Proceedings of the
44th IEEE Conference on Decision and Control, and
the European Control Conference 2005
Seville, Spain, December 12-15, 2005

Robust design Criteria for Integral Sliding Surfaces

Fernando Castaños and Leonid Fridman

Abstract— **The robustness properties of integral sliding mode controllers are studied. This note shows how to select the projection matrix in such a way that the euclidean norm of the resulting perturbation is minimal. It is also shown that when the minimum is attained, the resulting perturbation is not amplified. This selection is particularly useful if integral sliding mode control is to be combined with other methods to further robustify against unmatched perturbations. $\mathcal{H}_\infty$ is taken as a special case. Simulations support the general analysis and show the effectiveness of this particular combination.**

I. Introduction

Sliding mode control [1] is a robust technique, well known for its ability to withstand external disturbances and model uncertainties satisfying the *matching condition*, that is, perturbations that enter the state equation at the same point as the control input (e.g. the case of completely actuated systems). Sliding mode control (SMC) has other advantages as well, like ease of implementation and reduction in the order of the state equation. The latter property clearly simplifies the control design problem.

Roughly speaking, the conventional SMC design methodology comprises two steps: first design a sliding manifold such that the system's motion along the manifold meets the specified performance; second, design a (discontinuous) control law, such that the system's state is driven towards the manifold and stays there for all future time, regardless of disturbances or uncertainties. The resulting controller, although robust against matched perturbations, has some disadvantages. Among them we have: the need to measure the whole state; the lack of robustness against unmatched perturbations; and the *reaching phase*, i.e. an initial period of time in which the system has not yet reached the sliding manifold and it is sensitive, even to perturbations satisfying the matching condition.

Several strategies have been proposed to solve these problems. See for example [2], [3], [4], [5], [6] where the need to measure the whole state is relaxed. To address the issue of robustness against unmatched perturbations the main strategy has been the combination of SMC with other robust techniques, e.g. [7], [8], [9].

In order to solve the reaching phase problem, an *integral* sliding manifold was proposed [10], [11]. The basic idea is to define the control law as the sum of a continuous nominal control and a discontinuous control. The nominal control is responsible for the performance of the nominal system, i.e. without perturbations; and the discontinuous control is used to reject the perturbations. Finally, an integral term is included in the sliding manifold. The integral term allows to define the manifold in such a way that the system trajectories start in the manifold right at the beginning of the process. The later means that the closed-loop system is robust since the first time instant.

A. Motivation

To solve the problems of the reaching phase and of the robustness against unmatched perturbations simultaneously (e.g. in the case of sub-actuated systems), the main idea –as in the conventional sliding mode case– has been the combination of integral sliding mode control and other robust techniques. The particular combination depends of course on the specific nature of each problem, and each particular combination has a set of details that needs to be properly addressed. In the case of multi-model uncertain systems [12], [13] a multi-model decomposition becomes the essential problem; in the case of nonlinear systems with unknown unmatched uncertainties [14] Lyapunov's direct method becomes a key feature; if integral sliding mode control is to be combined with LMI based control techniques, the selection of the equivalent matched dynamics would be the main issue. For systems with time delay the essential problem is that the nominal control should contain a delayed component [15].

In all of the above mentioned cases the selection of the projection matrix plays a key role in the design of the sliding manifold. In this note we address the need for a universal choice of such matrix.

B. Main Contribution

In this work we show the following:

- At an integral sliding mode, the discontinuous control completely compensates the matched perturbations, but the unmatched ones are replaced by another (which we shall call equivalent) disturbance.
- There is a set of projection matrices for which the norm of the equivalent disturbance is minimal.
- For any projection matrix in this set, the gain of the discontinuous action is also minimal and the equivalent disturbance equals the unmatched one, i.e. there is no amplification of the unmatched disturbance.

All the above means that an integral sliding mode controller, if improperly designed, while eliminating the matched perturbations, could lead to amplification of the unmatched ones.

The main results are general and can be applied whenever ISMC is to be combined with other techniques to robustify

This work was supported by CONACyT (Consejo Nacional de Ciencia y Tecnología), grant no. 43807-Y, and by Programa de Apoyo a Proyectos de Investigación e Innovación Tecnológica (PAPIIT), UNAM, grant no. 117103.

The authors are at Universidad Nacional Autónoma de México (UNAM), Department of Control, Engineering Faculty. C.P. 04510. México D.F. lfridman@verona.fi-p.unam.mx

against unmatched disturbances. In this note $\mathcal{H}_\infty$ control is taken as a specific case. Simulations support the validity of the analysis developed and show that the performance of an $\mathcal{H}_\infty$ controller can be increased by this particular combination.

C. Paper's Structure

In the next section we present a short review of ISMC and state the problem formally. In section III the problem stated is solved and different interpretations are given to the results. In section IV we analyze the combination of ISM with $\mathcal{H}_\infty$ control. The conclusions are in section V.

II. Problem Statement

A. Preliminaries, ISMC

Consider a nonlinear system of the form

$$\dot{x} = f(x,t) + Bu(x,t) + \phi(x,t), \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $t \in \mathbb{R}$ represents time, $u(x,t) \in \mathbb{R}^m$ is the control action and $\phi(x,t)$ is a perturbation due to model uncertainties or external disturbances. The following assumptions are made:

Assumption 1: $\operatorname{rank} B = m$.

Assumption 2: The actual value of $\phi(x,t)$ is of course unknown, but it is bounded by a known function $\bar{\phi}(x,t) \in \mathcal{L}_\infty$, i.e. $\|\phi(x,t)\| \leq \bar{\phi}(x,t)$ for all x and t.

In the ISMC approach, a law of the form

$$u(x,t) = u_0(x,t) + u_1(x,t)$$

is proposed. The nominal control $u_0(x,t)$ is responsible for the performance of the nominal system; $u_1(x,t)$ is a discontinuous control action that rejects the perturbations by ensuring the sliding motion. The sliding manifold is defined by the set $\{x \mid s(x,t) = 0\}$, with

$$s(x,t) = G\Big[x(t) - x(t_0) - \int_{t_0}^{t} \big(f(x,\tau) + Bu_0(x,\tau)\big)\mathrm{d}\tau\Big]. \tag{2}$$

$G \in \mathbb{R}^{m\times n}$ is a projection matrix which must satisfy:

Assumption 3: The matrix product GB is invertible.

The term

$$x(t_0) + \int_{t_0}^{t} \big(f(x,\tau) + Bu_0(x,\tau)\big)\mathrm{d}\tau$$

in (2) can be thought as a trajectory of the system in the absence of perturbations and in the presence of the nominal control u_0, that is, as a nominal trajectory for a given initial condition $x(t_0)$. With this remark in mind, $s(x,t)$ can be considered a penalizing factor of the difference between the actual and the nominal trajectories, projected along G (hence the name *projection matrix*, not to be confused with a projection operator). Notice that at $t = t_0$, $s(x,t) = 0$, so the system always starts at the sliding manifold.

The discontinuous control u_1 is usually selected as

$$u_1(x,t) = -\rho(x,t)\frac{(GB)^T s(x,t)}{\|(GB)^T s(x,t)\|}, \tag{3}$$

where $\rho(x,t)$ is a gain high enough to enforce the sliding motion. To simplify notation we will omit some of the functions' arguments from now on.

B. Analysis of the Unmatched Perturbation

Before we analyse the effect of the unmatched perturbation it is convenient to introduce the following proposition

Proposition 1: For any matrix $B \in \Re^{n\times m}$ satisfying Assumption 1, the identity

$$I_n = BB^+ + B^\perp B^{\perp +}$$

holds, where B^+ is understood as the left inverse of B, that is $B^+ = (B^T B)^{-1} B^T$ and the columns of $B^\perp \in \mathbb{R}^{n\times(n-m)}$ span the null space of B^T.

Proof: Consider a matrix

$$P = \begin{bmatrix} B^+ \\ B^{\perp +} \end{bmatrix}.$$

This matrix is clearly non-singular since it's inverse is given by $P^{-1} = \begin{bmatrix} B & B^\perp \end{bmatrix}$, that is

$$P \cdot P^{-1} = \begin{bmatrix} B^+ B & 0 \\ 0 & B^{\perp +}B^\perp \end{bmatrix} = \begin{bmatrix} I_m & 0 \\ 0 & I_{n-m} \end{bmatrix}.$$

By reversing the order of the operands we get $P^{-1} \cdot P = BB^+ + B^\perp B^{\perp +} = I_n$. ■

Now we can project the perturbation ϕ into the matched and unmatched spaces

$$\phi = \phi_\mathrm{m} + \phi_\mathrm{u}, \quad \phi_\mathrm{m} \triangleq BB^+\phi, \quad \phi_\mathrm{u} \triangleq B^\perp B^{\perp +}\phi,$$

where ϕ_m and ϕ_u are the components that belong to the matched and unmatched spaces respectively.

To determine the motion equations at the sliding manifold we use the equivalent control method [1]. The derivative of s along time is

$$\begin{aligned}\dot{s} &= G\left[f + B(u_0 + u_1) + BB^+\phi + B^\perp B^{\perp +}\phi\right] - \\ &\quad - G\left[f + Bu_0\right] \\ &= GB(u_1 + B^+\phi) + G\phi_\mathrm{u}.\end{aligned}$$

The equivalent control is obtained by solving the equation $\dot{s} = 0$ for $u_{1\mathrm{eq}}$

$$u_{1\mathrm{eq}} = -B^+\phi - (GB)^{-1}G\phi_\mathrm{u}. \tag{4}$$

Remark 1: In the majority of the papers dealing with SMC, perturbations are assumed to be matched and the term on the far right is usually ignored.

By substituting $u_{1\mathrm{eq}}$ for u_1 in (1) we obtain the sliding dynamics

$$\begin{aligned}\dot{x}_\mathrm{eq} &= f + B(u_0 - B^+\phi - (GB)^{-1}G\phi_\mathrm{u}) + \quad (5)\\ &\quad + BB^+\phi + B^\perp B^{\perp +}\phi \\ &= f + Bu_0 + \left[I - B(GB)^{-1}G\right]\phi_\mathrm{u}. \quad (6)\end{aligned}$$

From the last equation we can draw several conclusions. First, the dynamics at the sliding manifold do not contain the matched perturbation: it has been successfully rejected. Second, with respect to conventional SMC, we have gained some extra degrees of freedom. We can use u_0 to stabilize the nominal system and to treat the unmatched perturbation. The projection matrix G can now be considered a free parameter. Third, the order of the equivalent dynamics is equal to that of the original system, that is, there is *no order reduction*. This is the "price" we pay in return for the extra degrees of freedom and the elimination of the reaching phase. And fourth, the unmatched perturbation is now multiplied by a matrix

$$\Gamma \triangleq \left[I - B(GB)^{-1}G\right].$$

Another way to look at this, is that we have traded the original perturbation $\phi_{\mathrm{m}} + \phi_{\mathrm{u}}$, for a new one: $\phi_{\mathrm{eq}} \triangleq \Gamma\phi_{\mathrm{u}}$.

C. Specific Questions

Matrix Γ is the main concern of this note. We would like to pose two specific questions regarding Γ:

1) Is there a G^*, such that norm of the equivalent perturbation ϕ_{eq} is minimal?
2) Does matrix Γ amplify the unmatched perturbation? i.e. is the norm of ϕ_{eq} greater than the norm of ϕ_{u}?

These questions make sense whenever we are considering unmatched perturbations and u_0 is to be designed with robustness against unmatched uncertainty in mind.

III. Main Results

In this section we answer the questions formulated in the problem statement and make some comments on the answers.

Proposition 2: B^T is a matrix which minimizes the norm of ϕ_{eq}, i.e.

$$G^* = B^T = \arg\min_{G\in\mathbb{R}^{m\times n}} \left\| \left[I - B\left(GB\right)^{-1} G\right]\phi_{\mathrm{u}}\right\|_2 \quad (7)$$

Proof: Notice first that

$$\left\| \left[I - B\left(GB\right)^{-1} G\right]\phi_{\mathrm{u}}\right\|_2 = \left\|\phi_{\mathrm{u}} - B\varphi\right\|_2$$

where $\varphi = \left(GB\right)^{-1} G\phi_{\mathrm{u}}$. Thus, problem (7) can be rewritten in the form:

$$\varphi^* = \arg\min_{\varphi\in\Re^m} \left\|\phi_{\mathrm{u}} - B\varphi\right\|_2,$$

which, according to the Projection theorem [16, p. 51] has $\varphi^* = B^+\phi_{\mathrm{u}}$ as a solution. Making $G = B^T$ we will have:

$$\varphi = \left(B^T B\right)^{-1} B^T\phi_{\mathrm{u}} = B^+\phi_{\mathrm{u}} = \varphi^*$$

which implies (7). ■

Notice that for $G = B^+$ we also have $\varphi = B^+\phi_{\mathrm{u}}$, so B^+ also minimizes ϕ_{eq}.

Proposition 3: For a minimizing G^*, the unmatched perturbation ϕ_{eq} is not amplified, i.e. for $m < n$ the following identity holds:

$$\|\Gamma^*\| = 1,$$

where $\Gamma^* = I - B\left(B^+B\right)^{-1} B^+ = I - BB^+$.

Proof: Notice first that

$$\begin{aligned} \Gamma^{*T}\Gamma^* &= \left[I - BB^+\right]\left[I - BB^+\right] \\ &= I - BB^+ - BB^+ + BB^+BB^+ \\ &= I - BB^+ = \Gamma^*, \end{aligned}$$

which means that Γ^* is a symmetric matrix and therefore all the eigen-values are real. Suppose that v is an eigen-vector associated to any eigen-value λ of Γ^*, that is,

$$\Gamma^* v = \lambda v \quad \Rightarrow \quad v^T\Gamma^{*T}\Gamma^* v = \lambda^2\|v\|^2. \quad (8)$$

But, since $\Gamma^{*T}\Gamma^* = \Gamma^*$ we have

$$v^T\Gamma^{*T}\Gamma^* v = v^T\Gamma^* v = \lambda\|v\|^2. \quad (9)$$

From (8) and (9), it is clear that the eigen-values of Γ must satisfy $\lambda^2 = \lambda$. The last equation has two solutions, $\lambda = 0$ and $\lambda = 1$.

Since $\mathrm{rank}(BB^+) < n$, the rank of $I - BB^+$ cannot be zero. This means that Γ^* must have at least one eigen-value different from zero, that is, the maximum eigen-value is one. The last statement implies that $\|\Gamma^*\| = 1$. ■

A possible interpretation of the previous theorems is that in order to avoid amplification, we should only penalize the difference between the actual and nominal trajectories, projected into the matched space. Notice that for $G = B^+$ the equivalent control (4) becomes $u_{1\mathrm{eq}} = -B^+\phi$, so, it should only contain the matched perturbations: any attempt to compensate the unmatched perturbations with the discontinuous control would only make matters worst.

The selection $G = B^+$ has other advantages. First, the discontinuous control (3) is simplified to

$$u_1 = -\rho\frac{s}{\|s\|}.$$

Second, the gain ρ can be reduced (and as a consequence the amplitude of the chattering can be reduced). Consider the candidate Lyapunov function $V = \|s\|^2/2$. At $G = B^+$ the derivative of s is

$$\dot{s} = -\rho\frac{s}{\|s\|} + B^+\phi,$$

and the derivative of V is

$$\dot{V} = s^T\left(-\rho\frac{s}{\|s\|} + B^+\phi\right) \leq -\|s\|(\rho - \|B^+\phi\|) \quad (10)$$

In order to guarantee the sliding motion the discontinuous action only has to major the matched disturbance. In the general case, the derivative of the Lyapunov function is

$$\begin{aligned} \dot{V} &= s^T\left(GB\left(-\rho\frac{(GB)^T s}{\|(GB)^T s\|} + B^+\phi\right) + G\phi_{\mathrm{u}}\right) \\ &\leq -\|(GB)^T s\|(\rho - \|B^+\phi - (GB)^{-1}G\phi_{\mathrm{u}}\|) \quad (11) \end{aligned}$$

So it is reasonable to select $G = B^+$.

Remark 2: Since $\phi_{\mathrm{u}} = B^\perp B^{\perp +}\phi$, at $G = B^+$ the product $G\phi_{\mathrm{u}}$ equals zero and we have $\phi_{\mathrm{eq}} = \phi_{\mathrm{u}}$. The system dynamics at the sliding manifold then becomes

$$\dot{x}_{\mathrm{eq}} = f + Bu_0 + \phi_{\mathrm{u}}. \quad (12)$$

IV. Example: ISM and $\mathcal{H}_\infty$ Control

In this section we analyze the specific combination of ISMC and another robust method. The main goal of this section is to support the previous analysis and propositions 2 and 3. For simplicity we have chosen a linear technique: $\mathcal{H}_\infty$ control.

A. Background, $\mathcal{H}_\infty$ control

Within the classical framework, when the full state is available the plants under consideration have the form

$$\dot{x} = Ax + B_w w + Bu \quad (13a)$$
$$z = Cx + Du, \quad z \in \Re^{n+m} \quad (13b)$$

where z is an artificial penalty variable, matrices C and D are of appropriate dimension and establish a compromise between the cost associated to the state and the cost of the control used to keep the state within some bounds. The goal is to minimize the $\mathcal{H}_\infty$ norm of the transfer matrix T_{zw} that goes from w to z.

The following assumption is typical:

Assumption 4: (A, B) is stabilizable, (C, A) is detectable and $D^T \begin{bmatrix} C & D \end{bmatrix} = \begin{bmatrix} 0 & I \end{bmatrix}$.

The first part of assumption 4 is obvious and the second guaranties the boundedness of the state. The last part means that z has no cross weighting between the state and control, and that the control weight matrix is the identity. The latter can be relaxed by a suitable coordinate transformation.

The following theorem (given without proof) is a standard result of $\mathcal{H}_\infty$ control [17].

Theorem 1 (Doyle et al.*):* Given assumption 4, there exist a controller satisfying

$$\|T_{zw}\|_\infty < \gamma$$

iff there exists a real, symmetric, positive semi-definite matrix X satisfying the Riccati equation

$$XA + A^T X - X(BB^T - \gamma^{-2} B_w B_w^T)X + C^T C = 0. \quad (14)$$

Moreover, when this condition holds, one such controller is

$$u = -B^T X x. \quad (15)$$

In [18], [19] it is shown that the $\mathcal{H}_\infty$ norm in the frequency domain and the (truncated) $\mathcal{L}_2$ induced norm of a linear system in the time domain are equivalent, i.e., if the conditions of Theorem 1 are satisfied, then

$$\int_{t_0}^{T} \|z\|^2 \mathrm{d}\tau \le \gamma^2 \int_{t_0}^{T} \|w\|^2 \mathrm{d}\tau \quad (16)$$

holds for all $T \ge t_0$. This equivalence allows to understand the $\mathcal{H}_\infty$ problem in terms of *disturbance attenuation*, to generalize the $\mathcal{H}_\infty$ control objective to nonlinear systems and to restate the $\mathcal{H}_\infty$ control problem in the following terms: minimize the system's performance index, where the performance index γ, is understood as a truncated $\mathcal{L}_2$ gain.

B. Proposed Methodology

The basic idea is to use an ISMC to reject the matched perturbation and design the nominal control using $\mathcal{H}_\infty$ techniques to attenuate the unmatched one. Suppose that a control is to be designed for system (13). In terms of (1) we have $f(x, t) = Ax$ and $\phi = B_w w$. According to (12), the system's dynamics at the sliding manifold is $\dot{x} = Ax + B^\perp B^{\perp+} B_w w + Bu_0$, where $\phi_{\mathrm{u}} = B^\perp B^{\perp+} B_w w$ was used to derive the previous equation. Notice that the discontinuous control u_1 is already fixed, so we need to replace u by u_0 in the definition of the penalty variable z, that is

$$z_0 = Cx + Du_0.$$

The problem now becomes that of finding a minimum γ and a semi-definite matrix X that satisfies (14), but with B_w substituted by $B^\perp B^{\perp+} B_w$.

The control u_1 is used to keep the state within some bounds and the cost of it should be taken into account if a comparison with the standard $\mathcal{H}_\infty$ control strategy is to be made, in other words: for comparison purposes the original definition of z should be used. Whether or not the discontinuous control u_1 improves the over all performance index is not an easy question to answer, for it depends mainly on the weight C assigned to the state. We can however, make a (rather informal) remark: notice that by orthogonality $\|B^\perp B^{\perp+} B_w w\|^2 = \|B_w w\|^2 - \|BB^+ B_w w\|^2$. Since the squared norm of the unmatched perturbation is not bigger than the original one, we should expect a better performance index if the weight given to the state is "high-enough".

We summarize the proposed methodology in the following algorithm:

1) Solve the Riccati equation

$$XA + A^T X - X\big(BB^T - \gamma^{-2} \bar{B}_w \bar{B}_w^T\big)X + C^T C = 0, \quad (17)$$

where $\bar{B}_w \triangleq B^\perp B^{\perp+} B_w$.

2) Set the sliding manifold as

$$s = B^+ \left[x(t) - x(t_0) - \int_{t_0}^{t} (A - BB^T X) x(\tau) \mathrm{d}\tau \right]$$

3) and the control as

$$u = -B^T X x - \rho \frac{s}{\|s\|}, \quad \rho > \|B^+ B_w w\|.$$

C. Numerical Example

Consider the following LTI system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -24 & -50 & -35 & -10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \quad (18)$$

$$+ \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}}_{B_w} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{B} u. \quad (19)$$

We define the error variable as

$$z_0 = \underbrace{\begin{bmatrix} 5I_4 \\ \mathbf{0} \end{bmatrix}}_{C} x + \underbrace{\begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix}}_{D} u_0$$

1) $\mathcal{H}_\infty$ control alone: Equation (14) has

$$X = \begin{bmatrix} 116.27 & 200.05 & 120.14 & 7.61 \\ 131.54 & 350.85 & 276.52 & 19.21 \\ -0.39 & 95.41 & 208.05 & 16.07 \\ -5.04 & -1.17 & 13.66 & 2.62 \end{bmatrix}$$

as a solution when $\gamma = 7.1$. The resulting controller is then,

$$u = -B^T X_\infty x = -\begin{bmatrix} -5.04 & -1.17 & 13.66 & 2.62 \end{bmatrix} x$$

2) ISMC plus $\mathcal{H}_\infty$: The disturbances are first decomposed as

$$B_w w = \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}}_{BB^+B_w} w + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}}_{B^\perp B^{\perp+} B_w} w.$$

The first part is matched and will be eliminated by the discontinuous control u_1; the second is unmatched and will be treated using the continuous control u_0, designed using the $\mathcal{H}_\infty$ technique.

The solution to (17) is given by

$$X = \begin{bmatrix} 24.77 & -36.36 & -60.69 & -8.76 \\ -8.37 & -89.89 & -183.66 & -24.26 \\ 81.03 & 159.07 & 93.54 & 1.83 \\ 5.15 & 10.16 & 5.85 & 1.53 \end{bmatrix},$$

and $\gamma = 5.6$. The nominal control is

$$u_0 = -B^T X_\infty x = -\begin{bmatrix} 5.15 & 10.16 & 5.85 & 1.53 \end{bmatrix} x$$

and the sliding manifold is

$$s(x,t) = B^+ \left[x(t) - x(t_0) - \int_{t_0}^{t} (Ax + Bu_0) \mathrm{d}\tau \right].$$

D. Simulation results

Three simulations were carried out. In all cases the system was perturbed by the signal

$$w = \sin(\pi t) \begin{bmatrix} 1 & -1 \end{bmatrix}^T \tag{20}$$

and the initial conditions were set at the origin. The first simulation was made using the $\mathcal{H}_\infty$ controller. The second, using the combination 'ISMC plus $\mathcal{H}_\infty$', but with G set different to B^+. To illustrate our point, a rather extreme case was used

$$G = 0.5B^+ + 10 \sum_{i=1}^{3} B_i^{\perp+} = \begin{bmatrix} -10 & -10 & 10 & 0.5 \end{bmatrix}$$

where $B_i^{\perp+}$ was taken as the ith row of $B^{\perp+}$. The third simulation was made using the optimal value $G = B^+$. The system's states are shown in Fig. 1.

In the second simulation the gain ρ needed to enforce the sliding mode was obtained using (11) and was set to 35. In the last simulation it was obtained using (10) and was set to 1.5. In both cases the discontinuous control was approximated by

$$u_1 = -35 \frac{s}{|s| + 0.0002} \quad \text{and} \quad u_1 = -1.5 \frac{s}{|s| + 0.0002}$$

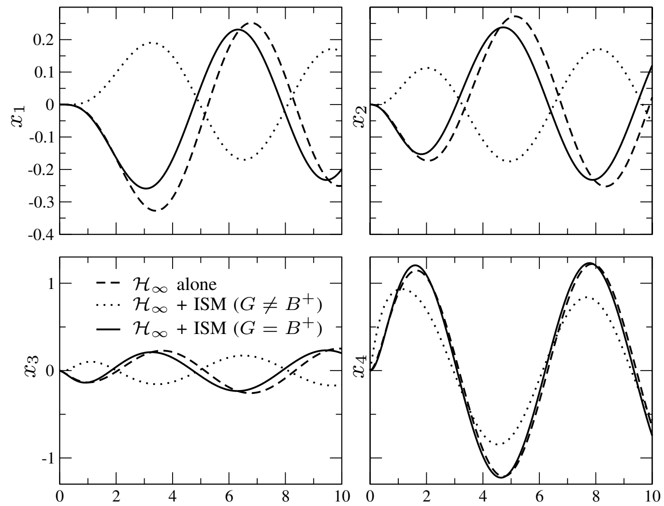

Fig. 1. System's states. Simulation results for the $\mathcal{H}_\infty$ controller (dashed-line) and the "ISM plus $\mathcal{H}_\infty$" controller for two cases: $G \neq B^+$ (dotted-line) and $G = B^+$ (solid-line)

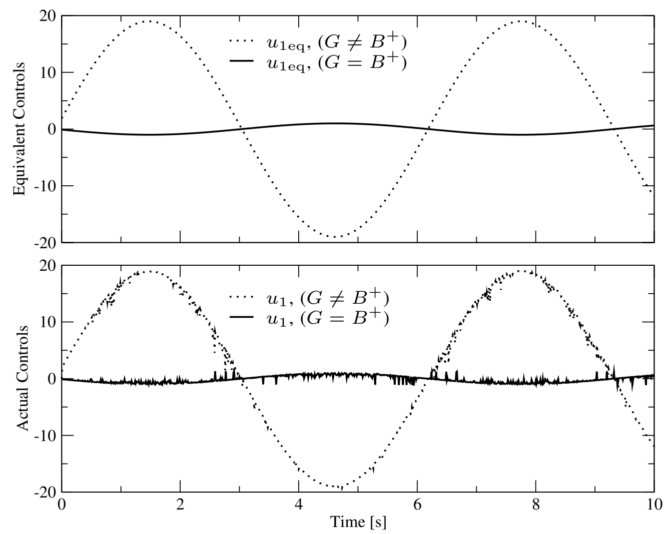

Fig. 2. High frequency controls. Equivalent controls (upper) for $G \neq B^+$ (dotted) and $G \neq B^+$ (solid). Actual controls (lower) $u_1 = -35s/(|s| + 0.0002)$ for $G \neq B^+$ (dotted) and $u_1 = -1.5s/(|s| + 0.0002)$ for $G = B^+$ (solid).

respectively. As shown in Fig. 2, these controls follow closely the equivalent controls obtained in (4). It can be seen, that when the matrix G achieving the minimal norm of the equivalent perturbation is not used, the control acts in the opposite direction, i.e. it's effect is *counter effective*.

For comparison purposes, we have in Fig. 3 a plot of $\|z\|_{\mathcal{L}_2}/\|w\|_{\mathcal{L}_2}$ for each controller. When G is selected improperly, the value is *increased* due to the amplification of ϕ_u. When G is selected properly, the value is lower than the one obtained by $\mathcal{H}_\infty$ alone.

V. Conclusions

In this note we analyze the effects that the projection matrix have on the resulting (equivalent) perturbation. In the presence of unmatched disturbances the projection matrix of an ISM controller should be selected carefully, for the resulting controller could amplify them. Two propositions provide a way for selecting the projection matrix correctly. The proposed parameters ensure that the effect of the unmatched disturbance will *not be amplified* by the discontinuous control. It is also shown that the discontinuous control can not attenuate the unmatched disturbances.

The analysis is aimed at combining ISMC with other robust techniques. $\mathcal{H}_\infty$ control was selected as a specific case, but other techniques could be used as well. Simulation results support the analysis developed.

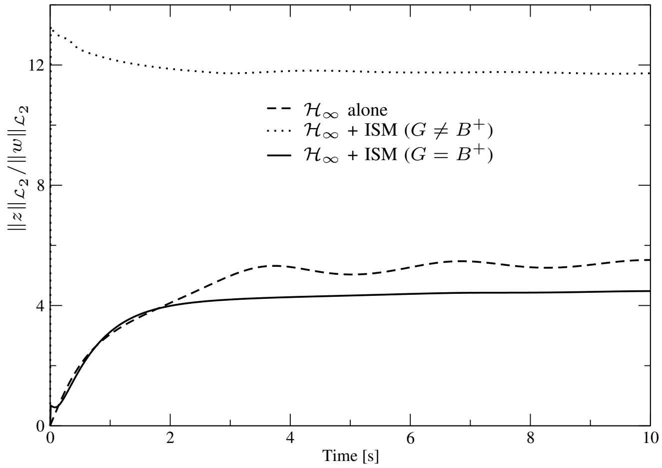

Fig. 3. Actual values of the $\mathcal{L}_2$ gains for perturbation (20).

References

[1] V. Utkin, J. Guldner, and J. Shi, *Sliding Modes in Electromechanical Systems*. London, U.K.: Taylor & Francis, 1999.

[2] C. Edwards and S. K. Spurgeon, "Sliding mode stabilisation of uncertain systems using only output information." *International Journal of Control*, vol. 62, pp. 1129–1144, 1995.

[3] K. Bag, S. K. Spurgeon, and C. Edwards, "Output feedback sliding mode design for linear uncertain systems," *IEE Proc.-Control Theory Appl.*, vol. 144, pp. 209–216, May 1997.

[4] C. Edwards, A. Akoachere, and S. K. Spurgeon, "Sliding-mode output feedback controller design using linear matrix inequalities," *IEEE Trans. Automat. Contr.*, vol. 46, pp. 115–119, Jan. 2001.

[5] A. Levant, "Higher-order sliding modes, differentiation and output-feedback control," *International Journal of Control*, vol. 76, pp. 924–941, 2003.

[6] Y. B. Stessel, "Nonlinear output tracking in conventional and dynamic sliding manifolds," *IEEE Trans. Automat. Contr.*, vol. 42, pp. 1282–1286, 1997.

[7] A. Poznyak, Y. B. Stessel, and C. Jiménez, "Mini-max sliding mode control for multimodel linear time varing systems," *IEEE Trans. Automat. Contr.*, vol. 48, pp. 2141–2150, Dec. 2003.

[8] Y. Orlov, W. Perruquetti, and J. Richard, "Sliding mode control synthesis of uncertain time-delay systems," *Asian Journal of Control*, vol. 5, pp. 568–577, 2003.

[9] H. H. Choi, "An LMI-based switching surface design method for a class of mismatched uncertain systems," *IEEE Trans. Automat. Contr.*, vol. 48, pp. 1634–1638, Sept. 2003.

[10] G. P. Matthews and R. A. DeCarlo, "Decentralized tracking for a class of interconnected nonlinear systems using variable structure control," *Automatica*, vol. 24, pp. 187–193, 1988.

[11] V. Utkin and J. Shi, "Integral sliding mode in systems operating under uncertainty conditions," in *Proc. Conference on Decision and Control*, Kobe, Japan, Dec. 1996, pp. 4591–4596.

[12] A. Poznyak, L. Fridman, and F. J. Bejarano, "Mini-max integral sliding mode control for multimodel linear uncertain systems," *IEEE Trans. Automat. Contr.*, vol. 49, pp. 97–102, Jan. 2004.

[13] L. Fridman, A. Poznyak, and F. J. Bejarano, "Mini-max multimodel optimal problem via integral sliding mode control," in *Proc. of the American Control Conference*, Boston, June 2004, pp. 620–625.

[14] J. Xu, Y. Pan, and T. Lee, "Analysis and design of integral sliding mode control based on Lyapunov's direct method," in *Proc. of the American Control Conference*, Denver, Colorado, June 2003, pp. 192–196.

[15] M. Basin, J. Rodríguez-González, L. Fridman, and P. Acosta, "Integral sliding mode design for robust filtering and control of linear stochastic time-delay systems," *Int. J. Robust Nonlinear Control*, vol. 15, pp. 407–421, 2005.

[16] D. G. Luenberger, *Optimization by Vector Space Methods*. New York: John Wiley & Sons, Inc., 1969.

[17] J. C. Doyle, P. P. Khargonekar, and B. A. Francis, "State-space solutions to $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control problems," *IEEE Trans. Automat. Contr.*, vol. 34, pp. 831–847, Aug. 1989.

[18] A. J. van der Schaft, "$\mathcal{L}_2$-gain analysis of nonlinear systems and nonlinear state feedback $\mathcal{H}_\infty$ control," *IEEE Trans. Automat. Contr.*, vol. 37, pp. 770–784, June 1992.

[19] A. Isidori and A. Astolfi, "Disturbance attenuation and $\mathcal{H}_\infty$-control via measurement feedback in nonlinear systems," *IEEE Trans. Automatic. Contr.*, vol. 37, pp. 1283–1293, Sept. 1992.