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Observer Design for Nonlinear Systems: An Approach Based on the Differential Mean Value Theorem.

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Abstract— **In this note, observers design for a class of non linear dynamical systems has been investigated. The main contribution lies in the use of the differential mean value theorem (DMVT) to transform the nonlinear error dynamics into a LPV system. The stability analysis is, therefore, performed using a standard Lyapunov function that leads to the solvability of a set of Linear Matrix Inequalities (LMIs), easily tractable. Numerical examples are provided to show high performances of the proposed approach and the large class of nonlinear dynamical systems that are concerned.**

I. INTRODUCTION

Over the last decades, tremendous research activities focus on observer design for nonlinear dynamical systems as can be shown through the vast literature in this field. The main motivations lie in the fact that state estimation may be used for control design, diagnosis or supervision. More recently, other applications like synchronization and input recovery, in communication systems, become one of the emerging and interesting research area [1]-[2]-[3]-[4].

For the lack of space here, we will mention some basic and standard observer methods. One of them consists in using nonlinear change of coordinates to bring the original system into a linear one (or pseudo-linear one). We refer the reader to the pioneering works and their extensions about this approach [5]-[6]-[7]-[8]-[9]-[10]-[11]. As can be expected, the main advantage behind the use of this approach is to simplify the observer design, however this requires strong sufficient conditions to be satisfied and therefore only a class of systems may be considered, see [12].

A second, and famous, approach consists in using the Extended Kalman Filter (EKF). The latter is frequently used as a deterministic observer for nonlinear systems. The proof of its exponential stability has been given in [13], and it has been proved in [14] that the EKF is locally stable when it is used as an observer for discrete-time nonlinear systems. In [15], the authors proposed an observer for continuous-time nonlinear systems where the observer gain is computed by a Riccati differential equation similar to the EKF. In spite of the large use of this method, only local convergence may be guaranteed.

The class of Lipschitz nonlinear systems has been widely investigated, since most physical processes can be described by nonlinear Lipschitz models. In [15], [16], [17], [18] and [19], the authors proposed specific solutions to this type of systems where the stability conditions are expressed in terms of algebraic Riccati equations. The same class of systems is investigated in [20] to construct an observer, where the convergence of the estimation error has been studied by using both Lyapunov functions and functionals, and stability conditions are expressed using LMIs. However, all these stability conditions are difficult to be satisfied for large values of the Lipschitz constant.

Recently, Arcak et *al.* [21], and Fan et *al.* [22] have presented a new observer for a class of systems with monotonic nonlinearities. This new design removes the global Lipschitz restriction and avoids *high-gain*. The single restriction is that the nonlinearities must be nondecreasing functions of linear combinations of unmeasured states. The stability conditions expressed as LMIs are not restrictive and are easily satisfied for many examples. However, the nondecreasing restriction excludes a broad variety of nonlinearities such as $x^2, \exp(-x)$, etc.

In this paper, we propose a new observer design for a large class of nonlinear systems. The basic idea of this work is to use the well known DMVT, which allows to write the dynamics of the observer error as a LPV system. The stability analysis is easy to investigate by using a classical quadratic Lyapunov function and convexity theory. The observer gain guaranteeing the global convergence of the proposed observer is computed by LMIs.

It should be noticed that the proposed approach can be applied in both continuous and discrete time nonlinear models even with non Lipschitz nonlinearities. Numerical examples are provided to show high performances and the large class of nonlinear dynamical systems that are concerned.

This paper is organized as follows. In section II, we recall the DMVT. The class of systems to be investigated and the proposed observer are presented in section III. In section IV, we introduce the main contribution of our paper. In section V we propose a generalization of our approach to a large class of systems and we give an extension to the discrete-time case. Three numerical examples are simulated in section VI to demonstrate the validity of this approach.

Finally, we give some conclusions in section VII.

Notations : In this note, the notation $(\star)$ is used for the blocks induced by symmetry. The matrix A^T represents the transposed matrix of A. The symbol $\sum_{i,j=1}^{q,n}$ means $\sum_{i=1}^{q}\sum_{j=1}^{n}$.

II. PRELIMINARIES

In this section, we present a mathematical tool which is important for the next section : the differential mean value theorem.

We first present the differential mean value theorem in one dimension and then we generalize it to a higher-dimension.

Theorem 2.1: (DMVT) Let $f : [a,b] \longrightarrow \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). Then, there is some c with $a < c < b$ such that

$$f(a) - f(b) = f^{'}(c)(a-b).$$

This theorem is a consequence of Rolle's theorem.

Now, we state the mean value theorem in higher-dimension which is important for the approach developed in this paper. Before stating this theorem, we introduce the following definition :

Definition 2.2: Let x, y be two elements in $\mathbb{R}^n$. We define by $Co(x,y)$ the convex hull of the set $\{x,y\}$, i.e.

$$Co(x,y) = \{\lambda x + (1-\lambda)y, \lambda \in [0,1]\}.$$

Theorem 2.3: (DMVT in $\mathbb{R}^n$) Let $f : \mathbb{R}^n \to \mathbb{R}$. Let a, b be two elements in $\mathbb{R}^n$. We assume that f is differentiable on $Co(a,b)$. Then, there is a constant $c \in Co(a,b), c \neq a, c \neq b$ such that :

$$f(a) - f(b) = f^{'}(c)(a-b)$$

where

$$f^{'} = [\frac{\partial f}{\partial x_1} ... \frac{\partial f}{\partial x_n}].$$

Proof: Let g: $[0,1] \to \mathbb{R}$ be the function defined by

$$g(t) = f(a + t(b-a)).$$

g is differentiable on $]0,1[$, continuous on $[0,1]$, and

$$g^{'}(t) = f^{'}(a + t(b-a))(b-a).$$

Using the Theorem 2.1, there exists $c_1 \in]0,1[$ such that :

$$g(1) - g(0) = g^{'}(c_1),$$

which is equivalent to

$$f(b) - f(a) = f^{'}(a + c_1(b-a))(b-a).$$

Then, there exists $c = a + c_1(b-a) \in Co(a,b), c \neq a, c \neq b$ such that :

$$f(a) - f(b) = f^{'}(c)(a-b).$$

■

Remark 2.4: Generally the differential mean value theorem is not true for higher-dimensional vector-valued functions. The following is a counter-example (see [23]).

Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be defined by :

$$f(x) = \begin{pmatrix} f_1(x_1,x_2) \\ f_2(x_1,x_2) \end{pmatrix} = \begin{pmatrix} x_1^2 \\ e^{x_1+x_2} \end{pmatrix}.$$

If we set $a = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $b = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$,

then,

$$f(a) = \begin{pmatrix} 1 \\ e^2 \end{pmatrix}, \ f(b) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and

$$\frac{\partial f}{\partial x}(a + c(b-a)) = \begin{pmatrix} 2(1-c) & 0 \\ e^{2-2c} & e^{2-2c} \end{pmatrix}.$$

If the mean value theorem exists, then we will have

$$\begin{pmatrix} 1 \\ e^2 - 1 \end{pmatrix} = \begin{pmatrix} 2(1-c) \\ 2e^{2-2c} \end{pmatrix}.$$

The first equation gives $c = \frac{1}{2}$, which contradicts the second equation.

Since, the DMVT is not correct for vector-valued function, we propose to proceed as follows :

Let

$$E_s = \left\{e_s(i) \ | \ e_s(i) = (0,...,0,1,0,...,0)^T, i = 1,...,s.\right\}$$

be the canonical basis of the vectorial space $\mathbb{R}^s$ for all $s \geq 1$.

Let

$$f \ : \mathbb{R}^n \to \mathbb{R}^q$$

be a vector function. Then,

$$f(x) = [f_1(x), ..., f_q(x)]^T,$$

where $f_i \ : \mathbb{R}^n \to \mathbb{R}$ is the i^{th} component of f.

We know that the vectorial space $\mathbb{R}^q$ is generated by the canonical basis E_q. Therefore, we can write :

$$f(x) = \sum_{i=1}^{q} e_q(i) f_i(x). \quad (1)$$

Now, we state the following proposition.

Proposition 2.5: Let $f : \mathbb{R}^n \to \mathbb{R}^q$. Let $a, b \in \mathbb{R}^n$. We assume that f is differentiable on $Co(a,b)$. Then, there are constant vectors $c_1, ..., c_q \in Co(a,b), c_i \neq a, c_i \neq b$ for $i = 1, ..., q$ such that :

$$f(a) - f(b) = \left(\sum_{i,j=1}^{q,n} e_q(i) e_n^T(j) \frac{\partial f_i}{\partial x_j}(c_i)\right)(a-b) \quad (2)$$

Proof: From (1), we have

$$f(a) - f(b) = \sum_{i=1}^{q} e_q(i)\left(f_i(a) - f_i(b)\right).$$

Now, we apply the DMVT on each $f_i, i = 1, ..., q$.

From the Theorem 2.3, there exists $c_i \in Co(a,b)$ such that

$$f_i(a) - f_i(b) = \frac{\partial f_i}{\partial x}(c_i)(a-b),$$

for all $i = 1, ..., q$.

As

$$\left(\frac{\partial f_i}{\partial x}\right)^T = \left(\frac{\partial f_i}{\partial x_1}, ..., \frac{\partial f_i}{\partial x_n}\right)^T \in \mathbb{R}^n,$$

then, we can write

$$\frac{\partial f_i}{\partial x}(c_i) = \sum_{j=1}^{n} e_n^T(j) \frac{\partial f_i}{\partial x_j}(c_i).$$

Therefore,

$$f(a) - f(b) = \left(\sum_{i,j=1}^{q,n} e_q(i) e_n^T(j) \frac{\partial f_i}{\partial x_j}(c_i) \right) (a - b).$$

■

III. PROBLEM FORMULATION

We consider the class of nonlinear systems described by the following nonlinear state equations :

$$\begin{cases} \dot{x}(t) = Ax(t) + Bf(x(t)) + g(y(t), u(t)) \\ y(t) = Cx(t) \end{cases} \tag{3}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the input vector and $y(t) \in \mathbb{R}^p$ is the output vector. *A, B, C* are constant matrices of appropriate dimensions. The functions f: $\mathbb{R}^n \mapsto \mathbb{R}$ and g: $\mathbb{R}^p \times \mathbb{R}^m \mapsto \mathbb{R}^n$ are nonlinear and f is assumed to be differentiable.
A state observer corresponding to (3) is given as follows :

$$\begin{cases} \dot{\hat{x}}(t) = A\hat{x}(t) + Bf(\hat{x}(t)) + g(y(t), u(t)) + \\ \qquad\qquad\qquad\qquad + L(y(t) - \hat{y}(t)) \\ \hat{y}(t) = C\hat{x}(t) \end{cases} \tag{4}$$

where $\hat{x}(t)$ denotes the estimate of the state $x(t)$. The observation problem consists in finding a *gain* L such that the observer error $\varepsilon(t) = x(t) - \hat{x}(t)$ converges exponentially and asymptotically towards zero.
The dynamics of the observer error is expressed as follows :

$$\dot{\varepsilon}(t) = (A - LC)\varepsilon(t) + B\big(f(x(t)) - f(\hat{x}(t))\big).$$

By the DMVT, there exists $z(t) \in Co(x(t), \hat{x}(t))$ such that

$$f(x(t)) - f(\hat{x}(t)) = \frac{\partial f}{\partial x}(z(t))(x(t) - \hat{x}(t)).$$

As $\left(\frac{\partial f}{\partial x}\right)^T \in \mathbb{R}^n$, then we can write

$$\frac{\partial f}{\partial x}(z(t)) = \sum_{i=1}^{n} e_n^T(i) \frac{\partial f}{\partial x_i}(z(t)).$$

With the notation

$$h_i(t) = \frac{\partial f}{\partial x_i}(z(t)),$$

the dynamics of the observer error becomes :

$$\dot{\varepsilon}(t) = \left(A + \sum_{i=1}^{n} h_i(t) B e_n^T(i) - LC \right) \varepsilon(t). \tag{5}$$

By setting

$$h(t) = (h_1(t), ..., h_n(t))$$

and

$$\mathcal{A}\left(h(t)\right) = A + \sum_{i=1}^{n} h_i(t) B e_n^T(i)$$

we have

$$\dot{\varepsilon}(t) = \big(\mathcal{A}(h(t)) - LC\big)\varepsilon(t). \tag{6}$$

The observer error system (6) is a LPV system, for which we can easily study the stability conditions.
Before introducing our main result, we introduce the following assumption :
Assumption : we assume that the functions h_i are bounded.

$$\max_t |h_i(z(t))| < +\infty.$$

Note that this assumption is not restrictive. Indeed, it is satisfied for a large class of nonlinear systems, namely the chaotic systems for which this assumption is always satisfied. By this assumption, the parameter $h(t)$ evolves in a bounded domain $\mathcal{H}_n$ of which 2^n vertices are defined by :

$$\mathcal{V}_{\mathcal{H}_n} = \big\{ \alpha = (\alpha_1, ..., \alpha_n) \mid \alpha_i \in \{\underline{h}_i, \bar{h}_i\} \big\}$$

where

$$\bar{h}_i = \max_t \big(h_i(t)\big) \text{ and } \underline{h}_i = \min_t \big(h_i(t)\big).$$

IV. MAIN RESULT

In this section, we introduce the main contribution of our work. We give sufficient conditions for the observer synthesis.

Theorem 4.1: The observer error $\varepsilon(t)$ converges exponentially towards zero if there exist matrices $P = P^T > 0$ and R of appropriate dimensions such that the following LMIs are feasible :

$$\mathcal{A}^T(\alpha)P - C^T R + P\mathcal{A}(\alpha) - R^T C < 0 \qquad \forall\ \alpha \in \mathcal{V}_{\mathcal{H}_n}. \tag{7}$$

When these LMIs are feasible, the *observer gain* L is given by $L = P^{-1}R^T$.

Proof: To study the exponential convergence of the observer error, we consider the following quadratic Lyapunov function

$$V(t) = V(\varepsilon(t)) = \varepsilon^T(t) P \varepsilon(t),$$

where P is a symmetric matrix, with $P > 0$.
The observer error converges exponentially towards zero if $V(t) > 0$ and $\dot{V}(t) < 0$ for all $\varepsilon(t) \neq 0$.
We have

$$\dot{V}(t) = \varepsilon^T(t) F(h(t)) \varepsilon(t)$$

where

$$F(h(t)) = \big(\mathcal{A}(h(t)) - LC\big)^T P + P\big(\mathcal{A}(h(t)) - LC\big).$$

The condition $V(t) > 0$ is satisfied because the matrix P is positive definite. Note that the condition $\dot{V}(t) < 0$ is satisfied if we have

$$F(h(t)) < 0 \text{ for all } h(t) \in \mathcal{H}_n.$$

Since the matrix function F is affine in $h(t)$, then using the convexity principle (see [24] for more details) we deduce that $\dot{V}(t) < 0$ if the following condition is satisfied :

$$F(\alpha) < 0,\ \forall\ \alpha \in \mathcal{V}_{\mathcal{H}_n}. \tag{8}$$

If we use the notation $R = L^T P$, the condition (8) is equivalent to (7). Thus, if (7) holds, then the inequality (8) is also verified, which imply that $\dot{V}(t) < 0$. This ends the proof of theorem 4.1. ■

V. EXTENSIONS

In this section, we generalize our main result to a broader class of systems and we extend it to the discrete-time case.

A. Generalization to a Broader Class of Systems

Consider the nonlinear system described by :

$$\begin{cases} \dot{x}(t) = Ax(t) + Bf(x(t)) + g(y(t), u(t)) \\ y(t) = Cx(t) \end{cases} \quad (9)$$

where $x(t) \in \mathbb{R}^n$ is the state vector,$u(t) \in \mathbb{R}^m$ is the input vector and $y(t) \in \mathbb{R}^p$ is the output vector. *A, C* are constant matrices of appropriate dimensions.
The functions f: $\mathbb{R}^n \mapsto \mathbb{R}^q$ and g: $\mathbb{R}^p \times \mathbb{R}^m \mapsto \mathbb{R}^n$ are nonlinear and f is assumed to be differentiable.
Consider also the corresponding state observer as follows :

$$\begin{cases} \dot{\hat{x}}(t) = A\hat{x}(t) + Bf(\hat{x}(t)) + g(y(t), u(t)) \\ \qquad\qquad + L(y(t) - \hat{y}(t)) \\ \hat{y}(t) = C\hat{x}(t) \end{cases} \quad (10)$$

The dynamics of the observer error is expressed as follows :

$$\dot{\varepsilon}(t) = (A - LC)\varepsilon(t) + B(f(x(t)) - f(\hat{x}(t))).$$

From the Proposition 2.5, there exist $z_i(t) \in Co(x(t), \hat{x}(t))$, for all $i = 1, ..., q$, such that :

$$f(x(t)) - f(\hat{x}(t)) = \left(\sum_{i,j=1}^{q,n} e_q(i) e_n^T(j) \frac{\partial f_i}{\partial x_j}(z_i(t)) \right) \varepsilon(t).$$

Using the notations

$$h_{ij}(t) = \frac{\partial f_i}{\partial x_j}(z_i(t)),$$

$$h(t) = (h_{11}(t), ..., h_{1n}(t), ..., h_{qn}(t))$$

and

$$\mathcal{A}(h(t)) = A + \sum_{i,j=1}^{q,n} h_{ij}(t) B e_q(i) e_n^T(j),$$

the observer error dynamics can be rewritten as follows :

$$\dot{\varepsilon}(t) = \big(\mathcal{A}(h(t)) - LC\big)\varepsilon(t). \quad (11)$$

As previously, we assume that the functions h_{ij} are bounded for all $i = 1, ..., q$ and $j = 1, ..., n$.
Then, the parameter vector $h(t)$ remains in a bounded domain $\mathcal{H}_{q,n}$ of which 2^{qn} vertices are defined by :

$$\mathcal{V}_{\mathcal{H}_{q,n}} = \big\{\alpha = (\alpha_{11}, ..., \alpha_{1n}, ..., \alpha_{qn}) \mid \alpha_{ij} \in \{\underline{h}_{ij}, \bar{h}_{ij}\}\big\}$$

where

$$\bar{h}_{ij} = \max_t \big(h_{ij}(t)\big) \text{ and } \underline{h}_{ij} = \min_t \big(h_{ij}(t)\big).$$

Now, we can state the following theorem.

Theorem 5.1: The observer error converges exponentially towards zero if there exist matrices $P = P^T > 0$ and R of appropriate dimensions such that the following LMIs are feasible :

$$\mathcal{A}^T(\alpha)P - C^T R + P\mathcal{A}(\alpha) - R^T C < 0 \qquad \forall\ \alpha \in \mathcal{V}_{\mathcal{H}_{q,n}}. \quad (12)$$

When these LMIs are feasible, the *observer gain* L is given by $L = P^{-1}R^T$.

Proof: The proof of this theorem is similar to that of theorem 4.1. The same Lyapunov function and the same reasoning are used. ■

B. Extension to the Discrete-Time Case

Now, we consider the discrete-time nonlinear system in the general form described by :

$$\begin{cases} x(k+1) = Ax(k) + Bf(x(k)) + g(y(k), u(k)) \\ y(k) = Cx(k) \end{cases} \quad (13)$$

where $x(k) \in \mathbb{R}^n$ is the state vector, $u(k) \in \mathbb{R}^m$ is the input vector and $y(k) \in \mathbb{R}^p$ is the output vector. *A,C* are constant matrices of appropriate dimensions. The functions
f: $\mathbb{R}^n \mapsto \mathbb{R}^q$ and g: $\mathbb{R}^p \times \mathbb{R}^m \mapsto \mathbb{R}^n$ are nonlinear and f is assumed to be differentiable.
A state observer corresponding to (13) is given as follows:

$$\begin{cases} \hat{x}(k+1) = A\hat{x}(k) + Bf(\hat{x}(k)) + g(y(k), u(k)) + \\ \qquad\qquad + L(y(k) - \hat{y}(k)) \\ \hat{y}(k) = C\hat{x}(k) \end{cases} \quad (14)$$

The dynamics of the observer error $\varepsilon(k) = x(k) - \hat{x}(k)$ is governed by the following equation :

$$\varepsilon(k+1) = (A - LC)\varepsilon(k) + B\big(f(x(k)) - f(\hat{x}(k))\big)$$

We proceed as in the continuous-time case. From the Proposition 2.5, there exists $z_i(k) \in Co(x(k), \hat{x}(k))$ for all $i = 1, ..., q$, such that :

$$f(x(k)) - f(\hat{x}(k)) = \left(\sum_{i,j=1}^{q,n} e_q(i) e_n^T(j) \frac{\partial f_i}{\partial x_j}(z_i(k)) \right) \varepsilon(k).$$

Then, using the notations :

$$h_{ij}(k) = \frac{\partial f_i}{\partial x_j}(z_i(k)),$$

$$h(k) = (h_{11}(k), ..., h_{1n}(k), ..., h_{qn}(k))$$

and

$$\mathcal{A}(h(k)) = A + \sum_{i,j=1}^{q,n} h_{ij}(z_j(k)) B e_q(i) e_n^T(j),$$

the equation of the observer error dynamics can be rewritten as follows :

$$\varepsilon(k+1) = \big(\mathcal{A}(h(k)) - LC\big)\varepsilon(k). \quad (15)$$

Our aim is to design the matrix $L \in \mathbb{R}^{n \times p}$ that guarantees the exponential convergence of the observer error to zero.
As in the previous section, we assume that the functions h_{ij} are bounded for all $i = 1, ..., q$ and $j = 1, ..., n$.
Then, the vector $h(k)$ evolves in a bounded domain $\bar{\mathcal{H}}_{q,n}$ of

which 2^{qn} vertices are defined by :

$$\bar{\mathcal{V}}_{\bar{\mathcal{H}}_{q,n}} = \{\alpha = (\alpha_{11}, ..., \alpha_{1n}, ..., \alpha_{qn}) \mid \alpha_{ij} \in \{\underline{h}_{ij}, \bar{h}_{ij}\}\}$$

where

$$\bar{h}_{ij} = \max_k \big(h_{ij}(k)\big) \text{ and } \underline{h}_{ij} = \min_k \big(h_{ij}(k)\big).$$

Under the above assumption, we can state the following theorem.

Theorem 5.2: The observer error $\varepsilon(k)$ converges exponentially towards zero if there exist matrices $P = P^T > 0$ and R of appropriate dimensions such that the following linear matrix inequalities (LMIs) are feasible :

$$\begin{bmatrix} -P & \mathcal{A}^T(\alpha)P - C^T R \\ (\star) & -P \end{bmatrix} < 0$$

$$\forall\ \alpha \in \bar{\mathcal{V}}_{\bar{\mathcal{H}}_{q,n}}. \quad (16)$$

When these LMIs are feasible, the *gain-matrix* L is given by $L = P^{-1}R^T$.

Proof: Consider the following usual Lyapunov function

$$V(k) = V(\varepsilon(k)) = \varepsilon^T(k)P\varepsilon(k).$$

The variation of this Lyapunov function is :

$$\Delta V = \varepsilon(k)^T \big((\mathcal{A}(h(k)) - LC)^T P(\mathcal{A}(h(k)) - LC) - P\big)\varepsilon(k).$$

Based on the Lyapunov stability theory, ΔV must be negative-definite in order to guarantee the convergence of the estimation error. Using the Schur complement, this implies

$$F(h(k)) = \begin{bmatrix} -P & \mathcal{A}^T(h(k))P - C^T R \\ (\star) & -P \end{bmatrix} < 0.$$

Finally, using the convexity theory as in the proof of Theorem 4.1, we deduce that $\Delta V < 0$ if F is negative definite on $\bar{\mathcal{V}}_{\bar{\mathcal{H}}_{q,n}}$, which is equivalent to (16). This ends the proof of Theorem 5.2. ■

VI. NUMERICAL EXAMPLES

Our approach has been tested successfully on several examples. We can mention the Rossler chaotic system, the Van Der Pol oscillator and all the examples presented in [22], [1], [9], [20] and others discrete-time nonlinear systems. In this paper we present three numerical examples. The first one is the Duffing nonlinear system. The second one is the model of a flexible joint robot link and the third one is the discrete version of the Lorenz chaotic system.

A. Example 1

Consider the following continuous-time nonlinear system of Duffing under the form (3) :
$A = \begin{bmatrix} 0 & 1 \\ 0 & -\delta \end{bmatrix}$, $C = \begin{bmatrix} 0.2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$,
$f(x(t)) = x_1^3$, $g(y(t), u(t)) = \begin{bmatrix} 0 \\ \sigma\cos(t) \end{bmatrix}$,

where $\delta = 0.1,\ \sigma = 11$ and $x = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T$.
Applying our approach, we obtain
$\frac{\partial f}{\partial x}(z(t)) = h_1(z(t))e_2^T(1)$, with $h_1(z(t)) = 2z_1(t)$
and $e_2^T(1) = \begin{bmatrix} 1 & 0 \end{bmatrix}$. Since the state of the system is bounded, then the parameter $z(t)$ is bounded. Then also $h_1(z(t))$ is bounded and we have after simulations, $\underline{h}_1 = 0$ and $\bar{h}_1 = 45.86$.
Then, the theorem 4.1 gives $L = \begin{bmatrix} 0.9901 & -0.0056 \end{bmatrix}^T$. The estimation error is showed in figure 1.

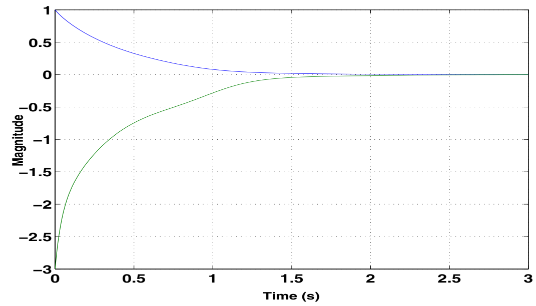

Fig. 1. The estimation error behavior.

B. Example 2

Consider the model of a flexible joint robot link. Joint flexibility is modeled as a stiffening torsional spring. The dynamic equations are given by :

$$\begin{cases} \dot{\theta}_m = \omega_m \\ \dot{\omega}_m = \frac{1}{J_m}\tau - \frac{b}{J_m}\omega_m + \frac{K_\tau}{J_m}u \\ \dot{\theta}_l = \omega_l \\ \dot{\omega}_l = -\frac{1}{J_l}\tau - \frac{Mgh}{J_l}sin(\theta_l) \end{cases} \quad (17)$$

where $\theta_m, \omega_m, \theta_l$ and ω_l are the motor and link position and velocities respectively. J_m and J_l are the inertia of the motor and link respectively, $2h$ and M represent the length and mass of the link, b is the viscous friction, and K_τ is the amplifier gain. The torque due to the stiffening spring is

$$\tau = \kappa_1(\theta_l - \theta_m) + \kappa_2(\theta_l - \theta_m)^3,$$

where κ_1 and κ_2 are positive constants.
By setting $x = \begin{bmatrix} \theta_m & \omega_m & \theta_l & \omega_l \end{bmatrix}^T$, the set of equations (17) can be rewritten under the form (9), with

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{\kappa_1}{J_m} & -\frac{b}{J_m} & \frac{\kappa_1}{J_m} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{\kappa_1}{J_l} & 0 & -\frac{\kappa_1}{J_l} & 0 \end{bmatrix},\ g(u(t), y(t)) = \begin{bmatrix} 0 \\ \frac{K_\tau}{J_m} \\ 0 \\ 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 \\ \frac{\kappa_2}{J_m} & 0 \\ 0 & 0 \\ -\frac{\kappa_2}{J_l} & -\frac{Mgh}{J_l} \end{bmatrix} \text{ and } f(x(t)) = \begin{bmatrix} (\theta_l - \theta_m)^3 \\ sin(\theta_l) \end{bmatrix}.$$

The output matrix is : $C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$.
Then, with the physical values $J_m = 3.7 \times 10^{-3}kgm^2$, $J_l = 9.3 \times 10^{-3}kgm^2, h = 1.5 \times 10^{-1}m, M = 0.21kg$, $b = 4.6 \times 10^{-2}m, K_\tau = 8 \times 10^{-2}NmV^{-1}, \kappa_1 = 1, \kappa_2 = 1$ and $u(t) = \sin(t)$, our approach gives the following *matrix-gain*, L, guaranteeing the exponential convergence of the proposed observer :

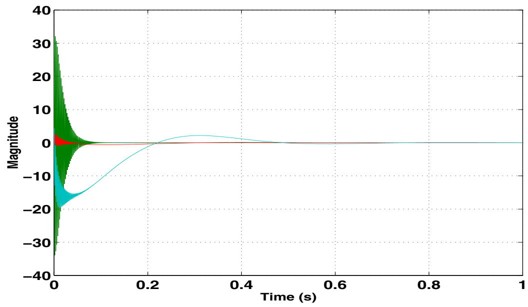

Fig. 2. The estimation error behavior.

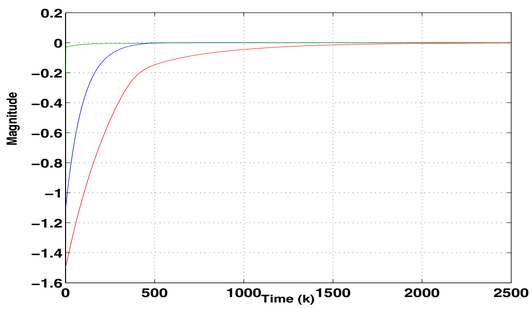

Fig. 3. The estimation error behavior.

$$L = \begin{bmatrix} 191 & 456 & 42 & 335 \\ -322 & -49032 & -2358 & -7659 \end{bmatrix}^T.$$

The exponential convergence of the state estimates is illustrated with simulations in figure 2.

C. Example 3

Consider the following discrete-time version of Lorenz chaotic system under the form (13) :

$$A = \begin{bmatrix} 1-10T & 10T & 0 \\ 28T & 1-T & 0 \\ 0 & 0 & 1-\frac{8}{3}T \end{bmatrix}, C = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix},$$

$$f(x(k)) = \begin{bmatrix} x_1(t)x_3(t) & x_1(t)x_2(t) \end{bmatrix}^T,$$

$$B = T \begin{bmatrix} 0 & 0 \\ -1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } g(u(k), y(k)) = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T,$$

where $T = 0.001$ is the sampling period.
Then, using our approach, we obtain by Theorem 5.2

$$L = \begin{bmatrix} 0.0252 & 0.9994 & 0.0567 \end{bmatrix}^T.$$

We give in figure 3 the observer error behavior which shows that the observer error converges exponentially towards zero.

VII. CONCLUSION

In this paper, an observer design problem for a large class of nonlinear systems has been considered. We used the DMVT which allows to write the dynamics of the observer error as a LPV system. New sufficient conditions are obtained. These conditions are expressed as a LMIs solvability problem which is easily tractable by convex optimization techniques. The convergence conditions presented in this paper are not restrictive. We have tested it on several examples and we proposed in this paper three examples to show the good performances of our method.

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