Proceedings of the
44th IEEE Conference on Decision and Control, and
the European Control Conference 2005
Seville, Spain, December 12-15, 2005

On Fractional Systems H_∞ -Norm Computation

J. Sabatier, M. Moze, A. Oustaloup

***Abstract*— Two methods are proposed in this paper for fractional system H∞-norm computation. These methods are extensions to fractional systems of well-known methods for integer systems. The first is based on singular value properties of a linear system and is applied on an academic example. In the second, two extensions of the real bounded lemma derived directly from Lyapunov's theory are deduced. The first method is applied on an academic example.**

I. Introduction

Fractional differentiation is now a well known tool for controller synthesis. Several presentations and applications of the fractional PID controller [1], [2], [3] [4] and of CRONE control [5] demonstrate their efficiency. Fractional differentiation also permits a simple representation of some high order complex integer systems [6]. Consequently, basic properties of fractional systems have been investigated these last ten years and criteria and theorems are now available in the literature concerning stability [7], observability, and controllability [8] of fractional systems.

Lyapunov based methods have also been developed for stability analysis and control law synthesis of integer linear systems, and for more complex systems such as nonlinear, time-varying, and LPV [1]. This has been possible, thanks to the development of efficient numerical methods to solve convex optimization problems [10], by resolving Lyapunov stability conditions or quadratic robust control problems [11] [12] defined by Linear Matrix Inequalities (LMI).

Paradoxically, only few studies deal with Lyapunov based synthesis of control laws for fractional systems. The most advanced method for such purposes consists of considering the fractional behaviour as perturbation while controlling the integer behaviour [13]. As analytical energy computation of fractional systems becomes available [14], methods considering the whole behaviour of fractional systems are now to be developed.

In this paper, we propose two tools for fractional systems H∞-norm computation, based on an extension of two widely used tools designed for integer systems. The first is certainly the most common way to evaluate H∞-norm, which is by manipulating properties of singular values. The other is the real bounded lemma and derived directly from Lyapunov's theory. The main point is that the methods presented here can easily be formulated in the form of Linear Matrix Inequalities (LMI), thus be easily solved thanks to the development of efficient numerical methods to solve convex optimization problems. The first tool is applied to an academic example.

Manuscript received March 7, 2005.

J. Sabatier is with the Laboratoire d'Automatique, Productique, et Signal, LAPS, 351 cours de la Libération, F33405 TALENCE cedex, France (+33(0)540 006 607, e-mail: jocelyn.sabatier@laps.u-bordeaux1.fr).

M. Moze is with the Laboratoire d'Automatique, Productique, et Signal, LAPS, 351 cours de la Libération, F33405 TALENCE cedex, France (e-mail: mathieu.moze@laps.u-bordeaux1.fr).

A. Oustaloup is with the Laboratoire d'Automatique, Productique, et Signal, LAPS, 351 cours de la Libération, F33405 TALENCE cedex, France (e-mail: alain.oustaloup@laps.u-bordeaux1.fr).

II. Notations And Definitions

A. Fractional Calculus

Riemann-Liouville fractional differentiation is used and the fractional integral of a function *f(t)* is defined by

$$I_0^\nu f(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-\xi)^{\nu-1} f(\xi) d\xi \qquad (1)$$

where $\nu \in \mathbb{R}^+$ denotes the fractional integration order, and where

$$\Gamma(\nu) = \int_0^\infty e^{-x} x^{\nu-1} dx . \qquad (2)$$

The fractional derivative of a function f of order $\nu \in \mathbb{R}^+$ can consequently be defined by [15]

$$D^\nu f(t) = D^m [I^{m-\nu} f(t)] , \qquad (3)$$

where m is the smallest integer that exceeds ν .

B. H_∞ -Norm

The H_∞ -norm of a continuous, Linear, Time-Invariant (LTI) system whose transfer function is $G(s)$, is defined through the L_2 -norm by:

$$\|G(s)\|_\infty = \sup_{U(s)\in H_2} \frac{\|Y(s)\|_2}{\|U(s)\|_2} , \qquad (4)$$

where $Y(s)$ and $U(s) \in H_2$ denote respectively Laplace transform of the output signal and of the input signal. L_2 -norm of signal $x(t)$ is

$$\begin{aligned}\|x(t)\|_2 &= \left(\int_0^{+\infty} x(t)^T x(t) dt \right)^{1/2} \\ &= \left(\frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega)^H X(j\omega) d\omega \right)^{1/2} = \|X(j\omega)\|_2\end{aligned} . \qquad (5)$$

Consequently, H_2 is the set of functions $f(s)$, analytic on $\mathrm{Re}(s) \geq 0$, and whose L_2 -norm is bounded.

It can also be shown that

$$\|G(s)\|_\infty = \sup_{\omega \in R} \sigma_{\max}(G(j\omega)), \quad (6)$$

where $\sigma_{\max}$ denotes the greatest value of the singular values of $G(j\omega)$ defined as

$$\sigma(G(j\omega)) = \sqrt{spec\left(G(-j\omega)^T G(j\omega)\right)} . \quad (7)$$

III. Computation of Fractional System H_∞-Norm Based on Stability Analysis

A. Method description

Let us consider a stable Multi-Input, Multi-Output (MIMO) LTI fractional system S_f whose input $u(t) \in \mathbb{R}^p$ and output $y(t) \in \mathbb{R}^m$ are linked by the fractional differential equation:

$$(S_f): \sum_{i=0}^{M} b_i \left(D^\nu\right)^i y(t) = \sum_{i=0}^{N} a_i \left(D^\nu\right)^i u(t). \quad (8)$$

It is supposed that $N \le M$, $(N, M) \in \mathbb{N}^2$, $a_i \in \mathbb{R}^{m \times p}$, $b_i \in \mathbb{R}^{p \times m}$, and that all the differentiation orders are multiples of the commensurate order ν.

It is also assumed that system S_f is relaxed at $t = 0$, so the Laplace transforms of $D^\alpha u(t)$ and of $D^\alpha y(t)$ are respectively considered as $s^\alpha U(s)$ and $s^\alpha Y(s)$ for any $\alpha \in \mathbb{R}$.

Given commensurate order hypothesis, system S_f also admits the state-space description [7]:

$$(S_f): \begin{cases} D^\nu x(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}, \quad (9)$$

where $\nu \in \mathbb{R}^+$ denotes the fractional order of the system, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{p \times 1}$, $C \in \mathbb{R}^{m \times n}$, and $D \in \mathbb{R}^{m \times p}$.

Based on this representation, transfer matrix $G(s)$ is given by

$$G(s) = C\left(s^\nu I - A\right)^{-1} B + D . \quad (10)$$

For simplicity, the form (A, B, C, D, ν) is used in the paper to refer to state space description of relation (9).

Let γ denotes a real positive number satisfying

$$\gamma > \sigma_{\max}(D). \quad (11)$$

From relation (6), system S_f H_∞-norm is bounded by γ if and only if

$$\sup_{\omega} \sigma_{\max}(G(j\omega)) < \gamma , \quad (12)$$

namely, if and only if

$$\forall \omega,\ \forall i \in \{1, \ldots, \min(m, p)\}, \quad \sigma_i(G(j\omega)) < \gamma , \quad (13)$$

where $\sigma_i(G(j\omega))$ is the i^{th} singular value of matrix $G(j\omega)$. Hence, from relation (7), H_∞-norm of system S_f is bounded by γ if and only if

$$\forall \omega,\ \forall i,\ \lambda_i\left(G(-j\omega)^T G(j\omega)\right) < \gamma^2 , \quad (14)$$

Due to eigenvalue properties, relation (14) can be rewritten as:

$$\forall \omega, \quad \forall i, \quad \lambda_i\left(\gamma^2 I - G(-j\omega)^T G(j\omega)\right) > 0 , \quad (15)$$

which is equivalent to the Linear Matrix Inequality (LMI):

$$\forall \omega,\ \gamma^2 I - G(-j\omega)^T G(j\omega) > 0 . \quad (16)$$

As

$$\lim_{\omega \to \infty} \gamma^2 I - G(-j\omega)^T G(j\omega) = \gamma^2 I - D^T D , \quad (17)$$

which is positive from condition (11), relation (16) is satisfied if and only if

$$\forall \omega,\ \gamma^2 I - G(-j\omega)^T G(j\omega) \text{ is non-singular}, \quad (18)$$

that is if and only if

$$\gamma^2 I - G(-s)^T G(s) \text{ has no zero on the imaginary axis.} \quad (19)$$

Hence the H_∞-norm of system S_f is bounded by γ if and only if system S_γ whose transfer matrix is

$$G_\gamma(s) = \left(\gamma^2 I - G(-s)^T G(s)\right)^{-1} \quad (20)$$

is asymptotically stable.

H_∞-norm of fractional system S_f can thus be computed using a dichotomous algorithm on variable γ, stability of system S_γ being analyzed for each value of γ produced by the algorithm. This stability analysis can be done using LMI tools recently developed [16]. Application of these tools requires a state space description for S_γ.

According to [16], note that $t^{-\alpha}$ stability is used to refer to the asymptotic stability of linear fractional systems.

From relation (10),

$$G(-s) = C\left((-s)^\nu I - A\right)^{-1} B + D , \quad (21)$$

or, using exponential form of -1,

$$G(-s) = C\left(e^{\nu j \pi} s^\nu I - A\right)^{-1} B + D , \quad (22)$$

and then

$$G(-s)=C\left(s^{\nu}I-e^{-\nu j\pi}A\right)^{-1}e^{-\nu j\pi}B+D\,. \qquad (23)$$

State space description associated with $G(-s)^T$ is thus $\left(e^{-\nu j\pi}A^T, C^T, e^{-\nu j\pi}B^T, D^T, \nu\right)$.

State space description associated with transfer matrix $\gamma^2 I-G(-s)^T G(s)$ is thus (A', B', C', D', ν), were $A'=\begin{pmatrix} A & 0 \\ C^TC & e^{-\nu j\pi}\end{pmatrix}$, $B'=\begin{pmatrix} B \\ C^TD\end{pmatrix}$, $D'=\begin{pmatrix} D^TC & e^{-\nu j\pi}B^T\end{pmatrix}$, and $D'=D^TD$.

Finally, from dynamic inversion rule, state space description of system S_γ is $(A_\gamma, B_\gamma, C_\gamma, D_\gamma, \nu)$ where

$$A_\gamma=\begin{pmatrix} A+B\left(\gamma^2I-D^TD\right)^{-1}D^TC & e^{-\nu j\pi}B\left(\gamma^2I-D^TD\right)^{-1}B^T \\ C^T\left(I+D\left(\gamma^2-D^TD\right)^{-1}D^T\right)C & e^{-\nu j\pi}\left(A^T+C^TD\left(\gamma^2I-D^TD\right)^{-1}B^T\right)\end{pmatrix},$$

$$B_\gamma=\begin{pmatrix} B\left(D^TD-\gamma^2I\right)^{-1} \\ C^TD\left(D^TD-\gamma^2I\right)^{-1}\end{pmatrix},\ C_\gamma^{\ T}=\begin{pmatrix} \left(\gamma^2I-D^TD\right)^{-1}D^TC \\ e^{-\nu j\pi}\left(\gamma^2I-D^TD\right)^{-1}B^T\end{pmatrix}^T$$

and $D_\gamma=\left(\gamma^2I-D^TD\right)^{-1}$. (24)

Given conclusion directly after relation (20), and using Matignon's stability theorem [7], the following theorem can hence be stated.

Theorem 1: *H_∞-norm of fractional system S_f, whose state space description is given by relation (9), is bounded by a real positive number γ if and only if the eigenvalues of matrix A_γ given by relation (24) lie in the stable domain defined by* $\left\{s\in\mathbb{C}:\left|\arg(s)\right|>\nu\frac{\pi}{2}\right\}$.

□

B. Application

Consider a DC motor whose transfer function is

$$G(s)=\frac{K}{s\left(1+\tau_e s\right)\left(f_m+J_m s\right)}, \qquad (25)$$

where $K=2.34\,N.m.V^{-1}$, $\tau_e=4.7\times10^{-3}s$. The viscous forces are modeled by $f_m=2\times10^{-3}\,N.m.s.rad^{-1}$, and J_m is the system inertia. By adding masses on the rotated disk, J_m can vary such that $J_m=J_0+\Delta_J$, where $J_0=0.066\,kg.m^2$ and $\left|\Delta_J\right|<0.042\,kg.m^2$.

The dynamic performances of the motor are imposed using a fractional controller whose transfer function is

$$C(s)=K_c\left(\frac{s^\nu}{\omega_b}+1\right)\Big/\left(\frac{s^\nu}{\omega_h}+1\right), \qquad (26)$$

where $\omega_b=0.1\,rad.s^{-1}$, $\omega_h=10\,rad.s^{-1}$, $\nu=0.6$ and $K_c=2.82\times10^{-3}$ to ensure a nominal gain cross-over frequency $\omega_{cg}=1\,rad.s^{-1}$ as can be seen in the open loop Bode diagrams in figure 1.

As controller $C(s)$ has been designed using nominal plant $G_0(s)$ defined by

$$G_0(s)=\frac{K}{s\left(1+\tau_e s\right)\left(f_m+J_0 s\right)}, \qquad (27)$$

stability of the closed loop for the entire plant set defined by (25) is to be verified. Small gain theorem introduced in [17] can be used for this purpose.

Fig.1. Nominal open loop Bode diagrams

Theorem 2: *(Small gain theorem) If $M(s)$ and $\Delta(s)$ are stable, the system depicted on figure 2 is stable for every $\Delta(s)$ such that $\left\|\Delta(s)\right\|_\infty<\alpha$ if $\left\|M(s)\right\|_\infty\le\alpha^{-1}$.*

□

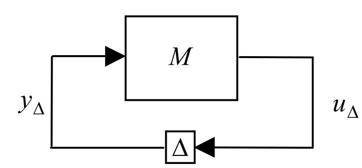

Fig. 2. Standard system for small gain theorem application

Noticing that

$$\frac{1}{f_m+J_m s}=\frac{\dfrac{1}{f_m+J_0 s}}{1+\dfrac{J_d s}{f_m+J_0 s}}, \qquad (28)$$

transfer function $G(s)$ can be rewritten in the form

$$G(s)=\frac{K}{s\left(1+\tau_e s\right)}\frac{\dfrac{1}{f_m+J_0 s}}{1+\dfrac{\Delta_J s}{f_m+J_0 s}}. \qquad (29)$$

The closed loop defined by

$$\beta_c=\frac{C(s)G(s)}{1+C(s)G(s)} \qquad (30)$$

is then given by block diagram presented on figure 3, where $J_d = 0.042$ and $\|\Delta\|_\infty < 1$.

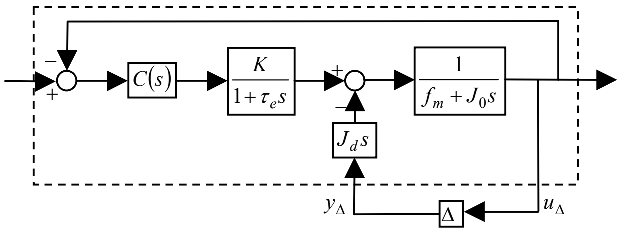

Fig. 3. Closed loop block diagram

It is then straightforward to apply small gain theorem by deriving system depicted on figure 3 using transfer matrix $M(s)$ defined by

$$M(s) = \begin{pmatrix} M_{11}(s) & M_{12}(s) \\ M_{21}(s) & M_{22}(s) \end{pmatrix}, \tag{31}$$

where

$$M_{22}(s) = \frac{-J_d s}{f_m + J_0 s}\left(\frac{1}{1 + CG_0}\right) \tag{32}$$

is the direct term from y_Δ to u_Δ. Closed loop is then stable if $\|M_{22}(s)\|_\infty \leq 1$.

Using a dichotomous method associated to theorem 1, it can be verified that $\|M_{22}(s)\|_\infty < 0.882$. Stability of closed loop system is thus ensured for the entire plant set.

The method introduced in this section is well known for integer systems. However new tools presented in the paper extend the method to fractional systems. Note that the method needs only slight changes to deal with time varying or parameter varying systems, simple case of uncertainty being treated here only for academic purposes.

IV. Computation of Fractional System H_∞-Norm Based on an Extension of Real Bounded Lemma

A. Real Bounded Lemma

According to its definition [17], H_∞-norm of a system S is bounded by γ if and only if

- S is asymptotically stable,
- for any bounded energy input, its output energy is always bounded and verify the following relation:

$$\forall T \geq 0,\ \int_0^T y(t)^T y(t)dt \leq \gamma^2 \int_0^T u(t)^T u(t)dt, \tag{33}$$

where $y(t)$ and $u(t)$ respectively denote output and input of system S. H_∞-norm of system S is then defined as the lowest γ satisfying relation (33).

Note that relation (33) holds for any system S (linear or not), including fractional systems.

Among the various methods that can be used for the H_∞-norm computation of a system S, real bounded lemma permits its computation through an LMI resolution.

Lemma 1 [17]*: (real bounded lemma) Integer system S_i whose state space description is given in (9) with $\nu = 1$, is stable and its H_∞-norm is bounded by γ if and only if there exists a symmetric positive definite matrix P such that*

$$\begin{pmatrix} A^T P + PA & PB & C^T \\ B^T P & -\gamma I & D^T \\ C & D & -\gamma I \end{pmatrix} < 0. \tag{34}$$

□

A rigorous proof of this theorem can be found in [17].

For a fractional system, real bounded lemma can not be directly applied to state space description (9) as $\dot{x}(t)$ is not explicitly given in this state space description.

B. Extension to general MIMO fractional systems

As shown by figure 4, it is supposed that the studied fractional system is decomposed into two sub-systems, a purely integer one S_i (whose output is $y_i(t)$) and a fractional one S_{ni} (whose output is $y_{ni}(t)$).

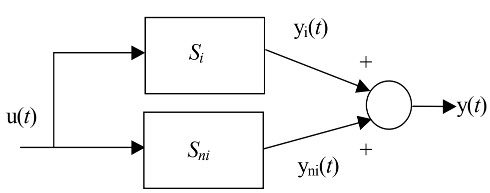

Fig. 4. A MIMO fractional system decomposition

For instance, such a decomposition can be obtained using the methods presented in [18] and [19].

It is also supposed that the L_2 gain of the fractional sub-system S_{ni} is bounded by a known real number r_2 (as it is in [18] and [19]),

$$\left\|S_f - S_i\right\|_{L_2} = \left\|S_{ni}\right\|_{L_2} \leq r_2, \tag{35}$$

and that no pure fractional integration appears in S_{ni}.

From (33), H_∞-norm of a fractional system S_f is bounded by γ if and only if S_f is stable and $\forall T \geq 0$,

$$\int_0^T (y_i(t) + y_{ni}(t))^T (y_i(t) + y_{ni}(t))dt \leq \gamma^2 \int_0^T u(t)^T u(t)dt, \tag{36}$$

or $\forall T \geq 0$,

$$\int_0^T y_i(t)^T y_i(t) + y_{ni}(t)^T y_{ni}(t) + 2 y_{ni}(t)^T y_i(t) dt \leq \gamma^2 \int_0^T u(t)^T u(t) dt. \quad (37)$$

As

$$\begin{aligned}&(y_{ni}(t) - y_i(t))^T (y_{ni}(t) - y_i(t)) \\ &\quad = y_{ni}(t)^T y_{ni}(t) + y_i(t)^T y_i(t) - 2 y_{ni}(t)^T y_i(t),\end{aligned} \quad (38)$$

and considering that

$$(y_{ni}(t) - y_i(t))^T (y_{ni}(t) - y_i(t)) \geq 0, \quad (39)$$

it is straightforward to note that

$$y_{ni}(t)^T y_{ni}(t) + y_i(t)^T y_i(t) \geq 2 y_{ni}(t)^T y_i(t), \quad (40)$$

and thus:

$$\begin{aligned}&y_i(t)^T y_i(t) + y_{ni}(t)^T y_{ni}(t) + 2 y_{ni}(t)^T y_i(t) dt \\ &\quad \leq 2 y_i(t)^T y_i(t) + 2 y_{ni}(t)^T y_{ni}(t)\end{aligned}. \quad (41)$$

H_∞-norm of a fractional system S_f is thus bounded by γ if S_f is stable and

$$\forall T \geq 0, \int_0^T 2 y_i(t)^T y_i(t) + 2 y_{ni}(t)^T y_{ni}(t) dt \leq \gamma^2 \int_0^T u(t)^T u(t) dt, \quad (42)$$

namely if $\forall T \geq 0$,

$$2\int_0^T y_i(t)^T y_i(t) dt + 2\int_0^T y_{ni}(t)^T y_{ni}(t) dt \leq \gamma^2 \int_0^T u(t)^T u(t) dt. \quad (43)$$

From relation (35),

$$\forall T \geq 0, \int_0^T y_{ni}(t)^T y_{ni}(t) dt \leq r_2^2 \int_0^T u(t)^T u(t) dt. \quad (44)$$

The H_∞-norm of a fractional system S_f is thus bounded by γ if S_f is stable and

$$\forall T \geq 0, 2\int_0^T y_i(t)^T y_i(t) dt \leq \left(\gamma^2 - 2r_2^2\right) \int_0^T u(t)^T u(t) dt. \quad (45)$$

A procedure similar to the one used to obtain lemma 1 can then be employed to derive lemma 2.

***Lemma 2**: (Extended lemma) Fractional system S_f is stable and its H_∞-norm is bounded by γ if there exists a symmetric positive definite matrix P such that*

$$\begin{pmatrix} A_i^T P + P A_i & P B_i & C_i^T \\ B_i^T P & -\left(\gamma^2 - 2r_2^2\right)^{1/2} I & D_i^T \\ C_i & D_i & -\left(\gamma^2 - 2r_2^2\right)^{1/2} I \end{pmatrix} < 0,$$

where $(A_i, B_i, C_i, D_i, 1)$ is the state space description of the integer subsystem S_i, and r_2 is a bound of the L_2 – norm of the fractional subsystem S_{ni} of figure 4.

□

C. Particular case of SISO fractional systems

Lemma 2 holds for MIMO and thus for SISO systems. If only the SISO case is considered, another extension of the real bounded lemma can be stated. As described in [13], any stable SISO linear fractional system S_f can be decomposed as the sum of three subsystems:

- an integer linear system denoted E,
- pure fractional integrators,
- and a stable fractional system denoted P.

Under some conditions, this decomposition is said "structural". This decomposition is not presented here for brevity purposes but is exhaustively given in [20]. Note that this decomposition can be easily extended to MIMO systems described by transfer matrix. Demonstration is in MIMO case based on partial fraction expansion of each element of fractional system S_f transfer matrix $G(s)$ given by (10).

It is shown in [20] that the L_1 – norm of the impulse response of fractional subsystem P is bounded by a calculable real number r_1 such that

$$\|P\|_{L_1} \leq r_1. \quad (46)$$

Note that as in SISO case, L_2 – gain of subsystem P is a lower bound of its L_1 – gain, H_∞ – norm of P is bounded by r_1.

In this extension, studied fractional system S_f is such that no pure fractional integration appears in the decomposition described by figure 5.

Let $y(t)$ denote the output of S_f. If $y_P(t)$ and $y_E(t)$ respectively denote the integer subsystem P output and the stable subsystem E output, then

$$y(t) = y_P(t) + y_E(t). \quad (47)$$

From relation (46),

$$\forall T \geq 0, \int_0^T y_P(t)^T y_P(t) dt \leq r_1^2 \int_0^T u(t)^T u(t) dt, \quad (48)$$

where r_1 is a bound of the L_1 – norm of the impulse response of fractional subsystem P.

By analogy with relations (36) to (45), the H_∞-norm of a fractional system S_f is thus bounded by γ if S_f is stable and

$$\forall T \geq 0, 2\int_0^T y_E(t)^T y_E(t) dt \leq \left(\gamma^2 - 2r_1^2\right) \int_0^T u(t)^T u(t) dt. \quad (49)$$

A procedure similar to the one used to obtain lemma 2 can then be employed to derive lemma 3.

***Lemma 3**: (Extended lemma, a second version for SISO systems) SISO Fractional system S_f is stable and its H_∞-norm is bounded by γ if there exists a symmetric positive definite matrix P such that*

$$\begin{pmatrix} A_E^T P + P A_E & P B_E & C_E^T \\ B_E^T P & -\left(\gamma^2 - 2r_1^2\right)^{1/2} I & D_E^T \\ C_E & D_E & -\left(\gamma^2 - 2r_1^2\right)^{1/2} I \end{pmatrix} < 0,$$

where $(A_E, B_E, C_E, D_E, 1)$ is the state space description of the integer subsystem, and r is a bound of the L_1-norm of the impulse response of fractional subsystem.

□

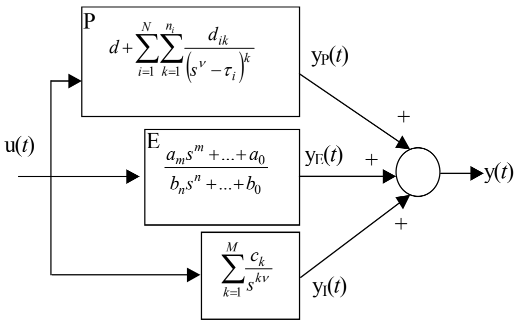

Fig. 5. A decomposition of a SISO fractional system

V. Conclusion

Fractional PID regulator and CRONE robust regulators are now well known in the field of fractional differentiation application in control theory. Synthesis strategies for these two classes of regulators are usually done in the frequency domain and are mainly based on the application of Nyquist criterion and its extensions. Paradoxically, no method based on more powerful tools such as Lyapunov stability or small gain theorem has been investigated for fractional systems. However, such methods are now essential for the extension of the existing control methods to time-varying or/and nonlinear fractional systems. In order to develop control methods for more complex fractional systems than the linear one, this paper proposes two tools for the computation of a fractional system H_∞-norm. The first one is based singular value properties of a linear system. Computing the H_∞-norm with this method consists in stability analysis of a fractional system and can be easily implemented. The second is an extension of the real bounded lemma. It is based on the decomposition of a fractional system into two sub-systems: an integer one and a fractional one. This method requires the computation of the fractional subsystem L_2-gain upper bound in the MIMO case or the computation of the fractional subsystem L_1-gain upper bound (see [20] for this gain estimation). Implementation of this method is thus more complex but until now, no other real bounded lemma extension exists for fractional systems. Our goal is thus now to improve this second method in order to show its efficiency for CRONE control of time-varying systems.

References

[1] Podlubny, I., 1999, "Fractional-Order systems and PID-Controllers", in *IEEE Trans. on Aut. Cont.*, vol. 44, no. 1, pp. 208-214.

[2] Monje, C., A., Vinagre, B., M., Chen, Y., Q., Feliu, V., Lanusse, P., Sabatier, J., "Proposals for fractional PID tuning", *FDA 04*, Bordeaux, France, 2004.

[3] Caponetto, R., Fortuna, L., Porto, D., "A new tuning strategy for a non integer order PID controller", *FDA 04*, Bordeaux, France, 2004.

[4] Chen, Y., Q., Moore, K., L., Vinagre, B., M., Podlubny, I., "Robust PID controller auto tuning with a phase shaper", *First IFAC workwhop on Fractional Derivative and its Application*, *FDA 04*, Bordeaux, France, 2004.

[5] Oustaloup, A., Mathieu, B., 1999, *La commande CRONE du scalaire au multivariable*, Hermes Science Publications, Paris.

[6] Battaglia, J-L., Cois, O., Puissegur, L., Oustaloup, A., "Solving an inverse heat conduction problem using a non-integer identified model", *International Journal of Heat and Mass Transfer*, Vol 44, n°14, pp 2671-2680, juillet 2001.

[7] Matignon, D., July 1996, "Stability results on fractional differential equations with applications to control processing", in *Computational Engineering in Systems and Application multiconference*, vol. 2, pp. 963-968, IMACS, IEEE-SMC.

[8] Matignon, D., D'Andrea-Novel, B., "Some results on controllability and observability of finite-dimensional fractional differential systems", in *Computational Engineering in Systems Applications*, vol. 2, pp. 952-956., July 1996, IMACS, IEEE-SMC.

[9] Biannic, J., M., "Commande Robuste des Systèmes à Paramètres Variables", Thesis, Ecole Nationale Supérieure de l'Aéronautique et de l'Espace, France.

[10] Boyd, S., Vandenberghe, L., 2004, Convex Optimization, Cambridge University Press.

[11] Balakrishnan, V. and Kashyap, R. L., March 1999, "Robust Stability and Performance Analysis of Uncertain Systems Using Linear Matrix Inequalities", In *Journal of Optimization Theory and Applications,* vol. 100, no. 3, pp. 457-478.

[12] Balakrishnan, V. , August 2002, "Linear Matrix Inequalities in Robust Control: A Brief Survey" In *Proc. Math. Thy of Networks and Systems*, Notre Dame, Indiana.

[13] Hotzel, R., "Contributions à la Théorie Structurelle et la Commande des Systèmes Linéaires Fractionnaires", Thesis, Université de Paris-Sud, centre d'Orsay, France.

[14] Malti, R., Cois O., Aoun, M., Levron, F., Oustaloup, A., "Computing impulse response energy of fractional transfer function", in the 15th IFAC World Congress 2002, Barcelona, Spain, July 21-26, 2002.

[15] Miller, K.S., Ross, B., 1993, *An Introduction To The Fractional Calculus and Fractional Differential Equation*, John Wiley & Sons, Inc., New York.

[16] Moze, M., Sabatier, J., Oustaloup, A., " LMI Tools for Stability Analysis of Fractional Systems", to be presented in the ASME 2005 International Design Engineering Technical Conferences (submitted).

[17] Boyd, S., El Ghaoui, L., Feron, E. Balakrishnan, V., June 1994, "Linear Matrix Inequalities in System and Control Theory", Vol 15 of *Stud. in Applied Math.*, Philadelphia.

[18] Gu, G., Khargonekar, P., P., Lee, E., B., "Approximation of Infinite-Dimensional Systems", in *IEEE Trans. On Automatic Control*, vol. 34, no. 6, June 1989.

[19] Raynaud, H., F., and Zergaïnoh, A., "State-Space Representation for Fractional Order Controllers", in Automatica 36 (2000), pp. 1017-1021.

[20] Hortzel, R., Fliess, M., 1998, "Gain Estimation for Fractional Linear Systems", in *Proc. IFAC Conf. System Structure and Control*, Nantes, pp. 237-242.