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On Revenue Generation When Auctioning Network Resources

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Abstract

While efficiency of mechanisms for control of communication networks has been extensively investigated, little attention has been paid to the critical metric of revenue generation. In this paper, we pursue such an investigation within a class of allocation schemes with the minimal signaling and computation costs necessary in communication network domains. We show that, within this space, linear cost rules for proportional allocation mechanisms are optimal for symmetric agent populations and reserving a portion of the resource can increase revenue even though less of the resource is being sold. While nonlinear cost functions can be better for asymmetric populations, intelligent agents can undermine this via signal splitting. We show how a resource can counter this phenomenon by declaring a linear cost.

Most current approaches to control of communication networks incorporate economic models to deal with the decentralization necessitated by the domain. Ideas such as "smart markets" where packets bid for service [3] and proportionally fair pricing [2] promoted market-based control to induce efficient use of network resources. While the success of various schemes at achieving efficiency has been studied extensively [1,5,7,8], little attention has been paid to revenue generation as a performance metric. Clearly, for many network resource owners, revenue generation is a critical motivation as illustrated by the emergence of bandwidth auctions and exchanges.

In this paper, we investigate various aspects of revenue generation through analysis of a class of mechanisms with minimal signaling and computational costs for the resource, which is vital in communication network settings and obtain several key results. We show that for symmetric populations of buyers (e.g. bandwidth set aside for one type of traffic - voice, video or data), we show that mechanisms with linear costs and proportional allocations are optimal. We see that it does not take many buyers for treating a network resource as a divisible good to yield greater revenue than by treating it as an indivisible good. Under certain conditions, service providers can obtain greater revenue by reserving a portion of the resource and selling less than the entire available capacity. We show that while strictly convex cost functions can outperform linear cost functions in settings with asymmetric buyers, intelligent agents (buyers or their proxies) can undermine such schemes by submitting multiple signals that gain the same allocation at a reduced cost. Finally, we illustrate how a resource can alleviate some of the revenue loss by declaring a linear cost function, which shifts up agents' demand functions.

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1. Models and Equilibrium

The problem set-up described in this section follows that of [5], which we provide here also in detail so that the paper would be self-contained. Transparent mechanisms (or auctions) are characterized by an allocation rule $x(s)$ and a cost rule $c(s)$, where $s = [s_1 \cdots s_N]$ represents the signals from a population of N agents and $x_i(s)$ and $c_i(s)$ are respectively the allocation and cost to the i^{th} agent. We work in the one-dimensional signaling space where $s_i \in I\!R^+$. One subset of this space of auctions is the collection of those that can be characterized by the following allocation rule: $x_i(s) = w_i(s)/\left(\sum_{j=1}^{N} w_j(s) + \epsilon\right)$ where ϵ is a parameter controlled by the resource (the resource's signal). Signals are translated to weights, denoted by the functions $\{w_i(\cdot)\}_{i=1}^{N}$, which determine the proportions of the allocation. Allocation rules of this form fit nicely with Generalized Processor Sharing models for flow control in networks [6]. We restrict our analysis to the case for which $w_i(s) = w(s_i)$ and $c_i(s) = c(s_i)$ as it incorporates the notion of fairness where each agent is given the same weight and pays the same cost as any other agent who makes the same signal. We assume that the weights and costs are strictly increasing functions of their arguments and a signal of zero will yield a weight and cost of zero as well. We consider this class of rules to be a reasonable and tractable initial expansion around the prevalent proportionally fair auction [2], which is the "point" in mechanism space characterized by $w(s_i) = s_i$ and $c(s_i) = s_i$. It can be shown that we do not need to express both $w(s_i)$ and $c(s_i)$. By making the substitutions $t_i = c(s_i)$ and $\tilde{w}(t_i) := w(c^{-1}(t_i))$ (c is invertible if it is monotonically increasing), we can express this class of mechanisms with the rules $x_i(t) = \tilde{w}(t_i) / \left(\sum_{j=1}^{N} \tilde{w}(t_j) + \epsilon \right)$ and $c_i(t) = t_i$. With similar substitutions, we can equivalently express this class with the rules

$$x_i(s) = \frac{s_i}{\sum_{j=1}^{N} s_j + \epsilon} \qquad c_i(s) = c(s_i). \tag{1}$$

We choose to work with the characterization described in (1) where $c(s_i) \in C^2$ is a twice differentiable increasing function of s_i. These mechanisms have the minimal signaling and computational costs for allocation determination that we desire in many communication network contexts.

Agents are typically modeled with quasilinear utility functions: $U_i(s) = v_i(x_i(s)) - c_i(s)$ where v_i is a twice differentiable concave increasing function ($v_i'(\cdot) > 0,\ v_i''(\cdot) \leq 0$). Let $s_{-i} = \sum_{j \neq i} s_j$ be the bids of all except the i^{th} agent. We have the derivatives $U_i'(s) = v_i'(x_i(s))x_i'(s) - c'(s_i)$ and $U_i''(s) = v_i''(x_i(s))[x_i'(s)]^2 + v_i'(x_i(s))x_i''(s) - c''(s_i)$ where $x_i'(s) = (s_{-i} + \epsilon) / (s_i + s_{-i} + \epsilon)^2 > 0$, and $x_i''(s) = -2(s_{-i} + \epsilon)/(s_i + s_{-i} + \epsilon)^3 < 0$. If $c_i''(s_i) \geq 0$, then we have $U_i''(s) < 0$. The strict concavity of the i^{th} agent's utility implies that it will have a unique optimal response to each opponent state $s_{-i} + \epsilon$. For the optimal response to be nonzero, we need the marginal utility when bidding zero to be positive. This occurs when $v_i'(0)/(s_{-i} + \epsilon) - c_i'(0) > 0 \ \Rightarrow\ v_i'(0)/c_i'(0) > s_{-i} + \epsilon$. The i^{th} agent's response can then be determined from

$$v_i'\left(\frac{s_i}{s_i + s_{-i} + \epsilon}\right) \frac{s_{-i} + \epsilon}{(s_i + s_{-i} + \epsilon)^2} - c'(s_i) = 0 \tag{2}$$

which yields the unique optimal s_i when facing $s_{-i} + \epsilon$.

Let us define $p := \sum_j s_j + \epsilon$. Then, p serves as a measure of demand for the resource and allows us to characterize agents' optimal responses with respect to a parameter which is identical for all agents at equilibrium. Let us call this characterization a demand function, $d(p)$, which captures an agent's allocation as a function of p when it uses the strategy obtained from (2). Thus, the demand function captures that $s_i = d(p)p$ is the optimal response to $s_{-i} + \epsilon = d(p)(1 - p)$.

Proposition 1 *If the cost function is concave, then there exists a valuation function for which the optimal response is not unique.*

Proof. By making the substitutions $p = \sum_j s_j + \epsilon$ and $x_i = s_i / (\sum_j s_j + \epsilon)$ into (2), we can express the first-order necessary condition for the optimal response as $v'(x) = pc'(px)/(1 - x) =: f(x|p)$. The derivative of the RHS of the previous equation, $f(x|p)$, as a function of allocation x is $\left[(1-x)p^2c''(px) + pc'(px)\right] / (1-x)^2$. The sign of $f'(x|p)$ is determined by the quantity $c''(px) + c'(px)/(p(1-x))$. If $c''(s) < 0$, then $c''(s) < -\delta,\ \ \forall s \in [s_1, s_2]$, for some $\delta > 0$ sufficiently small and some $s_1, s_2 > 0$. Then, $c''(px) + c'(px)/(p(1-x)) < -\delta + c'(\hat{s})/(p - s_2) < 0$ for p sufficiently large, where $\hat{s} = \arg\max_{s \in [s_1, s_2]} c'(s)$. Specifically, if $p > c'(\hat{s})/\delta + s_2$, we know that $f(x|p)$ is decreasing in x on $[s_1/p, s_2/p]$. Clearly, there are many functions v with $v''(\cdot) < 0$, where $v'(x)$ will intersect $f(x|p)$ more than once, which implies that there is more than one extremal point. By choosing a pair of points on $f(x|p)$ where $x_1 < x_2$ but $f(x_1|p) > f(x_2|p)$ and letting $v'(x)$ be the line that intersects these two points, yields at least two solutions. ■

Thus, we restrict our analysis to allocation mechanisms described by (1) where the cost function $c(s_i)$ is convex. We denote this class of mechanisms by $\mathcal{C}$. The intuition behind convex cost functions is that agents who receive larger allocations (due to greater signals) pay a higher cost per unit resource obtained. Such mechanisms are classified as *discriminatory price* auctions. Mechanisms in $\mathcal{C}$ with linear cost functions such as the proportionally fair auction are *uniform price* auctions. For games played by agents attempting to gain access to a resource allocated through a mechanism from $\mathcal{C}$, it is important to know whether we can obtain a unique operating point, i.e., a unique Nash equilibrium.

Proposition 2 *For every mechanism in $\mathcal{C}$, there is a unique Nash equilibrium.*

Proof. Making the substitutions $p = \sum_j s_j + \epsilon$ and $x_i = s_i / (\sum_j s_j + \epsilon)$ into (2), we can express the first order necessary condition for the optimal response as

$$v_i'(x_i)(1 - x_i) = pc_i'(px_i)\ . \tag{3}$$

Every pair (p, x_i) that satisfies the previous equation represents an optimal state for the i^{th} agent. We can interpret these states as demand functions (where x_i is a function of p). By treating the previous equation as an identity, we obtain $[v_i''(x_i)(1 - x_i) - v_i'(x_i)]\ \frac{\partial x_i}{\partial p} =$

$c_i'(px_i) + pc_i''(px_i)\left[x_i + p\frac{\partial x_i}{\partial p}\right]$ which implies $\frac{\partial x_i}{\partial p} = \frac{c_i'(px_i)+px_ic_i''(px_i)}{v_i''(x_i)(1-x_i)-v_i'(x_i)-p^2c_i''(px_i)}$. Because $c''(s_i) \geq 0$ for all mechanisms in $\mathcal{C}$, and the valuations are increasing concave functions, we have that $\partial x_i/\partial p < 0$. This implies that the demand functions $\{d_i(p)\}_{i=1}^N$ which characterize the optimal responses of agents are decreasing, where $d_i(p) := x_i(p)$ is obtained from the unique value of x_i which solves (3) for a particular p. We note that $d_i(0) = 1\ \forall i$. Following the reasoning in [4], since all agents are characterized by decreasing demand functions, the total demand will be a decreasing function. The Nash equilibrium point is defined by total demand being one, which occurs at only one value of p^*. Thus, there is a unique Nash equilibrium with signals $s_i^* = d_i(p^*)p^*$.■

Given that we have a class of auctions that yield the desirable property of a unique Nash equilibrium, a natural question is how we go about choosing a mechanism within $\mathcal{C}$. We had considered this question in [5] with social welfare maximization taken as a metric. In this paper, we consider the same, in the next two sections, but now with revenue generation as a metric.

2. Revenue for Symmetric Populations

Here, we investigate revenue generation for mechanisms from the class $\mathcal{C}$ for a population of symmetric agents, such as in situations where bandwidth is dedicated for one type of traffic. We first consider mechanisms where, in the allocation rule $x(s)$, we have $\epsilon = 0$.

Proposition 3 *For a resource allocated with a mechanism from the class $\mathcal{C}$ having an allocation rule with $\epsilon = 0$, the optimal revenue generated from a population of N symmetric agents is $v'(1/N)(N-1)/N$. Furthermore, this is achieved by using any linear cost function, $c(s_i) = \beta s_i$, $\beta \in I\!R^+$.*

Proof. Since we have a symmetric agent population with the entire resource allocated to all the agents, we know that at equilibrium, each agent will be allocated $x^* = 1/N$. Incorporating this into (3), which characterizes the optimal response, we have $v'\left(\frac{1}{N}\right)\left(\frac{N-1}{N}\right) = pc'\left(\frac{p}{N}\right)$ which can be rewritten as $c'\left(\frac{p}{N}\right) = v'\left(\frac{1}{N}\right)\left(\frac{N-1}{N^2}\right)\left(\frac{N}{p}\right)$. At equilibrium, we have $p/N = s^*$, which yields $c'(s^*) = \frac{k}{s^*}$ where $k = v'\left(\frac{1}{N}\right)\left(\frac{N-1}{N^2}\right)$. The equilibrium signal s^* is the value at which the functions $c'(s)$ and k/s intersect and consequently depends on the choice of the cost function. The cost paid by each agent is $c(s^*) = \int_0^{s^*} c'(s)\, ds \leq s^*c'(s^*)$ for all cost functions in $\mathcal{C}$, as $c'(s)$ must be an increasing function of s. The amount paid by the agent is maximized by having $c'(s) = c'(s^*)$, which implies a linear cost function. Thus, any cost function that has constant derivative will yield maximum revenue generation. We note that each agent pays $s^*c'(s^*) = k = v'\left(\frac{1}{N}\right)\left(\frac{N-1}{N^2}\right)$ which is independent of s^*. For a population of N agents, the total revenue generated will be $Ns^*c'(s^*) = v'\left(\frac{1}{N}\right)\left(\frac{N-1}{N}\right)$. ■

As the number of agents becomes large, the revenue will approach $v'(0) \geq v(1)$, where $v(1)$ is the revenue generated by a second price auction for the good as a single unit. Thus, for every type of agent (characterized by its valuation function) there is a threshold N^* above which the resource owner has an incentive to sell his good as a divisible commodity. The threshold value will be $N^* = \min\{N \in \mathbb{Z}^+ : v'\left(\frac{1}{N}\right)\left(\frac{N-1}{N}\right) \geq v(1)\}$ for a symmetric agent population with valuation $v(x)$.

We now consider the more general case where $\epsilon \geq 0$ in the allocation rule $x(s)$, where ϵ represents a constant bid made by the resource owner. One might think that by choosing $\epsilon > 0$, one would be diminishing one's ability to generate revenue as the quantity of resource being allocated to agents is reduced. However, we see that this is not the case as the optimal allocations for revenue generation may involve allocating less than the entire resource and may be independent of the number of agents participating.

Proposition 4 *For a symmetric agent population of size N with valuations $v(x)$, the optimal revenue is $\max_{x\in[0,1]} Nxv'(x)(1-x)$. Let $x^* = \arg\max_{x\in[0,1]} Nxv'(x)(1-x)$. If $x^* = 1/N$, we obtain maximum revenue with any linear cost function. Otherwise, the optimal cost function is $c(s) = v'(x^*)(1-x^*)^2 s/\epsilon$, where $\epsilon > 0$ is arbitrary but must be identical in the allocation rule $x(s)$.*

Proof. If we choose any $\epsilon > 0$, equilibrium allocation for a set of symmetric agents will be $x^* = \frac{1}{N}\left(1-\frac{\epsilon}{p}\right)$. Incorporating this into (3), we have $v'\left(\frac{1}{N}\left(1-\frac{\epsilon}{p}\right)\right)\left(1-\frac{1}{N}\left(1-\frac{\epsilon}{p}\right)\right) = pc'\left(\frac{p-\epsilon}{N}\right)$ which characterizes the relation between the cost function and valuation at equilibrium. Letting $p = Ns+\epsilon$, we have $c'(s) = v'\left(\frac{1}{N}\left(1-\frac{\epsilon}{Ns+\epsilon}\right)\right)\left(1-\frac{1}{N}\left(1-\frac{\epsilon}{Ns+\epsilon}\right)\right)\left(\frac{1}{Ns+\epsilon}\right)$. The equilibrium bid s^* will be the unique solution to the previous equation. As in Proposition 3, the cost paid by each agent is maximized over the mechanisms in $\mathcal{C}$ by having $c'(s)$ be constant, i.e., $c'(s) = c'(s^*)$ where s^* is the equilibrium bid. However, in this case, the value of the equilibrium bid is not irrelevant to profit. We now maximize revenue over s^* to find the slope of the optimal cost function. With a linear cost $c(s) = c'(s^*)s$

and N agents, the revenue generated at equilibrium will be $Nc'(s^*)s^*$
$= \frac{Ns^*}{Ns^*+\epsilon}\left[1-\frac{1}{N}\left(1-\frac{\epsilon}{Ns^*+\epsilon}\right)\right]\gamma$
$= \left[1-\frac{\epsilon}{Ns^*+\epsilon}\right]\left[1-\frac{1}{N}\left(1-\frac{\epsilon}{Ns^*+\epsilon}\right)\right]\gamma$
$= N\frac{r(s^*)}{N}\left(1-\frac{r(s^*)}{N}\right)v'\left(\frac{r(s^*)}{N}\right) = Nx^*\left(1-x^*\right)v'\left(x^*\right)$
$=: \quad R(x^*) \quad$ where $\gamma \quad = \quad v'\left(\frac{1}{N}\left(1-\frac{\epsilon}{Ns^*+\epsilon}\right)\right)$, $r(s^*) = 1-\epsilon/(Ns^*+\epsilon)$ is the fraction of the resource allocated to the agents and $x^* = \frac{r(s^*)}{N} = \frac{s^*}{Ns^*+\epsilon}$ is the equilibrium allocation. We see that the optimal revenue is a function of equilibrium allocation. Candidate solutions are obtained by taking the partial derivative of the revenue with respect to allocation and solving $f(x^*) := (1-2x^*)v'(x^*) + x^*(1-x^*)v''(x^*) = 0$. We know $f(0) = v'(0) > 0$ and $f(x^*) < 0, \forall x^* > 1/2$. Thus, the revenue has a maximum at $x^* \in (0, 1/2)$. We note that the candidate solutions for optimal allocations are independent of the number of agents. However, the number of agents has implications on feasibility as only allocations $x^* \in [0, 1/N]$ are achievable. Thus, we maximize $R(x^*)$ over $[0, 1/N]$. If the maximum is achieved at $x^* = 1/N$, we must set $\epsilon = 0$ and from Proposition 3, we know that any linear cost function may be applied. However, if $x^* \in (0, 1/N)$, we need the equilibrium bid to be $s^* = \epsilon x^*/(1-Nx^*)$. Substituting this into the expression for $c'(s)$, we get $c'(s^*) = v'(x^*)(1-x^*)\left(\frac{1}{Ns^*+\epsilon}\right) = \frac{v'(x^*)(1-x^*)^2}{\epsilon}$. If the revenue is maximized with $x^* < 1/N$, we obtain it by choosing an arbitrary $\epsilon > 0$ with a linear cost function $c(s) = \left(\frac{v'(x^*)(1-x^*)^2}{\epsilon}\right)s$. ■

What does the previous result mean intuitively? It states that to maximize revenue, the resource attempts to choose ϵ and the cost function to assign every agent a particular equilibrium allocation x^*, which is independent of the number of agents requesting service from the resource. Since all the agents are symmetric, a resource can choose its allocation mechanism based on how to extract the maximum payment out of a single agent. The cost paid by a single agent is $c(s^*) = c(x^*p)$, where s^* is the equilibrium bid and p is the sum of all equilibrium bids (including that of the resource). Proposition 4 as well as Proposition 3 state that the revenue is maximized when the cost function is linear. We can express the payment of a single agent as $c(s^*) = s^*c'(s^*) = x^*pc'(px^*)$. One might suppose that since p is affected by all agents, the number of agents would have an effect on the revenue generated from a single agent. However, incorporating the optimality condition in (3), we can express the cost as $c(s^*) = x^*(1-x^*)v'(x^*)$, which is independent of p. Thus, optimal revenue generation seems independent of the size of the agent population. The number of agents does affect the feasibility of this optimal allocation, as only $x^* \in [0, 1/N]$ can be enforced by any mechanism. Thus, a revenue maximizing resource owner would attempt to allocate x^* to all agents except in the cases where the number of agents makes it infeasible, in which case the owner must apply some constraints to the optimization. For valuation functions that yield a single extremal point in maximizing single-agent cost, the infeasibility of x^* would lead to an allocation of $1/N$.

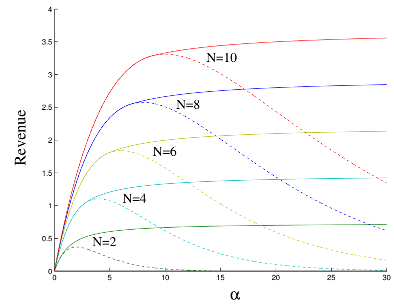

Figure 1. Revenue for Agent Population Sizes $N \in \{2,4,6,8,10\}$ Where $v(x) = 1-e^{-\alpha x}$ when $\epsilon = 0, x = 1/N$ (Dotted) and $\epsilon \geq 0, x = \min\{x^*(\alpha), 1/N\}$ (Solid)

Example 1 *We obtain the optimal allocations and revenue for a symmetric population of N agents with $v(x) = 1-e^{-\alpha x}$.*

The optimal allocation is obtained by solving $\frac{\partial}{\partial x}x(1-x)v'(x) = (1-2x)v'(x) + x(1-x)v''(x) = 0 \Rightarrow (1-2x)\alpha e^{-\alpha x} - x(1-x)\alpha^2 e^{-\alpha x} = 0 \Rightarrow (1-2x) - \alpha x(1-x) = 0 \Rightarrow 1-2x-\alpha x+\alpha x^2 = 0 \Rightarrow \alpha x^2-(2+\alpha)x+1 = 0 \Rightarrow x = \frac{2+\alpha\pm\sqrt{(2+\alpha)^2-4\alpha}}{2\alpha}$. Simplifying the determinant, $\sqrt{(2+\alpha)^2-4\alpha} = \sqrt{4+4\alpha+\alpha^2-4\alpha} = \sqrt{4+\alpha^2}$ and substituting into x, we have $\frac{1}{\alpha}+\frac{1}{2}+\sqrt{\frac{1}{\alpha^2}+\frac{1}{4}} > 1 \Rightarrow x^* = \frac{2+\alpha-\sqrt{4+\alpha^2}}{2\alpha}$. The revenue gain of enforcing $x = \min\{x^*, 1/N\}$ versus $x = 1/N$ is shown in Figure 1. □

3. Revenue for Asymmetric Populations

From our analysis so far, we can state that linear cost functions seem to yield the highest revenues. However, we were only considering symmetric agent populations. Does this hold if we extend to asymmetric agent populations? The following example with two asymmetric agents shows that it does not.

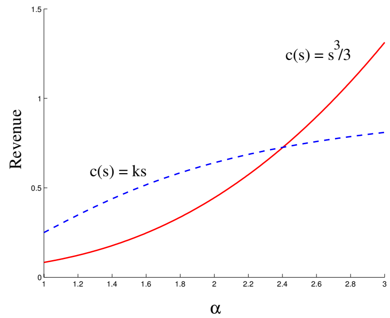

Figure 2. Revenue Generated in Two Agent Game of Example 2 for $c(s) = ks$ (Dotted) and $c(s) = s^3/3$ (Solid)

Example 2 *Comparison of revenue when using mechanisms with the cost functions $c(s) = ks$ vs. $c(s) = s^3/3$ for a two-agent game where $v_1'(x) = \alpha^4(1-x)$, $v_2'(x) = 1-x$, and $\epsilon = 0$ in $x(s)$.*

Because $\epsilon = 0$, we have $x_1 + x_2 = 1$. We first investigate the case where $c(s) = ks$. From (3), we know that at equilibrium we have $v_1'(x_1)(1-x_1) = pc'(px_1) \Rightarrow \alpha^4(1-x_1)^2 = pk, v_1'(x_2)(1-x_2) = pc'(px_2) \Rightarrow (1-x_2)^2 = pk \Rightarrow x_1^2 = pk$. Equating pk from above, we have $pk = \alpha^4(1-x_1)^2 = x_1^2 \Rightarrow \quad x_1 = \frac{\alpha^2}{1+\alpha^2} \quad \Rightarrow pk = \left(\frac{\alpha^2}{1+\alpha^2}\right)^2$. The revenue generated for any linear cost function $c(s) = ks$ for the two agents in this example is $k(s_1+s_2) = k(px_1+px_2) = pk = \left(\frac{\alpha^2}{1+\alpha^2}\right)^2$. We now consider the case where $c(s) = s^3/3$, which is also a mechanism in $\mathcal{C}$. From (3), we know at equilibrium, we have $v_1'(x_1)(1-x_1) = pc'(px_1) \quad \Rightarrow \quad \alpha^4(1-x_1)^2 = p^3x_1^2, v_1'(x_2)(1-x_2) = pc'(px_2) \quad \Rightarrow \quad (1-x_2)^2 = p^3x_2^2 \Rightarrow x_1^2 = p^3(1-x_1)^2$. Equating p^3 from above, we have $p^3 = \frac{\alpha^4(1-x_1)^2}{x_1^2} = \frac{x_1^2}{(1-x_1)^2} \Rightarrow \quad x_1 = \frac{\alpha}{1+\alpha} \quad \Rightarrow \quad p^3 = \alpha^2$. The revenue generated when using $c(s) = s^3/3$ for the two agents in this example is $\frac{s_1^3}{3} + \frac{s_2^3}{3} = \frac{(px_1)^3}{3} + \frac{(px_2)^3}{3} = \frac{p^3}{3}\left(x_1^3 + x_2^3\right) = \frac{\alpha^2}{3}\left[\left(\frac{\alpha}{1+\alpha}\right)^3 + \left(\frac{1}{1+\alpha}\right)^3\right]$. A comparison of revenue generated by each mechanism as a function of α is plotted in Figure 2. We see that for $\alpha > 2.5$, the cost function $c(s) = s^3/3$ outperforms linear cost functions. □

The previous example illustrates that there exist scenarios where strictly convex cost functions can generate more revenue than linear cost functions. However, we can show that strictly convex cost functions can be undermined by intelligent agents who split their signals. Assume that an agent (denoted by the index i) participates in auction for a resource allocated by a mechanism in $\mathcal{C}$ with a cost function $c(s)$ where $c''(\cdot) > 0$. At equilibrium, the agent obtains an allocation $x_i^* = s_i^*/(s_i^* + s_{-i}^* + \epsilon)$ for a cost $c(s_i^*)$. Now suppose that the i^{th} agent sends two signals of magnitude $s_i^*/2$. This will be transparent to the other agents and to the allocation rule, leaving the agent with the same equilibrium allocation x_i^*. The i^{th} agent's cost is now $2c(s_i^*/2)$. Because of the strict convexity of $c(s)$, we know $c'(s_1) < c'(s_2)$, $\forall s_1 < s_2$, which implies $2c(s_i^*/2) = \int_0^{s_i^*/2} c'(s)\ ds + \int_0^{s_i^*/2} c'(s)\ ds < \int_0^{s_i^*/2} c'(s)\ ds + \int_{s_i^*/2}^{s_i^*} c'(s)\ ds = c(s_i^*)$. By sending two signals at half the equilibrium value, the agent maintains the same allocation but reduces its cost which leads to a greater utility. By similar reasoning, if the i^{th} agent submits N bids of s_i^*/N, its utility will increase as N increases. Taking the limit as N gets arbitrarily large, the i^{th} agent's cost becomes $\lim_{N\to\infty} Nc(s_i^*/N) \approx N\left(c'(0)(s_i^*/N)\right) = c'(0)s_i^*$ which is linear with respect to signal. Intelligent agents using signal splitting will pay linear costs even though the declared cost used to compute and arrive at the original equilibrium signal was strictly convex. If $c'(0) = 0$, agents can essentially obtain service without paying anything. To address this phenomenon, an alternative for the resource to obtain more revenue is to alter the equilibrium bid value through mechanism design. We show that this can be achieved by declaring the linear cost function that the agents are ultimately paying.

Proposition 5 *Given an agent with valuation $v(x)$, let $d_c(p)$ be the demand function elicited by a mechanism in $\mathcal{C}$ with a strictly convex cost function $c_c(s)$ where $c_c'(0) > 0$. Let $d_l(p)$ be the demand function elicited by the linear cost function $c_l(s) = c_c'(0)s$. For all $p > 0$, $d_l(p) > d_c(p)$.*

Proof. We obtain $d_c(p)$ by solving (3) for x given each $p \in [0, v'(0)]$. Let us say that for some arbitrary p, we have $v'(x_c)(1-x_c) = pc_c'(px_c)$. Then, $d_c(p) = x_c$. To find the demand at the same price when applying $c_l(s)$, we need to solve $v'(x)(1-x) = pc_l'(px)$. Let us denote the solution as x_l. We have $c_l'(px_l) = c_c'(0) < c_c'(px_c)$ if $x_c > 0$. Since $v'(x)(1-x)$ is a decreasing function of x, we have $x_l > x_c$, which implies that $d_l(p) = x_l > x_c = d_c(p)$. ■

What does this imply in terms of revenue generation? If a resource owner declares a linear cost function $c_l(s) = c_c'(0)s$ instead of a strictly convex cost function $c_c(s)$, the equilibrium value of p will be higher as the demands under the linear costs are higher. If the same allocations are maintained, each agent will have a larger signal at equilibrium which leads to a larger payment to the resource. Simply put, if intelligent agents are going to split their signals to pay a linear cost regard-

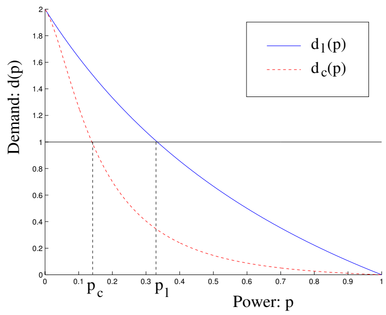

Figure 3. Demand Functions for Convex Cost Function $c(s) = 10s^2 + s$ and Linear Cost Function $c(s) = s$.

less of the declared cost function, the resource might as well declare the linear cost function to move the equilibrium to a point where more revenue can be generated. This is illustrated in the following example.

Example 3 *Comparison of revenue generated between a mechanism with strictly convex cost $c(s) = 10s^2 + s$ which leads to agents splitting signals and a mechanism with linear cost $c(s) = s$, for two agents with $v(x) = \log(1 + x)$.*

We first obtain the demand function for the convex cost function $c(s) = 10s^2 + s$. From (3), we have $\frac{1-x}{1+x} = p(20px + 1) \Rightarrow 1 - x = 20p^2x + 20p^2x^2 + p + px \Rightarrow [20p^2]x^2 + [20p^2 + p + 1]x + [p - 1] = 0 \Rightarrow x = \frac{-(20p^2+p+1)+\sqrt{(20p^2+p+1)^2-80p^2(p-1)}}{40p^2} =: d_c(p)$. Again applying (3) with $c(s) = s$, we have $\frac{1-x}{1+x} = p \Rightarrow 1-x = p + px \Rightarrow x = \frac{1-p}{1+p} =: d_l(p)$. The demand functions and resulting equilibrium values of p are shown in Figure 3. We see that the demand function elicited by the linear cost dominates the one elicited from the convex cost and $p_l \approx 0.333 > 0.139 \approx p_c$. If the two agents did not split bids, the resource would obtain $2[10(p_c/2)^2 + (p_c/2)] \approx 0.2346$ in revenue. However, if the agents split their bids in an arbitrarily large manner the resource would only receive $2(0.139/2) = 0.139$. If the resource instead declared $c(s) = s$, the resulting revenue would be $2(0.333/2) = 0.333$. $\square$

What have we learned from our analysis of revenue generation? For symmetric agent populations, mechanisms with linear cost functions are optimal within the class proportional allocations where costs are convex functions of signal. We also know there are cases where a service provider can obtain greater revenue when it does not allocate the entire resource to agents. We know that strictly convex cost functions may outperform linear cost functions in settings with asymmetry. However, strictly convex cost functions can be undermined by agents that split their signals. The resource can alleviate some of the revenue loss by declaring a linear cost function, which increases demand and hence the signal magnitude at equilibrium.

The structure of the allocation mechanism has implications on whether agents split or combine their process streams. The splitting phenomenon might be addressed by the resource if it were to add a small cost for making any bid. This may not completely eliminate the incentive to split, but could add a threshold beyond which an agent may choose not to split. It is also important to note that knowing the valuation structure of the agent population is often necessary to construct an optimal revenue generating mechanism. In general, this information is not available, and makes revenue a difficult metric to optimize, however, as it is critical to many communication network domains, maximizing revenue under this uncertainty becomes a key area for further research.

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