Proceedings of the
44th IEEE Conference on Decision and Control, and
the European Control Conference 2005
Seville, Spain, December 12-15, 2005

On the differential/difference representation of sampled dynamics

S. Monaco * **and D. Normand-Cyrot** **

* Dipartimento di Informatica e Sistemistica, Università di Roma "La Sapienza",
Via Eudossiana 18, 00184 Rome, Italy.
** Laboratoire des Signaux et Systèmes, CNRS-ESE,
Plateau de Moulon, 91190 Gif-sur-Yvette, France.

Abstract: In this paper, the differential/difference representation (DDR) of an input-affine dynamics under sampling is characterized. Making use of combinatoric identities and formal calculus techniques, it is shown how to compute the sampled dynamics at any desired degree of approximation.

Keywords: Nonlinear systems, sampled systems, formal series expansions.

I. INTRODUCTION

As well known, given a controlled dynamics described by a forced differential equation and assuming the control constant over time intervals of amplitude δ, the solutions at times $t = k\delta$ for $k \geq 0$ satisfy a first order difference equation; such a difference equation defines the sampled equivalent model. The sampled equivalent model of a continuous-time linear dynamics is easily computed and maintains linearity so recovering a discrete-time linear dynamics [2]. The situation is more difficult in a nonlinear context even in presence of "small" nonlinearities ([8],[9]).

The computation of the sampled equivalent model as well as the study of its properties in relation with those of the continuous-time dynamics are widely investigated in the systems and control literature due to their high impact in digital control ([5],[12],[14],[7]). For dynamics generally nonlinear, only approximated solutions of the sampled equivalent model can be computed. These solutions correspond to the truncation of the Taylor-type series expansions describing the solutions of the differential equations defining the continuous-time dynamics.

In [11], an alternative to the usual repesentation of nonlinear controlled discrete-time dynamics in the form of maps was proposed. Such a representation, referred to as the differential/difference representation (DDR) of discrete-time dynamics, puts in light a set of vector fields suitable for characterizing the dynamical behaviour. In this framework, several classical control properties admit elegant geometric formulations and the analysis of the effects of sampling is facilitated. Given a discrete-time dynamics smoothly depending on the control, the basic idea is to describe the evolution at each step by means of a difference equation parametrized by a fixed control value together with a differential equation which models the variations with respect to the fixed control value. It results that the DDR is in fact composed with a set of differential equations with initial conditions fixed at each step by the discrete evolution. In this sense, a link can be established with hybrid systems representation.

Open questions about the DDR remain its practical efficiency for modeling real phenomena and its existence when starting from discrete-time dynamics described in the form of maps. However, when dealing with sampled dynamics, the first question does not apply and due to the invertibility of the flow associated with a differential equation, the DDR of the sampled equivalent model always exists. Moreover, as detailed in the sequel, because combinatoric expansions describe the sampled equivalent model, approximated solutions can easily be defined and computed through formal computational tools. The approach here proposed so provides efficient tools for explicit computation and, at the same time, makes easier the understanding of the properties of the sampled dynamics. The paper treats single-input-affine dynamics but the results can be generalized to multi-input and more general differential dynamics. Interpreating multirate sampling as a special multi-input case, the results can be used in multirate digital control [12]. Partial related results can be found in [10],[11].

The paper is organized as follows. Section 2 is concerned with the two equivalent representations of a single-input-affine continuous-time dynamics under sampling. The differential/ difference representation is specified in terms of computable formal series expansions in section 3. Section 4 develops an academic example. Some insights about the generalization to the multi-input case and to generally nonlinear dynamics are given in section 5.

Some notations

Throughout the paper $x \in R^n$, $t \in R$, $u \in \mathcal{U}$, a neighborhood of 0 in R. All the objects, maps, vector fields, control systems are analytic on their domains of definition, that is infinitely differentiable and admitting convergent Taylor series expansions. The vector fields are complete, that is the associated flow is defined at any time and for any initial condition. Given a generic map on R^n, its evaluation at a point x is denoted either by "(x)" or "$\big|_x$". Given a function $\lambda : R^n \rightarrow R$ and a vector field θ on R^n, the differential (or Lie derivative) operator L_θ acts on λ as $L_\theta \lambda := \frac{\partial \lambda}{\partial x}\theta$ so expressing the Lie derivative of λ along θ. The iterative action of L_θ gives for $k > 0$, $L_\theta^k \lambda := \frac{\partial L_\theta^{k-1}\lambda}{\partial x}\theta$ with $L_\theta^k := L_\theta \circ \cdots \circ L_\theta$; $k-$times and $L_\theta^0 = 1$, the identity operator. Given another vector field σ on $\mathcal{X}$, $[\theta, \sigma]$ denotes the Lie bracket of θ and σ; $L_{[\theta,\sigma]} = (L_\theta \circ L_\sigma - L_\sigma \circ L_\theta)$; we note $ad_\theta^0(\sigma) := \sigma$, and for $k \geq 0$, $ad_\theta^{k+1}(\sigma) := [\theta, ad_\theta^k(\sigma)]$. Analogously, we note $e^\theta = e^{L_\theta} = 1 + \sum_{k\geq 1} \frac{L_\theta^k}{k!}$. Denoting by I_n the identity function on R^n, the following property holds true $e^\theta \lambda\big|_x = \lambda(x) + \sum_{k\geq 1} \frac{1}{k!} L_\theta^k \lambda(x) = \lambda(e^\theta I_n\big|_x)$. Finally, given a family of formal variables ζ_i and a formal law of composition "$\zeta_{i_1} \ldots \zeta_{i_m}$" with neutral element 1, the *shuffle product*, [15], denoted by "$\sqcup\!\sqcup$", is defined in a recursive way on objects of increasing length

$$1 \sqcup\!\sqcup \zeta_i \;=\; \zeta_i \sqcup\!\sqcup 1 = \zeta_i, \quad \zeta_i \sqcup\!\sqcup \zeta_j = \zeta_j \sqcup\!\sqcup \zeta_i = \zeta_i.\zeta_j + \zeta_j.\zeta_i$$

and iteratively

$$\begin{aligned} \zeta_{i_1} \ldots \zeta_{i_m} \sqcup\!\sqcup \zeta_{j_1} \ldots \zeta_{j_p} \;&:=\; \zeta_{i_1}(\zeta_{i_2} \ldots \zeta_{i_m} \sqcup\!\sqcup \zeta_{j_1} \ldots \zeta_{j_p}) \\ &+\; \zeta_{j_1}(\zeta_{i_1} \ldots \zeta_{i_m} \sqcup\!\sqcup \zeta_{j_2} \ldots \zeta_{j_p}). \end{aligned}$$

We note that the shuffle product formally reproduces the rule of integration by parts.

II. CONTINUOUS-TIME DYNAMICS UNDER SAMPLING

Let us preliminarily recall how the DDR is defined in [11] in the context of discrete-time dynamics. We have

Proposition 2.1: The first order difference equation

$$x_{k+1} = F(x_k, u_k) \tag{1}$$

and the two differential/difference equations below

$$\begin{aligned} x^+ &= F(x) && (2)\\ \frac{dx^+(u)}{du} &= G(x^+(u), u); \quad x^+(0) = x^+ && (3) \end{aligned}$$

with

$$\begin{aligned} F(x) &:= F(x, 0) && (4)\\ G(F(x,u), u) &:= \frac{\partial F(x,u)}{\partial u}. && (5) \end{aligned}$$

describe the same discrete-time dynamics provided the existence of $G(x, u)$ satisfying (5) is ensured.

Proof: The integration of (3) with respect to u between 0 and u_k with initial condition specified by (2), $x^+(0) = F(x_k)$ restitutes (1); i.e.

$$x_{k+1} = x^+(0) + \int_0^{u_k} G(x^+(v), v)dv = F(x_k, u_k). \tag{6}$$

On the other hand, (4) and (5) specify the DDR starting from (1). ■

In the above proposition, the notation $x^+(u)$ specifies that $x^+(u)$ is a curve in R^n parameterized by u. More precisely, for a given $x \in R^n$, $x^+(u)$ is the trajectory in R^n, parameterized by u, which passes through $x^+(0) = x^+ = F(x)$.

Let us now go back to the problem of sampling. Let Σ be a *single-input-affine continuous-time dynamics*

$$\dot{x}(t) \;=\; f(x(t)) + u(t)g(x(t)) \tag{7}$$

and assume the control variable $u(t)$ constant over small time intervals of amplitude $\delta > 0$; let u_k be its value over $[k\delta, (k+1)\delta[$ and let x_k be the value of $x(t)$ at time $t = k\delta$. The sampled equivalent dynamics to (7) is defined so that its state evolution coincides with that of (7) at the sampling instants for any $x_0 = x(t = 0)$.

A. Two equivalent representations

Two representations of the sampled equivalent dynamics can be given, either in the usual form of a map or as a DDR. The map corresponds to express the solution to (7) at $t = (k+1)\delta$ under $u(t) = u_k$, with initial condition $x(t = k\delta) = x_k$ so obtaining

$$x_{k+1} = F^\delta(x_k, u_k) := e^{\delta(f+u_k g)} I_n|_{x_k} \tag{8}$$

with $F^\delta(., u) = F_0^\delta(.) + \sum_{p\geq 1} \frac{u^p}{p!} F_p^\delta(.)$. The F_p^δ's can be iteratively computed through Poincaré integro-differential formulae based on the expansion of the flow associated with (7). More precisely, integrating the successive kernels of the Volterra series expansion ([1],[3],[8],[16],[6]) characterizing over $[0, \delta[$ the solution to (7) when $u = u_k$ and setting $F_0^\delta(x) = F^\delta(x)$, one has for $p \geq 1$

$$F_p^\delta(x) = p \int_0^\delta e^{(\delta - s)f} L_g F_{p-1}^s(x) ds. \tag{9}$$

Regarding the DDR, we note that the invertibility of $F^\delta(x, u)$ in (8) for $u \in \mathcal{U}$ is a consequence of the the invertibility of $F^\delta(x) = e^{\delta f}(x)$ for small δ; $(F^\delta)^{-1}(x) = e^{-\delta f}(x)$. The existence of $G^\delta(x, u)$ satisfying (5) is thus guaranteed. Setting

$$\begin{aligned} x^+ &= F^\delta(x) && (10)\\ \frac{dx^+(u)}{du} &= G^\delta(x^+(u), u); \quad x^+(0) = x^+ && (11) \end{aligned}$$

with

$$\begin{aligned} F^\delta(x) &:= F^\delta(x, 0) && (12)\\ G^\delta(x, u) &:= \frac{\partial F^\delta(x,u)}{\partial u}\Big|_{x = F^{-\delta}(x,u)}. && (13) \end{aligned}$$

(8) and (10-11) are two equivalent representations of (7) under sampling.

Combinatoric relations between these two representations stated in [13] in the purely discrete-time context, are herafter detailed in the sampled context. To do so, expanding $G^\delta(., u)$ in (13) in powers of u

$$G^\delta(., u) = G_1^\delta + \sum_{p\geq 1} \frac{u^p}{p!} G_{p+1}^\delta \tag{14}$$

we define the vector fields $G_{p+1}^\delta(.) := \frac{\partial^p G^\delta(.,u)}{\partial u^p}|_{u=0}$. The theorem below details $F^\delta(., u)$ in (8) in terms of the $G_p^\delta(.)$. We have

Theorem 2.1:

$$F^\delta(x,u) = e^{u\mathcal{G}^\delta(.,u)} I_n \Big|_{F^\delta(x)} \tag{15}$$

where the exponent $u\mathcal{G}^\delta(.,u)$ is defined by the Lie series expansion

$$u\mathcal{G}^\delta(.,u) = \sum_{p\geq 1} u^p B_p(G_1^\delta,\ldots,G_p^\delta) \tag{16}$$

where $B_p(G_1^\delta,\ldots,G_p^\delta)$ stands for an homogeneous Lie polynomial of degree p in its arguments when G_p^δ is by convention of degree p. (16) can be computed according to the identity

$$\begin{aligned}\frac{\partial u\mathcal{G}^\delta(.,u)}{\partial u} &= \mathcal{Z}(-ad_{u\mathcal{G}^\delta(.,u)})G^\delta(.,u)\\ &= \sum_{p\geq 0}\frac{(-1)^p b_p}{p!} ad^p_{u\mathcal{G}^\delta(.,u)} G^\delta(.,u).\end{aligned}$$

The formal series $\mathcal{Z}(\zeta)$ is described by its expansion

$$\mathcal{Z}(\zeta) = \frac{\zeta}{e^\zeta - 1} = \sum_{p\geq 0}\frac{b_p}{p!}\zeta^p \tag{17}$$

with the Bernoulli numbers b_i as coefficients: $b_0 = 1$, $b_1 = -1/2$, $b_2 = 1/6$, $b_{2k+1} = 0$ for $k > 0$, $b_4 = -1/30$, $b_6 = 1/42$, $b_8 = -1/30$.
For the first terms in (16) we have

$$\begin{aligned}B_1 &= G_1^\delta; B_2 = \frac{1}{2!}G_2^\delta; B_3 = \frac{1}{3!}(G_3^\delta + 1/2[G_1^\delta, G_2^\delta])\\ B_4 &= \frac{1}{4!}(G_4^\delta + [G_1^\delta, G_3^\delta])\\ B_5 &= \frac{1}{5!}(G_5^\delta + 3/2[G_1^\delta, G_4^\delta] + [G_2^\delta, G_3^\delta] + 1/6[G_1^\delta[G_1^\delta, G_3^\delta]] -\\ &\quad 1/2[G_2^\delta[G_1^\delta, G_2^\delta]] - 1/6[G_1^\delta[G_1^\delta[G_1^\delta, G_2^\delta]]]).\end{aligned}$$

From (15) and (16), the F_p^δ's in (9) can now be expressed in terms of the G_p^δ's. We have

Proposition 2.2: [11]Setting $P_1^\delta := L_{G_1^\delta}$, one has for $p \geq 2$

$$F_p^\delta(x) = P_p^\delta I_n \Big|_{F^\delta(x)}$$

where the formal operator P_p^δ satisfies the decomposition

$$P_p^\delta = \sum_{l=1}^{p}\sum_{p_1+\ldots+p_l=p} a(p_1,\ldots,p_l) L_{G^\delta_{p_1}}\circ\ldots\circ L_{G^\delta_{p_l}}$$

with $1 \leq p_j \leq p$ and real coefficients $a(p_1,\ldots,p_l)$ computed iteratively according to the relation

$$P_{p+1}^\delta := P_1^\delta \circ P_p^\delta + (P_p^\delta)^{(+)}.$$

The formal rule $(.)^{(+)}$ which formally reproduces the rule of derivation of a composition of non commuting operators, is defined below setting $L_{(G_p^\delta)^{(+)}} := L_{G^\delta_{p+1}}$ and

$$(L_{G^\delta_{p_1}}\circ\ldots\circ L_{G^\delta_{p_l}})^{(+)} = \sum_{q=1}^{l} L_{G^\delta_{p_1}}\circ\ldots\circ L_{(G^\delta_{p_q})^{(+)}}\circ\ldots\circ L_{G^\delta_{p_l}}.$$

For the first terms one has

$$\begin{aligned}F_1^\delta(x) &= L_{G_1^\delta} I_n|_{F^\delta(x)}\\ F_2^\delta(x) &= (L_{G_1^\delta}\circ L_{G_1^\delta} + L_{(G_1^\delta)^{(+)}})I_n|_{F^\delta(x)}\\ &= (L^2_{G_1^\delta} + L_{G_2^\delta})I_n|_{F^\delta(x)}\\ F_3^\delta(x) &= (L^3_{G_1^\delta} + 2L_{G_1^\delta}\circ L_{G_2^\delta} + L_{G_1^\delta}\circ L_{G_2^\delta} + L_{G_3^\delta})I_n|_{F^\delta(x)}.\end{aligned}$$

III. The differential/difference representation

In this section we further specify $G^\delta(.,u)$ in (13). To do so, let us now consider the expansion of the G_p^δ in (14) in powers of δ, so defining the vector fields $G_{p,i}$; i.e.

$$G_p^\delta := \sum_{i\geq 1}\frac{\delta^i}{i!}G_{p,i} \tag{18}$$

and set by definition $X_i := (-1)^i ad_f^i(g)$.
Two characterizations can be given, the first one, more combinatoric, makes reference to the formal series $\mathcal{Z}$ again, while the second one is integro-differential.
Taking into account that the formal inverse of $\mathcal{Z}(\zeta)$ in (17), satifies the formal series expansion

$$\mathcal{Z}^{-1}(\zeta) = \int_0^1 e^{s\zeta}ds = \frac{e^\zeta - 1}{\zeta} = 1 + \sum_{i\geq 1}\frac{\zeta^i}{(i+1)!} \tag{19}$$

the next Theorem sets some equalities at the basis of the two proposed solutions.

Theorem 3.1: $G^\delta(.,u)$ satisfies both the series expansion

$$\begin{aligned}G^\delta(.,u) &= \int_0^\delta e^{-sad_{f+ug}}g ds = \delta g + \sum_{i\geq 1}\frac{(-1)^i\delta^{i+1}}{(i+1)!}ad^i_{f+ug}g\\ &= \mathcal{Z}^{-1}(-ad_{\delta f+\delta ug})\delta g\end{aligned} \tag{20}$$

and the integro-differential formula

$$\frac{\partial G^\delta(.,u)}{\partial u} = \int_0^\delta [\frac{\partial G^s(.,u)}{\partial s}, G^s(.,u)]ds. \tag{21}$$

Proof: As far as (20) is concerned, we note that from (9) we compute

$$\frac{\partial F^\delta(.,u)}{\partial u} = \int_0^\delta e^{(\delta-s)(f+ug)}\circ L_g\circ e^{s(f+ug)} I_n ds$$

and thus

$$\begin{aligned}e^{-\delta(f+ug)}\circ\frac{\partial F^\delta(.,u)}{\partial u} &= \int_0^\delta e^{-s(f+ug)}\circ L_g\circ e^{s(f+ug)}I_n ds\\ &= \int_0^\delta e^{-sad_{f+ug}}g ds\end{aligned}$$

which gives (20) because of (19).

As far as (21) is concerned, we deduce from the definition of $G^\delta(.,u)$ the equality of formal operators, $\frac{\partial e^{s(f+ug)}}{\partial u} = e^{s(f+ug)}\circ L_{G^s(.,u)}$ and thus $\frac{\partial e^{-s(f+ug)}}{\partial u} = -L_{G^s(.,u)}\circ e^{-s(f+ug)}$. It is now a matter of computations to recover (21) when applying these formal equalities to $G^\delta(.,u) = \int_0^\delta e^{-s(f+ug)}\circ L_g\circ e^{s(f+ug)}I_n ds$ with $\frac{\partial G^s(.,u)}{\partial s} = e^{-sad_{f+ug}}g = e^{-s(f+ug)}\circ L_g\circ e^{s(f+ug)}I_n$. ■

A. The combinatoric solution

The combinatoric solution is deduced from (20).

Proposition 3.1: G_1^δ in (14) is given by

$$G_1^\delta = \int_0^\delta e^{-sad_f} g ds = \mathcal{Z}^{-1}(-ad_{\delta f})\delta g := \sum_{i\geq 1} \frac{\delta^i}{i!} X_{i-1} \quad (22)$$

and the $G_{p+1,i}$'s can be recursively computed. For $p \geq 1$ and $i \geq p+1$, we have

$$G_{p+1,i+1} = -ad_f G_{p+1,i} - pad_g G_{p,i} \quad (23)$$

with $G_{p+1,i\leq p+1} = 0$ and $G_{p+1,p+2} = (-1)^p p! ad_{X_0}^p X_1$.

Proof: Setting $u = 0$ in (20), we get (22). The recurrent relation (23) is deduced from the shuffle product definition. ■

It is now possible from these results to define and compute approximated DDR through truncations of the series solutions at a fixed power in u and δ. More precisely, the G_p^δ's can be computed from the continuous-time dynamics accordingly to the next proposition.

Proposition 3.2: We have for $p \geq 1$

$$\begin{aligned} G_{p+1}^\delta &= p! \frac{1 + \sum_{i=1}^p \frac{(-1)^i \delta^i}{i!} ad_f^i - e^{-\delta ad_f}}{ad_f^{p+1}} \sqcup\!\sqcup\, ad_g^p g \\ &= p! \sum_{i\geq p+1} \frac{(-1)^i \delta^{i+1}}{(i+1)!} ad_f^{i-p} \sqcup\!\sqcup\, ad_g^p g \quad (24) \\ &= (-1)^p p! \int_0^\delta \int_0^{s_1} \ldots \int_0^{s_p} e^{-\tau ad_f} \sqcup\!\sqcup\, ad_g^p g d\tau ds_1 \ldots ds_p. \end{aligned}$$

Proof: (24) is obtained performing successive derivatives with respect to u of $G^\delta(.,u)$ in (20). Setting formally $\zeta_1 + u\zeta_2 = -\delta ad_f - u\delta ad_g$, we get (24) because $\frac{d(\zeta_1+u\zeta_2)^i}{du} = \zeta_1^{i-1} \sqcup\!\sqcup\, \zeta_2$ and $\frac{d^{p-1}(\zeta_1+u\zeta_2)^i}{du^{p-1}} = (p-1)! \zeta_1^{i-p+1} \sqcup\!\sqcup\, \zeta_2^{p-1}$. ■

Detailing the computations for the first terms, we have

$$\begin{aligned} G_2^\delta &= \frac{1 - \delta ad_f - e^{-\delta ad_f}}{ad_f^2} \sqcup\!\sqcup\, ad_g g \\ &= \sum_{i\geq 2} \frac{(-1)^i \delta^{i+1}}{(i+1)!} ad_f^{i-1} \sqcup\!\sqcup\, ad_g g \\ &= -\int_0^\delta \int_0^{s_1} e^{-\tau ad_f} \sqcup\!\sqcup\, ad_g g d\tau ds_1 \\ G_3^\delta &= 2 \frac{1 - \delta ad_f + \frac{\delta^2}{2!} ad_f^2 - e^{-\delta ad_f}}{ad_f^3} \sqcup\!\sqcup\, ad_g^2 g \\ &= 2 \sum_{i\geq 3} \frac{(-1)^i \delta^{i+1}}{(i+1)!} ad_f^{i-2} \sqcup\!\sqcup\, ad_g^2 g \\ &= 2 \int_0^\delta \int_0^{s_1} \int_0^{s_2} e^{-\tau ad_f} \sqcup\!\sqcup\, ad_g^2 g d\tau ds_1 ds_2. \end{aligned}$$

B. The integro-differential approach

An alternative solution can be obtained arguing as follows. Each G_p^δ for $p \geq 2$ can be expressed in terms of the X_p's and their Lie brackets applying the integro-differential formula (21). More precisely, we have

Proposition 3.3: G_2^δ is given by

$$G_2^\delta = \sum_{i\geq 2} \frac{\delta^{i+1}}{(i+1)!} \sum_{k=0}^{i-1} C_i^k [X_k, X_{i-k-1}] \quad (25)$$

where $C_i^k := \frac{i!}{k!(i-k)!}$ and the $G_{p+1,i+1}$'s can be recursively computed as follows for $p \geq 2$ and $i \geq p+1$

$$G_{p+1,i+1} = (G_{p,i+1})^{(+)} \quad (26)$$

with $G_{2,i+1} = \sum_{k=0}^{i-1} C_i^k [G_{1,k+1}, G_{1,i-k}]$.

Proof: Computing (21) at $u = 0$, we get $\frac{\partial G_2^\delta}{\partial \delta} = [\frac{\partial G_1^\delta}{\partial \delta}, G_1^\delta]$ from which we deduce (25). To prove (26), we note that the law $(.)^{(+)}$ formally reproduces the operation of derivation and that $G_{p+1,i+1}$ is deduced from $G_{p,i+1}$ through derivation with respect to u at $u = 0$. ■

Computing the first terms, we get

$$\begin{aligned} G_2^\delta &= -\frac{\delta^3}{3!}[X_0, X_1] - \frac{2\delta^4}{4!}[X_0, X_2] \\ &- \frac{3\delta^5}{5!}[X_0, X_3] - \frac{2\delta^5}{5!}[X_1, X_2] + \ldots. \end{aligned}$$

Similarly, from (26), we compute

$$\begin{aligned} & G_{3,i+1} = (G_{2,i+1})^{(+)}; \quad i \geq 3 \\ = & \sum_{k=0}^{i-1} C_i^k ([G_{1,k+1}, G_{1,i-k}])^{(+)} \\ = & \sum_{k=0}^{i-1} C_i^k ([G_{2,k+1}, G_{1,i-k}] + [G_{1,k+1}, G_{2,i-k}]) \\ & G_{4,i+1} = (G_{3,i+1})^{(+)}; \quad i \geq 4 \\ = & \sum_{k=0}^{i-1} C_i^k ([G_{2,k+1}, G_{1,i-k}] + 2[G_{2,k+1}, G_{2,i-k}] \\ & + [G_{1,k+1}, G_{3,i-k}]) \end{aligned}$$

The following comments further specify the G_p^δ.

- From Proposition 3.2, it is a matter of computation to verify

$$G_2^\delta = \sum_{i\geq 2} \frac{\delta^{i+1}}{(i+1)!} \sum_{p=0}^{i-2} (-1)^{p+1} ad_f^p \Big[X_0, X_{i-p-1}\Big]. \quad (27)$$

Comparing (27) with (25), we get the equality of formal series below which, more in general, specifies the action of the operator ad_f^p over any Lie bracket of the type $[X_0, X_i]$; we have

$$\sum_{k=0}^{i-1} C_i^k \Big[X_k, X_{i-k-1}\Big] = \sum_{k=0}^{i-2} (-1)^{k+1} ad_f^k \Big[X_0, X_{i-k-1}\Big]; i \geq 2.$$

For the first terms, we have

$$\begin{aligned}
[X_0,X_2] &+ ad_f[X_0,X_1]=0\\
2[X_0,X_3] &+ 2[X_1,X_2]+ad_f[X_0,X_2]-ad_f^2[X_0,X_1]=0\\
3[X_0,X_4] &+ 5[X_1,X_3]+ad_f[X_0,X_3]\\
&- ad_f^2[X_0,X_2]+ad_f^3[X_0,X_1]=0;\ldots
\end{aligned}$$

• Working out (26) and successive substitutions, we can express each G_p^δ, $p\geq 3$, as an homogeneous Lie polynomial of degree $p-1$ in the X_i. We obtain $G_3^\delta =$

$$\sum_{i\geq 2}\frac{\delta^{i+1}}{(i+1)!}\sum_{k=0}^{i-1}\sum_{l=0}^{i-k-1}(C_i^k-C_i^{i-k-1})C_{i-k-1}^l[X_k[X_l,X_{i-k-l-2}]]$$

$$=\frac{2\delta^4}{4!}[X_0[X_0,X_1]]+\frac{6\delta^5}{5!}[X_0[X_0,X_2]]+\frac{2\delta^5}{5!}[X_1[X_0,X_1]]....$$

C. The canonical set of vector fields

In both discrete-time and sampled contexts, the transport of the vector fields G_p^δ along the drift term F^δ, denoted by $F_*^\delta G_p^\delta$ play a fundamental role [11]. Let us recall that by definition they satisfy the equality

$$F_*^\delta G_p^\delta=\frac{\partial F^\delta(x)}{\partial x}G_p^\delta(x):=G_p^\delta(F^\delta(x)).$$

Identically, the transport of G_p^δ along F^δ q-times is defined as $F_*^{q\delta}G_p^\delta$ because $F^\delta\circ\ldots\circ F^\delta=(F^\delta)^q=F^{q\delta}$.

Proposition 3.4: The relations below hold for $q\geq 1$

$$F_*^{q\delta}(.,u)G^\delta(.,u)=\int_{q\delta}^{(q+1)\delta}e^{-s\,ad_{f+ug}}g\,ds \tag{28}$$

$$F_*^{q\delta}(.,u)G^\delta(.,u)=G^{(q+1)\delta}(.,u)-G^{q\delta}(.,u). \tag{29}$$

Proof: Due to the invertibility of F^δ, $F_*^{q\delta}(.,u)G^\delta(.,u)$ can be rewritten as $e^{-q\delta ad_{f+ug}}G^\delta(.,u)$ with $G^\delta(.,u)=\int_0^\delta e^{-s\,ad_{f+ug}}g\,ds$. Thanks to an elementary change of variables, we get (28) and (29). ∎

Setting $u=0$ in (28-29), we get (30) and (31) and then, by definition of the transport, we get (32) below.

Proposition 3.5: In terms of the X_p, we have

$$F_*^{q\delta}G_1^\delta=\sum_{p\geq 0}\int_{q\delta}^{(q+1)\delta}\frac{s^p}{p!}X_p\,ds=\sum_{p\geq 0}\left[\frac{s^{p+1}}{(p+1)!}\right]_{q\delta}^{(q+1)\delta}X_p \tag{30}$$

$$F_*^{q\delta}G_1^\delta=G_1^{(q+1)\delta}-G_1^{q\delta} \tag{31}$$

and for $p\geq 2$

$$F_*^{q\delta}G_q^\delta(.)=e^{-q\delta ad_f}G_p^\delta. \tag{32}$$

IV. An example

Let us consider the single input-affine continuous-time dynamics

$$\dot x_1=x_2+ax_3^2;\quad \dot x_2=x_3;\quad \dot x_3=u$$

with $f=(x_2+ax_3^2,x_3,0)^T;\quad X_0=g=(0,0,1)^T;$

$X_1=-ad_fg=(2ax_3,1,0)^T;X_2=ad_f^2g=(1,0,0)^T;X_{i\geq 3}=0.$

The sampled equivalent dynamics is described either by the nonlinear difference equations

$$\begin{aligned}
x_1(k+1) &= x_1(k)+\delta(x_2(k)+ax_3^2(k))\\
&\quad+\frac{\delta^2}{2!}(x_3(k)+2ax_3(k)u(k))+\frac{\delta^3}{3!}(u(k)+2au^2(k))\\
x_2(k+1) &= x_2(k)+\delta x_3(k)+\frac{\delta^2}{2!}u(k)\\
x_3(k+1) &= x_3(k)+\delta u(k)
\end{aligned}$$

or its DDR

$$\begin{aligned}
x_1^+ &= x_1+\delta(x_2+ax_3^2)+\frac{\delta^2}{2!}x_3\\
x_2^+ &= x_2+\delta x_3;\quad x_3^+=x_3\\
\frac{\partial x_1^+(u)}{\partial u} &= \delta^2ax_3^+(u)+\frac{\delta^3}{3!}(1-2ua)\\
\frac{\partial x_2^+(u)}{\partial u} &= \frac{\delta^2}{2!};\quad \frac{\partial x_3^+(u)}{\partial u}=\delta.
\end{aligned}$$

One easily verifies that

$$\begin{aligned}
F^\delta(x) &= (1+\delta L_f+\frac{\delta^2}{2!}L_f^2+\frac{\delta^3}{3!}L_f^3)I_n\Big|_x\\
G_1^\delta(x) &= \delta X_0+\frac{\delta^2}{2!}X_1+\frac{\delta^3}{3!}X_2\\
G_2^\delta(x) &= -\frac{\delta^3}{3!}[X_0,X_1];\quad G_{i\geq 3}^\delta(x)=0\\
F_*^\delta G_1^\delta(x) &= \delta X_0+\frac{3\delta^2}{2!}X_1+\frac{7\delta^3}{3!}X_2=G_1^{2\delta}(x)-G_1^\delta(x)\\
F_1^\delta(x) &= \int_0^\delta e^{(\delta-s)f}\circ L_g\circ F^s(x)ds=\int_0^\delta L_g\circ F^s(x)ds\\
&= G_1^\delta(e^{\delta f}(x))\\
F_2^\delta(x) &= 2\int_0^\delta e^{(\delta-s)f}\circ L_g\circ F_1^s(x)ds=2\int_0^\delta L_g\circ F_1^s(x)ds\\
&= (L_{G_2^\delta}+L_{G_1^\delta}\circ L_{G_1^\delta})I_n\Big|_{e^{\delta f}(x)}.
\end{aligned}$$

The sampled equivalent models are of finite degree both in u and δ.

V. About some extensions

A. The multi-input case

Consider the m-input dynamics

$$\begin{aligned}
\dot x(t) &= f(x(t))+\underline{u}(t)g(x(t))\\
&= f(x(t))+\sum_{i=1}^m u_i(t)g_i(x(t)).
\end{aligned}\tag{33}$$

Under piecewice constant controls, the sampled equivalent is described either by the mapping $x \rightarrow F^\delta(x, \underline{u}) = e^{\delta(f+\underline{u}g)} I_n\Big|_x$ or its DDR. Setting, for each u_i

$$ {}_iG^\delta(., \underline{u}) \quad := \quad \frac{\partial F^\delta(x, \underline{u})}{\partial u_i}\Big|_{x=F^{-\delta}(x,\underline{u})} \tag{34} $$

we can define the DDR as previously, making reference to partial differential equations with respect to each u_i; i.e.

$$ x^+ \quad = \quad F^\delta(x); \quad x^+(0) = x^+ \tag{35} $$

$$ \frac{\partial x^+(\underline{u})}{\partial u_i} \quad = \quad {}_iG^\delta(x^+(\underline{u}), \underline{u}); \quad i = (1, ..., m) \tag{36} $$

with $F^\delta := e^{\delta f} I_n$ and ${}_iG^\delta(., \underline{u})$ described by (34). The results set in Theorem 3.1 can easily be generalized so getting

Proposition 5.1: ${}_iG^\delta(., \underline{u})$ satisfies for $i = (1, ..., m)$

$$ \begin{aligned} {}_iG^\delta(., \underline{u}) \quad &= \quad \int_0^\delta e^{-sad_{(f+\underline{u}g)}} g_i ds \\ &= \quad \delta g_i + \sum_{l \geq 1} \frac{(-1)^l \delta^{l+1}}{(l+1)!} ad^l_{(f+\underline{u}g)} g_i \\ &= \quad \mathcal{Z}^{-1}(-ad_{\delta(f+\underline{u}g)}) \delta g_i \end{aligned} $$

and

$$ \frac{\partial_i G^\delta(., \underline{u})}{\partial u_j} \quad = \quad \int_0^\delta \Big[\frac{\partial_i G^s(., \underline{u})}{\partial s}, {}_jG^\delta(., \underline{u}) \Big] ds. $$

In this multi-input case, we deduce from Proposition 5.1 the equality (37) below which corresponds to the so-called *compatibility conditions* guaranteeing the integrability of the partial-derivatives equations (34). Under these conditions, directly satisfied under sampling, a discrete-time dynamics can always be represented as a DDR.

Corollary 5.1:

$$ \Big[{}_iG^\delta(., \underline{u}), {}_jG^\delta(., \underline{u}) \Big] \quad = \quad \frac{\partial_j G^\delta(., \underline{u})}{\partial u_j} - \frac{\partial_i G^\delta(., \underline{u})}{\partial u_i}. \tag{37} $$

Proof: The proof works out by equating the second order partial derivatives $\frac{\partial^2 e^{\delta(f+\underline{u}g)}}{\partial u_i \partial u_j}$ and $\frac{\partial^2 e^{\delta(f+\underline{u}g)}}{\partial u_j \partial u_i}$. ∎

B. The general case

Let a generally *nonlinear continuous-time dynamics*

$$ \dot{x}(t) \quad = \quad f^u(x(t)) = f_0(x(t)) + \sum_{i \geq 1} \frac{u^i(t)}{i!} f_i(x(t)) $$

where $u^i(t)$ indicates the control variable at power i. Its sampled equivalent is described either by a mapping $x \rightarrow F^\delta(x, u) = e^{\delta f^u} I_n\Big|_x$ or its DDR

$$ \begin{aligned} x^+ \quad &= \quad F^\delta(x); \quad x^+(0) = x^+ \\ \frac{dx^+(u)}{du} \quad &= \quad G^\delta(x^+(u), u) \end{aligned} $$

with $F^\delta(x) := e^{\delta f_0} I_n\Big|_x$; $G^\delta(x, u) := \frac{de^{\delta f^u} I_n}{du}\Big|_{x=e^{-\delta f^u}(x)}$. The results in Theorem 3.1 can still be generalized.

Proposition 5.2:

$$ G^\delta(., u) = \int_0^\delta e^{-sad_{f^u}} \frac{df^u}{du} I_n ds = \mathcal{Z}^{-1}(-ad_{\delta f^u}) \delta \frac{df^u}{du}. $$

and

$$ \frac{dG^\delta(., u)}{du} = \int_0^\delta [\frac{\partial G^s(., u)}{\partial s}, G^\delta(., u)] ds + \int_0^\delta e^{-sad_{f^u}} \frac{d^2 f^u(.)}{du^2} ds. $$

VI. Conclusion

In this work, the differential/difference representation of a continuous-time input-affine dynamics under sampling has been characterized in terms of combinatoric series. Formal expansions of these series solutions provide algorithmic techniques for computing approximated solutions more efficient in practice. It is shown that multi-input and generally nonlinear dynamics can be investigated according to the same lines.

References

[1] A.A. Agrachev and R.V. Gamkrelidze, *Exponential representations of flows and chronological calculus*, Math. USSR Sbornik, **35**, pp. 727–785, 1978.

[2] G. F. Franklin and J. D. Powell, *Digital Control of Dynamic Systems*, Reading, MA Addisov Wesley, 1977.

[3] R.V. Gamkrelidze, *Exponential representations solutions of ordinary differential equations*, Lect. Notes Math., **703**, 1979.

[4] W. Gröbner, *Serie di Lie e Loro Applicazioni*, Poliedro, Cremonese, Roma, 1973.

[5] L. Hou, A.N. Michel, and H. Ye, *Some qualitative properties of sampled-data systems*, IEEE Trans. AC, **42**, pp. 1721– 1725, 1997.

[6] M. Kawski and H.J. Sussmann, *Noncommutative power series and formal Lie algebraic techniques in control theory*, in Systems and Linear Algebra (U.Helmke and D. Pratzël Wolters, E. Zertz eds.), pp. 111–128, Studgart, 1997.

[7] H.K. Khalil, *Performance recovery under output feedback sampled-data stabilization of a class of nonlinear systems*, IEEE Trans. AC, **49**, pp. 2173–2185, 2004.

[8] K. Lesiak, A.J. Krener, *The existence and uniqueness of Volterra series for nonlinear systems*, IEEE Trans. AC **23**, pp. 1090–1095, 1978.

[9] S. Monaco, D. Normand-Cyrot, *On the sampling of a linear control system*, Proc. 24th IEEE-CDC, pp. 1457–1462, 1985.

[10] S. Monaco, D. Normand-Cyrot, *A combinatorial approach to the nonlinear sampling problem*, in Lect. Notes in Cont. and Info. Sci., **144**, (M.Thoma et A.Wymer, Eds.), Springer-Verlag, pp. 788–797, 1990.

[11] S. Monaco and D. Normand-Cyrot, *A unified representation for nonlinear discrete-time and sampled dynamics*, J. of Mathematical Systems, Estimation and C., pp. 101–103 (summary) 1995, **7**, pp. 477-503, 1997.

[12] S. Monaco and D. Normand-Cyrot, *Issues on nonlinear digital systems*, European Journal of Control, **7**, pp. 160–178, 2001.

[13] S. Monaco, D. Normand-Cyrot, C. Califano, *Exponential representations of Volterra-like expansions: an explicit computation of the exponent series*, Proc. 41st IEEE-CDC, pp. 2708–2713, 2002.

[14] D. Nesic, A.R. Teel, *A framework for stabilization of nonlinear sampled-data systems based on their approximate discrete-time models*, IEEE Trans. AC, **49**, pp.1103–1123, 2004.

[15] R. Ree, *Lie elements and an algebra associated with shuffles*, Ann. of Maths., 68, pp. 210–220, 1958.

[16] I.W. Sandberg, *Volterra-like expansions for solutions of nonlinear integral equations and nonlinear differential equations*, IEEE Trans. AC, **30**, pp. 68–77, 1983.