	0.0914
p_3	0.8450	0.5950	0.2959
p_4	0.2550	0.2324	0.0886
p_5	0.3720	0.4448	0.1957
p_6	0.2440	0.1235	0.4943
p_7	-3.0360	-2.9239	0.0370
τ_1	0.0120	0.0126	0.0500
τ_2	0.0440	0.0451	0.0250

TABLE III

ESTIMATION RESULTS WITH u_{opt}

Parameter	$\bar{p}$	$\hat{p}^{ls}_{u_{opt}}$	$\frac{\lvert\hat{p}^{ls}_{u_{opt}}-\bar{p}\rvert}{\lvert\bar{p}\rvert}$
p_1	0.2440	0.2641	0.0824
p_2	4.6260	4.7818	0.0337
p_3	0.8450	0.7960	0.0580
p_4	0.2550	0.2630	0.0314
p_5	0.3720	0.3394	0.0876
p_6	0.2440	0.2681	0.0988
p_7	-3.0360	-3.1426	0.0351
τ_1	0.0120	0.0121	0.0083
τ_2	0.0440	0.0443	0.0068

The results presented in Table II show a good estimation of the parameters $p_2,\ p_4,\ p_7$ and delays by using a non optimal input given by u_{nopt} [10]. The estimation of the parameters $p_1,\ p_3,\ p_5,\ p_6$ has been improved by using the optimal input u_{opt} (see Table III). Thus a significant increase in accuracy of the parameter estimation has been obtained with u_{opt}.
The trajectories presented in the figures 2, 3, 4, 5 are obtained by the following way: $p = \bar{p}$ (full line) and either $p = p_0$ (dotted line) (figures 2, 4) or $p = \hat{p}_{ls}$ (dotted line) (figures 3, 5). They show a good trajectory reconstruction and point out the efficiency of the method which is based on the approximated model D^p

C. Relation between the approximated system D^p and original system $\sum^p$

The theoretical comparison between the systems $\sum^p$ and D^p has to be done. Nevertheless it is important to compare the numerical solutions of both systems. It is presented in figures 6, 7, 8, 9 which give the trajectories of the four state variables corresponding to u_{opt} and $p = \hat{p}^{ls}_{u_{opt}}$. The obtained trajectories (figures 6, 7, 8, 9) with both systems are very closed: in fact they are indistinguishable relatively to the measure error of all states ($\sum^p$: full line and D^p: dotted line).

V. CONCLUSION

This paper presents an approach for the estimation of parameters and delay identifiability of aircraft dynamics which consist in a nonlinear and retarded system. An approximated model is used with a class of practicable inputs and an approximation of the retarded state based on the Padé approximants. The structural global identifiability is shown. By optimizing Fisher information, an optimal input is computed, then the parameters and delays are estimated. The use of an optimal input improves significantly the accuracy of the estimation. A comparison between numerical solutions of original and approximated systems shows the pertinence of the proposed method which is capable of handling estimation of parameters for augmented aircraft models for example.

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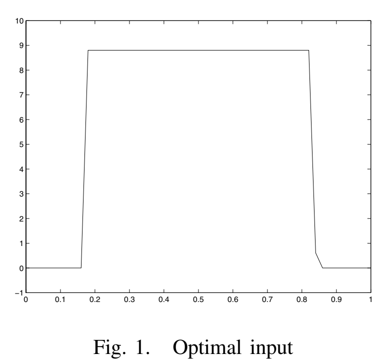

Fig. 1. Optimal input

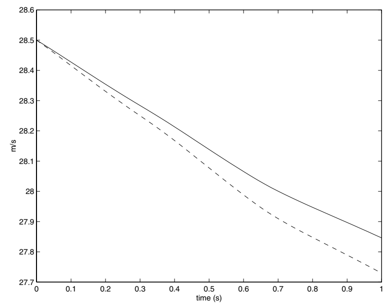

Fig. 2. Speed reconstruction with p_0.

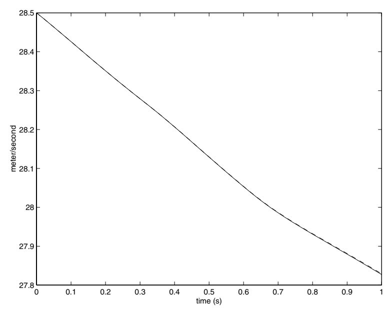

Fig. 3. Speed reconstruction with $\hat{p}_{ls}$.

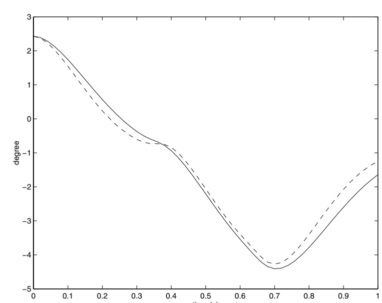

Fig. 4. Pitch angle reconstruction with p_0.

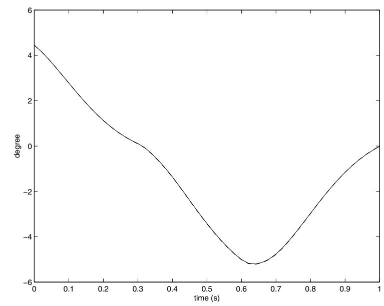

Fig. 5. Pitch angle reconstruction with $\hat{p}_{ls}$.

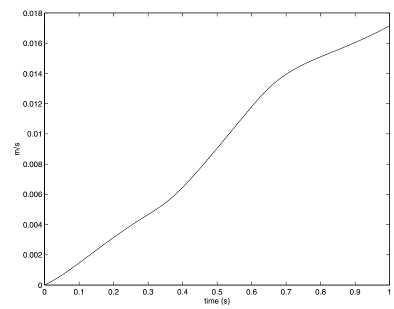

Fig. 6. Speed reconstruction (D^p and $\sum^p$ systems).

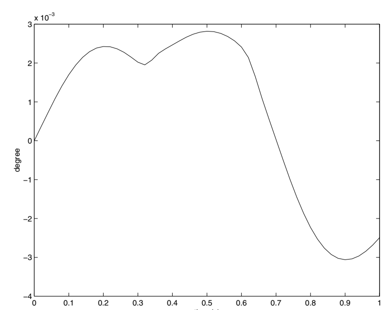

Fig. 7. Angle of attack reconstruction (D^p and $\sum^p$ systems).

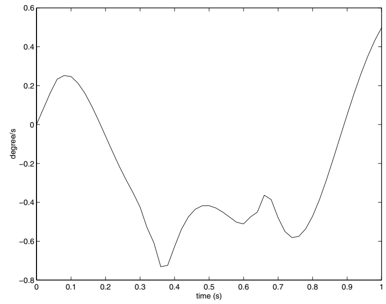

Fig. 8. Pitch rate reconstruction (D^p and $\sum^p$ systems).

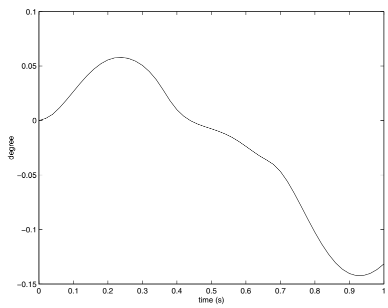

Fig. 9. Pitch angle reconstruction (D^p and $\sum^p$ systems).