Proceedings of the
44th IEEE Conference on Decision and Control, and
the European Control Conference 2005
Seville, Spain, December 12-15, 2005

Stabilization of Underactuated 3D Pendulum Using Partial Angular Velocity Feedback

Nalin A. Chaturvedi and N. Harris McClamroch
Department of Aerospace Engineering
University of Michigan
Ann Arbor, MI 48109

Abstract— **Models for a 3D pendulum, consisting of a rigid body that is supported at a frictionless pivot, were introduced in a 2004 CDC paper [1]. A subsequent paper, in the 2005 ACC [2], developed stabilizing controllers for a fully actuated 3D pendulum assuming feedback of the reduced attitude and angular velocities. In the present paper, stabilizing controllers are developed for an underactuated, 3D pendulum using partial feedback of the angular velocities. We propose controllers and provide sufficient conditions for stabilization of the hanging equilibrium of an underactuated, 3D pendulum. A controller is proposed to stabilize the inverted equilibrium of the 3D pendulum, using partial angular velocity and reduced attitude feedback.**

I. Introduction

Pendulum models have provided a rich source of examples that have motivated and illustrated many recent developments in nonlinear dynamics and control. Much of the published research treats 1D planar pendulum models or 2D spherical pendulum models or some multi-body version of these. In a recent paper [1], we summarized a large part of this published research, emphasizing control design results. In addition, we introduced a new 3D pendulum model.

A related paper [2], obtained controllers for a 3D, rigid pendulum. Controllers were introduced that provide asymptotic stabilization of a reduced attitude equilibrium. The reduced attitude of the 3D pendulum is defined as the attitude or orientation of the 3D pendulum, modulo rotation about a vertical axis. Stabilization results are provided for the hanging equilibrium and for the inverted equilibrium in [2]. In [2], it was assumed that the 3D pendulum is fully actuated and the controllers were designed with full angular velocity and reduced attitude feedback.

The present paper continues our research on control of 3D pendula assuming partial angular velocity feedback and underactuation. The 3D pendulum is supported at a pivot that is assumed to be frictionless and inertially fixed. The location of its center of mass is distinct from the location of the pivot. Forces that arise from uniform and constant gravity act on the pendulum.

We follow the development and notation introduced in [1]. In particular, the formulation of the models depend on construction of a Euclidean coordinate frame fixed to the pendulum with origin at the pivot and an inertial Euclidean coordinate frame with origin at the pivot. We also assume that the inertial coordinate frame is selected so that the first two axes lie in a horizontal plane and the "positive" third axis points down. The property of conservation of angular momentum about the vertical axis, allows development of equations of motion of the 3D pendulum, in terms of the angular velocity vector of the rigid body and the reduced attitude vector of the rigid body.

This research has been supported in part by NSF under grant ECS-0140053 and grant ECS-0244977.

The problem of stabilization of a rigid body has been studied extensively in the literature; for example, see [3], [4]. However, these papers make the assumption of full angular velocity feedback and a fully actuated system. Although studies on stabilization of underactuated rigid bodies have appeared in the literature [5], [6], these provide only local stabilization results and hence they fail if the initial conditions do not lie sufficiently close to the equilibrium.

In the present paper, we remove these limitations. We provide asymptotic stabilization with at least a computable domain of attraction. In [5], [6] it was shown that the asymptotic stability of an underactuated 3D rigid pendulum required a fundamentally nonlinear analysis. These complications were shown to arise since the coupling was essentially nonlinear, and hence the proof involved a careful analysis of the nonlinear polynomial dynamics of the rigid body. The results presented are necessarily local.

In the present paper, we use Lyapunov functions and LaSalle's invariance principle to prove asymptotic stability of the hanging equilibrium for an underactuated, 3D pendulum using output-feedback. Finally, we prove almost global asymptotic stability of the inverted equilibrium of a 3D pendulum using partial angular velocity and reduced attitude feedback.

II. Control Problem Formulation

In this section, we introduce the 3D pendulum model. The model is given as

$$\begin{cases} J\dot{\omega} = J\omega \times \omega + mg\rho \times \Gamma + Bu, \\ \dot{\Gamma} = \Gamma \times \omega, \\ y = C\omega, \end{cases} \tag{1}$$

where $\omega \in \mathbb{R}^3$ represents the angular velocity of the rigid body, $\Gamma \in S^2$ is a unit vector in the direction of gravity in the body reference frame, ρ represents the constant vector from the pivot to the center of mass of the pendulum, in body frame coordinates, $u \in \mathbb{R}^3$ denotes the control input and y represents the angular velocity output. The 3D pendulum has two natural equilibria [1], namely, the hanging equilibrium $(0, \Gamma_h)$ and the inverted equilibrium $(0, \Gamma_i)$, where $\Gamma_h = \frac{\rho}{\|\rho\|}$ and $\Gamma_i = -\frac{\rho}{\|\rho\|}$.

Consider first, the problem of stabilization of the hanging equilibrium. If B and C are assumed to be the identity matrix, then equation (1) reduces to the model studied in [2]. The control problem for stabilization of the hanging equilibrium can then be classified into the following cases:

1) If both B and C are full rank, the control problem corresponds to the selection of a fully actuated, angular velocity feedback controller. This control problem was studied in [2].
2) If B is rank deficient and C is full rank, then the control problem corresponds to the selection of an underactuated controller with complete angular velocity feedback.
3) If B is full rank and C is rank deficient, then the control problem corresponds to the selection of a fully actuated controller with partial angular velocity feedback.
4) In the case that both B and C are rank deficient, the control problem corresponds to the selection of an underactuated controller using partial angular velocity feedback.

Our first objective is to find a controller $u = -\mathbf{U}(y)$ that asymptotically stabilizes the hanging equilibrium of the 3D pendulum for case (4). Substituting for the control law in (1), we obtain the closed loop equation

$$\begin{cases} J\dot{\omega} = J\omega \times \omega + mg\rho \times \Gamma - B\mathbf{U}(C\omega), \\ \dot{\Gamma} = \Gamma \times \omega. \end{cases} \tag{2}$$

Next, we treat the problem of asymptotic stabilization of the inverted equilibrium. In this paper, we limit our presentation to the case corresponding to complete actuation, so that B is full rank and assume Γ to be available for feedback. The controller in this case can be written as $u = -\mathbf{U}(y, \Gamma)$; substituting in (1), we obtain the closed-loop equation

$$\begin{cases} J\dot{\omega} = J\omega \times \omega + mg\rho \times \Gamma - B\mathbf{U}(C\omega, \Gamma), \\ \dot{\Gamma} = \Gamma \times \omega. \end{cases} \tag{3}$$

III. Stabilization of the Hanging Equilibrium

We first review prior results in [2], where we presented a family of controllers with angular velocity feedback only, that asymptotically stabilize the hanging equilibrium. In [2], we assumed

$$\alpha(\|\omega\|) \geq \omega^{\mathrm{T}} B\mathbf{U}(C\omega) \geq \epsilon_1 \|\omega\|^2, \quad \forall \omega \in \mathbb{R}^3, \tag{4}$$

where $\epsilon_1 > 0$ and $\alpha(\cdot)$ is a class-$\mathcal{K}$ function. Since B and C were assumed to be identity, $\mathbf{U}(\omega)$ was necessarily positive definite and decrescent, and hence the damping generated by the controller in [2] is *complete*. Note that the controller in this case corresponds to case (1).

In this section, we present weaker conditions which guarantee that the hanging equilibrium of (2) is asymptotically stable. We assume that $\mathbf{U} : \mathbb{R}^3 \mapsto \mathbb{R}^3$ is a smooth function satisfying

$$\alpha(\|\omega\|) \geq \omega^{\mathrm{T}} B\mathbf{U}(C\omega) \geq 0, \quad \forall\ \omega \in \mathbb{R}^3, \tag{5}$$

where $\alpha(\cdot)$ is a Class-$\mathcal{K}$ function,

$$\omega^{\mathrm{T}} B\mathbf{U}(C\omega) \geq \omega^{\mathrm{T}} P\omega, \tag{6}$$

in some open neighborhood $\mathcal{U}$ of the origin containing the set where $\mathbf{U}(\cdot)$ vanishes, and P is positive semidefinite and, thus, not necessarily full rank. We denote the term *rank deficiency* of the triple $(B, C, \mathbf{U})$, as the rank deficiency of the matrix P.

Since P is rank deficient, the task of designing a controller becomes more complicated since damping is restricted to a proper subspace in the tangent space at each point of the base manifold of the tangent bundle. Hence, we utilize the nonlinear coupling in the dynamics of the rigid body to *disperse* damping in all directions of the tangent space. Note that we require the positive semi-definiteness condition (6) to hold only in a neighborhood of the set where it vanishes, and not globally. Thus, the feedback controller can accommodate saturation effects.

In [5], the closed-loop model given in (2) was studied assuming that $B\mathbf{U}(C\omega) = P\omega$, where P is positive semidefinite and has a rank deficiency of one. Sufficient conditions for *local* asymptotic stability of the hanging equilibrium were provided. Briefly, the results in the paper state that if the hanging equilibrium Γ_h is neither collinear nor orthogonal to the one dimensional subspace generated by the kernel of the matrix P, then the hanging equilibrium is locally asymptotically stable.

In [5], the authors observed that conditions required for asymptotic stability could not be obtained through linearization, since asymptotic stability of the hanging equilibrium depends essentially on nonlinear coupling. The sufficient conditions for local asymptotic stability of the hanging equilibrium presented in [5] were based on a careful study of the nonlinear polynomial dynamics of the rigid body. However, these sufficient conditions are only local and they fail to guarantee asymptotic stability for a solution that does not start sufficiently close to the equilibrium. Furthermore, the controller is restricted to be linear.

We now prove that conditions presented in (5) and (6), are sufficient to guarantee asymptotic stabilization of the hanging equilibrium with a large, computable domain of attraction. Furthermore, the controller is allowed to be nonlinear, and, in particular, it may include saturation.

Consider the model given in (1) and the control law

$$u = -\mathbf{U}(y), \tag{7}$$

satisfying conditions (5) and (6).

Lemma 1: *Consider the 3D pendulum given by (1). Let* $\mathbf{U} : \mathbb{R}^3 \mapsto \mathbb{R}^3$ *satisfy (5) and (6). Furthermore, let* P *have a rank deficiency of one and denote* $\mathrm{Ker}\{P\} = \mathrm{span}[\hat{v}]$*, where* $\hat{v} \in \mathbb{R}^3$ *is a unit vector. Suppose* $\hat{v}$ *satisfies* $\Gamma_h^{\mathrm{T}}(J\hat{v} \times \hat{v}) \neq 0$ *and* $\Gamma_h^{\mathrm{T}} J\hat{v} \neq 0$ *i.e.* $\Gamma_h \notin \mathrm{span}[\hat{v}, J\hat{v}]$ *and* Γ_h *is not orthogonal to* $J\hat{v}$*. Then, the hanging equilibrium of the closed-loop defined by (1) and (7) is asymptotically stable. Furthermore, for every* $\epsilon \in (0, 2mg\|\rho\|)$*, all solutions of the closed-loop system given by (1) and (7), such that* $(\omega(0), \Gamma(0)) \in \mathcal{H}_\epsilon$*, where*

$$\mathcal{H}_\epsilon = \left\{ (\omega, \Gamma) \in \mathbb{R}^3 \times S^2 : \frac{1}{2}\omega^{\mathrm{T}} J\omega + \frac{1}{2} mg\|\rho\| \, \|\Gamma - \Gamma_h\|^2 \leq 2mg\|\rho\| - \epsilon \right\} \quad (8)$$

satisfy $(\omega(t), \Gamma(t)) \in \mathcal{H}_\epsilon$*,* $t \geq 0$*, and* $\lim_{t\to\infty} \omega(t) = 0$ *and* $\lim_{t\to\infty} \Gamma(t) = \Gamma_h$*.*

Proof: Consider the closed loop system given by (1) and (7). We propose the following candidate Lyapunov function

$$V(\omega, \Gamma) = \frac{1}{2} \left[\omega^{\mathrm{T}} J\omega + mg\|\rho\| \|\Gamma - \Gamma_h\|^2 \right]. \quad (9)$$

Note that the above Lyapunov function is positive definite on $\mathbb{R}^3 \times S^2$ and $V(0, \Gamma_h) = 0$.

Differentiating the above Lyapunov function, we obtain that the derivative of V along a solution of the closed-loop given by (1) and (7) is

$$\dot{V}(\omega, \Gamma) = -\omega^{\mathrm{T}} B\mathbf{U}(C\omega) \leq 0,$$

where the last inequality follows from (5). Thus, $V(\cdot)$ is positive definite and $\dot{V}(\cdot)$ is negative semidefinite on $\mathbb{R}^3 \times S^2$.

Next, consider the sub-level set given by $\mathcal{H}_\epsilon = \{(\omega, \Gamma) \in \mathbb{R}^3 \times S^2 : V(\omega, \Gamma) \leq 2mg\|\rho\| - \epsilon\}$. Note that the compact set $\mathcal{H}_\epsilon$ contains the hanging equilibrium $(0, \Gamma_h)$ and does not contain the inverted equilibrium $(0, \Gamma_i)$. Since, $\dot{V}(\omega, \Gamma) \leq 0$, all solutions such that $(\omega(0), \Gamma(0)) \in \mathcal{H}_\epsilon$ satisfy $(\omega(t), \Gamma(t)) \in \mathcal{H}_\epsilon$ for all $t \geq 0$. Thus, $\mathcal{H}_\epsilon$ is an invariant set for solutions of (2) and (7).

Next, from LaSalle's invariant set theorem, we obtain that solutions satisfying $(\omega(0), \Gamma(0)) \in \mathcal{H}_\epsilon$ converge to the largest invariant set in $\mathcal{S} = \{(\omega, \Gamma) \in \mathcal{H}_\epsilon : B\mathbf{U}(C\omega) = 0\}$. Note that $(0, \Gamma_h)$ is a trivial solution contained in the largest invariant set in $\mathcal{S}$. Thus, we want to show that any other nontrivial solution contained in the largest invariant set in $\mathcal{S}$ converges to the hanging equilibrium.

Let $\mathcal{U}$ be an open neighborhood of $\mathcal{S}$ such that (6) holds. Since, for all $(\omega, \Gamma) \in \mathcal{S} \subseteq \mathcal{U}$, $0 \geq \omega^{\mathrm{T}} B\mathbf{U}(C\omega) \geq \omega^{\mathrm{T}} P\omega$ and P is positive semidefinite, hence $\mathcal{S} \subseteq \mathcal{S}_1 = \{(\omega, \Gamma) \in \mathcal{H}_\epsilon : P\omega = 0\}$. Thus, we consider the largest invariant set of the closed loop dynamics in $\mathcal{S}_1$. Clearly, if P is full rank, then, there are no nontrivial solutions and the result follows immediately from [2]. Thus, the difficult case is when P is positive semidefinite.

We next show that the largest invariant set for the solutions of (1) and (7) in the set $\mathcal{S}_1$ is a union of the trivial solution at the hanging equilibrium and nontrivial solutions that converge asymptotically to the hanging equilibrium. Thus, all solutions starting in $\mathcal{H}_\epsilon$ converge to the hanging equilibrium.

Next, we write the dynamical equations satisfied by the system in the set $\mathcal{S}_1$. They are obtained by requiring that $P\omega = 0$, thus yielding

$$\begin{cases} J\dot{\omega} = J\omega \times \omega + mg\rho \times \Gamma, \\ \dot{\Gamma} = \Gamma \times \omega, \\ P\omega = 0. \end{cases} \quad (10)$$

Now, since $\mathrm{Ker}\{P\} = \mathrm{span}[\hat{v}]$, where $\hat{v} \in \mathbb{R}^3$ is a unit vector, $\omega(t) = \alpha(t)\hat{v}$ where $\hat{v}$ is fixed in the body axis frame.

Consider the following two cases. First, suppose that $\alpha(t) \equiv 0$. Then $\dot{\Gamma}(t) \equiv 0$, which implies that Γ is a constant vector in the body axis frame. Furthermore, $\dot{\omega}(t) \equiv 0$ and hence substituting this in (10), we obtain that $\rho \times \Gamma = 0$. Thus, $\Gamma = \Gamma_i$ or $\Gamma = \Gamma_h$. Since $(0, \Gamma_i) \notin \mathcal{H}_\epsilon$, it implies that $(0, \Gamma_i) \notin \mathcal{S}_1$, and hence the solution converges to $\Gamma = \Gamma_h$. Thus, the invariant solution in this case, is given by $(\omega, \Gamma) = (0, \Gamma_h)$.

Next, suppose that $\alpha(t) \not\equiv 0$. Then substituting into (10) and writing in terms of Γ_h, we obtain

$$\begin{cases} \dot{\alpha}(t) J\hat{v} = \alpha^2(t) J\hat{v} \times \hat{v} + mg\|\rho\|(\Gamma_h \times \Gamma), \\ \dot{\Gamma}(t) = \alpha(t)\Gamma \times \hat{v}. \end{cases} \quad (11)$$

Next, pre-multiply both sides of the first equation in (11) by Γ_h^{T}. This yields

$$\dot{\alpha}(t)\Gamma_h^{\mathrm{T}} J\hat{v} = \alpha^2(t)\Gamma_h^{\mathrm{T}} \left(J\hat{v} \times \hat{v} \right).$$

Since, by assumption $\Gamma_h^{\mathrm{T}} J\hat{v} \neq 0$ and $\Gamma_h^{\mathrm{T}} \left(J\hat{v} \times \hat{v} \right) \neq 0$,

$$\dot{\alpha}(t) = k\alpha^2(t), \text{ where } k = \frac{\Gamma_h^{\mathrm{T}} \left(J\hat{v} \times \hat{v} \right)}{\Gamma_h^{\mathrm{T}} J\hat{v}} \quad (12)$$

is a non-zero constant. If $\alpha(0) = 0$, then $\alpha(t) = 0$ for all $t \geq 0$. However, since $\alpha(t) \not\equiv 0$, we obtain that any solution of the equation satisfies either $|\alpha(t)| \to \infty$ or $\alpha(t) \to 0$ as $t \to \infty$.

However, since all solutions are bounded in the invariant set $\mathcal{H}_\epsilon$, $|\alpha(t)|$ remains bounded for all $t \geq 0$. Thus, $\alpha(t) \to 0$ as $t \to \infty$. Thus, in the limit as $t \to \infty$,

$$\lim_{t\to\infty} \dot{\alpha}(t) = k \lim_{t\to\infty} \alpha^2(t) = 0.$$

Thus, since $\dot{\alpha}(t) \to 0$ and $\alpha(t) \to 0$ as $t \to \infty$, (11) yields

$$\lim_{t\to\infty} \dot{\alpha}(t) J\hat{v} = \lim_{t\to\infty} \alpha^2(t) J\hat{v} \times \hat{v} + \lim_{t\to\infty} mg\|\rho\|(\Gamma_h \times \Gamma(t)),$$

which implies $\lim_{t\to\infty} (\Gamma_h \times \Gamma(t)) = 0$.

Thus, either $\Gamma \to \Gamma_i$ or $\Gamma \to \Gamma_h$ as $t \to \infty$. Since, $(0, \Gamma_i) \notin \mathcal{H}_\epsilon \supseteq \mathcal{S}_1 \supseteq \mathcal{S}$, therefore, $\Gamma(t) \to \Gamma_h$ as $t \to \infty$. Thus, any nontrivial solution starting in the largest invariant set in $\mathcal{S}$, converges to the hanging equilibrium as $t \to \infty$. ■

In the above result we have shown that for every $\epsilon > 0$, there exists a compact set $\mathcal{H}_\epsilon$, such that all solutions starting in $\mathcal{H}_\epsilon$ converge to the hanging equilibrium. In the next Theorem, we extend this result to include the boundary of the set $\mathcal{H}_\epsilon$, when $\epsilon = 0$, in the domain of attraction of the hanging equilibrium.

Theorem 1: *Consider the 3D pendulum given by (1). Let* $\mathbf{U} : \mathbb{R}^3 \mapsto \mathbb{R}^3$ *satisfy (5) and (6). Furthermore, let* P *have a rank deficiency of one and denote* $\mathrm{Ker}\{P\} = \mathrm{span}[\hat{v}]$, *where* $\hat{v} \in \mathbb{R}^3$ *is a unit vector. Suppose* $\hat{v}$ *satisfies* $\Gamma_h^{\mathrm{T}}(J\hat{v} \times \hat{v}) \neq 0$ *and* $\Gamma_h^{\mathrm{T}} J\hat{v} \neq 0$ *i.e.* $\Gamma_h \notin \mathrm{span}[\hat{v}, J\hat{v}]$ *and* Γ_h *is not orthogonal to* $J\hat{v}$. *Then, all solutions of the closed-loop system given by (1) and (7), such that* $(\omega(0), \Gamma(0)) \in \mathcal{N} \backslash \{(0, \Gamma_i)\}$, *where*

$$\mathcal{N} = \left\{ (\omega, \Gamma) \in \mathbb{R}^3 \times S^2 : \frac{1}{2}\omega^{\mathrm{T}} J\omega + \frac{1}{2} mg\|\rho\| \, \|\Gamma - \Gamma_h\|^2 \leq 2mg\|\rho\| \right\} \tag{13}$$

satisfy $\lim_{t\to\infty} \omega(t) = 0$ *and* $\lim_{t\to\infty} \Gamma(t) = \Gamma_h$.

Proof: We present an outline of the proof. The idea of the proof is similar to the proof of Lemma 2 in [2]. It can be shown that all solutions of the closed loop (1) and (7), satisfying $(\omega(0), \Gamma(0)) \in \partial\mathcal{N} \backslash \{(0, \Gamma_i)\}$ enter the set $\mathcal{H}_\epsilon$ in Lemma 1, for some $\epsilon > 0$, in finite time. Then, from Lemma 1 and the definition of $\mathcal{N}$, we note that for every $\epsilon \in (0, 2mg\|\rho\|)$ and $(\omega(0), \Gamma(0)) \in \mathcal{H}_\epsilon \bigcup \left(\partial\mathcal{N} \backslash \{(0, \Gamma_i)\}\right)$, $\omega(t) \to 0$ and $\Gamma(t) \to \Gamma_h$ as $t \to \infty$. Next, since $\mathcal{N} = \bigcup_{\epsilon \in (0, 2mg\|\rho\|)} \left(\mathcal{H}_\epsilon \bigcup \partial\mathcal{N}\right)$, it follows that all solutions satisfying $(\omega(0), \Gamma(0)) \in \mathcal{N} \backslash \{(0, \Gamma_i)\}$ converge to the hanging equilibrium. ■

Remark 1: It is worth noting that the analysis based on linearization of the closed-loop dynamics given by (1) and (7) yields eigenvalues with zero real part and hence, does not yield asymptotic stability. Thus, analysis based on linearization fails to be conclusive regarding the stability of the hanging equilibrium. Indeed, asymptotic stability of the hanging equilibrium arises from the nonlinear coupling, which is necessarily of higher order. In [5], local asymptotic stability was shown to arise from these terms in the polynomial dynamics of the closed-loop system. In contrast, we analyze the non-local closed loop dynamics.

Remark 2: Note that the controller (7) stabilizes the hanging equilibrium asymptotically, but not exponentially. This is clearly seen from proof of Lemma 1, where possible non-exponential solutions of the form $\omega(t) = \alpha(t)\hat{v}$ are identified, where $\alpha(t)$ evolves as in (12).

Remark 3: The above controller can be viewed as providing a mechanism for generating nonlinear dissipation along the free dynamics of the 3D pendulum. The nonlinear dissipation that this controller generates is independent of the attitude of the pendulum.

Theorem 1 applies to the case when both underactuation and partial angular velocity feedback are assumed. Thus, in the particular case of only underactuation as in case (2), or only partial angular velocity feedback, as in case (3), we obtain a corresponding result for the controller. For simplicity, we assume the controller is linear: $\mathbf{U}(y) = Ky$. We obtain the following two corollaries.

The result given below holds for an underactuated controller of rank deficiency one.

Corollary 1: *Consider the 3D pendulum given by (1) where* C *is the identity. Let* $\mathbf{U}(y) = Ky$, *such that* BK *is positive semidefinite, and* $\mathrm{Ker}\{BK\} = \mathrm{span}[\hat{v}]$, *where* $\hat{v} \in \mathbb{R}^3$ *is a unit vector such that* $\hat{v}$ *satisfies* $\Gamma_h^{\mathrm{T}}(J\hat{v} \times \hat{v}) \neq 0$ *and* $\Gamma_h^{\mathrm{T}} J\hat{v} \neq 0$. *Then, all solutions of the closed-loop system given by (1) and (7), such that* $(\omega(0), \Gamma(0)) \in \mathcal{N} \backslash \{(0, \Gamma_i)\}$, *where*

$$\mathcal{N} = \left\{ (\omega, \Gamma) \in \mathbb{R}^3 \times S^2 : \frac{1}{2}\omega^{\mathrm{T}} J\omega + \frac{1}{2} mg\|\rho\| \, \|\Gamma - \Gamma_h\|^2 \leq 2mg\|\rho\| \right\} \tag{14}$$

satisfy $\lim_{t\to\infty} \omega(t) = 0$ *and* $\lim_{t\to\infty} \Gamma(t) = \Gamma_h$.

Proof: : The proof easily follows by noting that $B\mathbf{U}(C\omega) = BK\omega$. Thus, choosing $P = BK$, and noting that $\mathrm{Ker}\{P\} = \mathrm{span}[\hat{v}]$ satisfies $\Gamma_h^{\mathrm{T}}(J\hat{v} \times \hat{v}) \neq 0$ and $\Gamma_h^{\mathrm{T}} J\hat{v} \neq 0$, the result follows follows from Theorem 1. ■

A similar result for partial angular velocity feedback is next presented.

Corollary 2: *Consider the 3D pendulum given by (1) where* B *is the identity. Let* $\mathbf{U}(y) = Ky$, *such that* KC *is positive semidefinite, and* $\mathrm{Ker}\{KC\} = \mathrm{span}[\hat{v}]$, *where* $\hat{v} \in \mathbb{R}^3$ *is a unit vector such that* $\hat{v}$ *satisfies* $\Gamma_h^{\mathrm{T}}(J\hat{v} \times \hat{v}) \neq 0$ *and* $\Gamma_h^{\mathrm{T}} J\hat{v} \neq 0$. *Then, all solutions of the closed-loop system given by (1) and (7), such that* $(\omega(0), \Gamma(0)) \in \mathcal{N} \backslash \{(0, \Gamma_i)\}$, *where*

$$\mathcal{N} = \left\{ (\omega, \Gamma) \in \mathbb{R}^3 \times S^2 : \frac{1}{2}\omega^{\mathrm{T}} J\omega + \frac{1}{2} mg\|\rho\| \, \|\Gamma - \Gamma_h\|^2 \leq 2mg\|\rho\| \right\} \tag{15}$$

satisfy $\lim_{t\to\infty} \omega(t) = 0$ *and* $\lim_{t\to\infty} \Gamma(t) = \Gamma_h$.

Proof: The proof is similar to the proof given in Corollary 1. ■

IV. Stabilization of the Inverted Equilibrium

In this section, we desire to stabilize the reduced configuration corresponding to the inverted equilibrium for a 3D pendulum. In [2], we designed a controller to achieve the above objective assuming complete angular velocity and reduced attitude feedback. Here we assume that B is full rank, and that the rank deficiency of C is one. Thus, we want to find a feedback controller of the form $u = -\mathbf{U}(y, \Gamma)$ that stabilizes the inverted equilibrium of the 3D pendulum.

Let $\Phi : [0,1) \mapsto \mathbb{R}$ be a C^1 monotonically increasing function such that $\Phi(0) = 0$ and $\Phi(x) \to \infty$ as $x \to 1$. Let

$$u = -B^{-1}\left[\Phi'\left(\frac{1}{4}(\Gamma_i^{\mathrm{T}}\Gamma - 1)^2\right)(\Gamma_i^{\mathrm{T}}\Gamma - 1)(\Gamma_i \times \Gamma)\right] - B^{-1}\Big(mg\rho \times \Gamma\Big) - B^{-1}Ky, \tag{16}$$

where KC is a positive semidefinite matrix. Due to the assumption that C is rank deficient, it is not possible to satisfy the conditions given in [2]. Therefore, we next prove conditions for which the controller (16) stabilizes the inverted equilibrium of the 3D pendulum almost globally.

Theorem 2: *Consider the 3D pendulum given by (1). Suppose that KC is positive semidefinite and $\mathrm{Ker}\{KC\} = \mathrm{span}[\hat{v}]$ satisfies $\Gamma_i^{\mathrm{T}}(J\hat{v} \times \hat{v}) \neq 0$ and $\Gamma_i^{\mathrm{T}}J\hat{v} \neq 0$ i.e. $\Gamma_i \notin \mathrm{span}[\hat{v}, J\hat{v}]$ and Γ_i is not orthogonal to $J\hat{v}$. Then $(0, \Gamma_i)$ is an equilibrium of the closed loop system (1) and (16) that is globally asymptotically stable with domain of attraction $\mathbb{R}^3 \times (S^2 \backslash \{\Gamma_h\})$.*

Proof: The proof for this theorem is related to the proof of Theorem 3 in [2] and Lemma 1 in this paper. We propose the candidate Lyapunov function

$$V(\omega, \Gamma) = \frac{1}{2}\omega^{\mathrm{T}}J\omega + 2\Phi\left(\frac{1}{4}(\Gamma_i^{\mathrm{T}}\Gamma - 1)^2\right). \tag{17}$$

Note that the Lyapunov function is positive definite and proper on $\mathbb{R}^3 \times S^2$ and $V(0, \Gamma_i) = 0$. Hence, every sub-level set of the Lyapunov function in $\mathbb{R}^3 \times S^2$ is compact, and the closed-loop vector field given by (1) and (16) has only one equilibrium in each sub-level set, namely $(0, \Gamma_i)$.

Next, computing the derivative of the Lyapunov function along the solution of (1) and (16), it can be shown that $\dot{V}(\omega, \Gamma) = -\omega^{\mathrm{T}}KC\omega \leq 0$, since KC is assumed to be positive semidefinite. Thus, $V(\cdot)$ is positive definite and $\dot{V}(\cdot)$ is negative semidefinite on $\mathbb{R}^3 \times S^2$.

Next, consider the sub-level set given by $\mathcal{K} = \{(\omega, \Gamma) \in \mathbb{R}^3 \times S^2 : V(\omega, \Gamma) \leq V(\omega(0), \Gamma(0))\}$. Note that the compact set $\mathcal{K}$ contains the equilibrium $(0, \Gamma_i)$ and does not contain the equilibrium $(0, \Gamma_h)$. Since, $\dot{V}(\omega, \Gamma) \leq 0$, all solutions such that $(\omega(0), \Gamma(0)) \in \mathcal{K}$ satisfy $(\omega(t), \Gamma(t)) \in \mathcal{K}$ for all $t \geq 0$. Thus, $\mathcal{K}$ is an invariant set for solutions of (1) and (16).

Then, from LaSalle's invariant set theorem, we obtain that solutions satisfying $(\omega(0), \Gamma(0)) \in \mathcal{K}$ converge to the largest invariant set in $\mathcal{S} = \{(\omega, \Gamma) \in \mathcal{K} : KC\omega = 0\}$. Thus, we next consider the largest invariant set of the closed loop dynamics in $\mathcal{S}$.

Then using similar arguments as in Lemma 1, we can show that the equations in the largest invariant set in $\mathcal{S}$ are given by

$$\begin{cases} J\dot{\omega} = J\omega \times \omega + (1 - \Gamma_i^{\mathrm{T}}\Gamma)\Phi'\left(\frac{(\Gamma_i^{\mathrm{T}}\Gamma - 1)^2}{4}\right)(\Gamma_i \times \Gamma), \\ \dot{\Gamma} = \Gamma \times \omega, \\ KC\omega = 0, \end{cases}$$

where, $\mathrm{Ker}\{KC\} = \mathrm{span}[\hat{v}]$, $\hat{v} \in \mathbb{R}^3$ is a unit vector, $\omega(t) = \alpha(t)\hat{v}$ and $\hat{v}$ is fixed in the body axis frame.

Then proceeding as in Lemma 1, it can be shown that, if $\alpha(t) \not\equiv 0$, it evolves as

$$\begin{cases} \dot{\alpha}(t)J\hat{v} = \alpha^2(t)J\hat{v} \times \hat{v} \\ \qquad\qquad + (1 - \Gamma_i^{\mathrm{T}}\Gamma)\Phi'\left(\frac{(\Gamma_i^{\mathrm{T}}\Gamma - 1)^2}{4}\right)(\Gamma_i \times \Gamma), \\ \dot{\Gamma}(t) = \alpha(t)\Gamma \times \hat{v}. \end{cases} \tag{18}$$

Next, pre-multiplying both sides of the first equation in (18) by Γ_i^{T} yields

$$\dot{\alpha}(t)\Gamma_i^{\mathrm{T}}J\hat{v} = \alpha^2(t)\Gamma_i^{\mathrm{T}}\left(J\hat{v} \times \hat{v}\right).$$

Since, by assumption $\Gamma_i^{\mathrm{T}}J\hat{v} \neq 0$ and $\Gamma_i^{\mathrm{T}}\left(J\hat{v} \times \hat{v}\right) \neq 0$,

$$\dot{\alpha}(t) = k\alpha^2(t), \text{ where } k = \frac{\Gamma_i^{\mathrm{T}}\left(J\hat{v} \times \hat{v}\right)}{\Gamma_i^{\mathrm{T}}J\hat{v}}$$

is a non-zero constant. Finally, arguing as in Lemma 1, we obtain that $\alpha(t) \to 0$ as $t \to \infty$. Thus, in the limit as $t \to \infty$,

$$\lim_{t\to\infty}\dot{\alpha}(t) = k\lim_{t\to\infty}\alpha^2(t) = 0,$$

and hence

$$\lim_{t\to\infty}\dot{\alpha}(t)J\hat{v} = \lim_{t\to\infty}\alpha^2(t)J\hat{v} \times \hat{v} + \lim_{t\to\infty}(1 - \Gamma_i^{\mathrm{T}}\Gamma)\Phi'\left(\frac{(\Gamma_i^{\mathrm{T}}\Gamma - 1)^2}{4}\right)(\Gamma_i \times \Gamma),$$

which implies that $\lim_{t\to\infty}(\Gamma_i \times \Gamma(t)) = 0$.

Thus, either $\Gamma \to \Gamma_i$ or $\Gamma \to \Gamma_h$ as $t \to \infty$. Since, $(0, \Gamma_h) \notin \mathcal{K} \supseteq \mathcal{S}$, therefore, $\Gamma(t) \to \Gamma_i$ as $t \to \infty$. Thus, the solution in the largest invariant set in $\mathcal{S}$, converges to the equilibrium $(0, \Gamma_i)$ as $t \to \infty$. ■

Theorem 2 proposes a family of output feedback controllers which asymptotically stabilizes the inverted equilibrium of the 3D pendulum, even when all angular velocity measurements are not fed back. Feedback of the reduced attitude vector Γ and partial angular velocity are required for feedback. Knowledge of the moment of inertia and location of the center of mass are also assumed.

Remark 4: The domain of attraction of the stabilized inverted equilibrium is almost global. By almost global asymptotic stabilization of an equilibrium, we mean that for every initial condition in the phase-space contained in the complement of a set of Lebesgue measure zero, the solution converges to this equilibrium. Thus, the domain of attraction of the equilibrium is the whole of the phase-space, excluding a set of Lebesgue measure zero.

V. SIMULATION RESULTS

We present simulation results for specific controllers that stabilize the hanging equilibrium of the underactuated 3D pendulum with partial angular velocity feedback, and the inverted equilibrium of the fully actuated 3D pendulum, with partial angular velocity and reduced attitude feedback.

Consider the stabilization of the hanging equilibrium. Assume $\tau_x \equiv 0$ with

$$B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, C = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, J = \begin{bmatrix} 30 & 10 & 0 \\ 10 & 40 & 0 \\ 0 & 0 & 50 \end{bmatrix},$$

where units for J are kg-m^2, $m = 140$ kg, and $\rho = (0, 0, 0.5)^{\mathrm{T}}$ m. Note that B and C are rank deficient. Choose a linear controller as $u = -Ky$ with

$$K = \begin{pmatrix} 0 & 0 & 0 \\ 10 & 50 & 10 \\ 10 & 10 & 10 \end{pmatrix}.$$

Initial conditions are given by $\Gamma(0) = [0.5\ 0.8047\ -0.32]^{\mathrm{T}}$ and $\omega(0) = (1, 3, 1)$ rad/sec. The controller and the chosen initial conditions satisfy the conditions given in Theorem 1. Figure 1 describes the path of the center of mass of the 3D pendulum in the inertial frame and illustrates that $\omega_x \to 0$ and $\Gamma \to \Gamma_h$ as $t \to \infty$.

We next present results for the stabilization of the inverted equilibrium. The values for the mass and the vector from the pivot to the center of mass are chosen as before. Assume that

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, J = \begin{bmatrix} 30 & 20 & 7 \\ 20 & 40 & 0 \\ 7 & 0 & 50 \end{bmatrix},$$

and choose a controller of the form (16) with $K = \mathrm{diag}(0, 10, 20)$ and $\Phi(x) = -5\ln(1 - x)$. This controller satisfies the conditions given in Theorem 2. The initial conditions are $\omega(0) = 0$ and $\Gamma(0) = [0.1\ 0.9487\ 0.3]^{\mathrm{T}}$. Figure 2 describes path of the center of mass of the 3D pendulum in the inertial frame and illustrates that $\omega \to 0$ and $\Gamma \to \Gamma_i$ as $t \to \infty$.

Finally, we present simulation results for an interesting case wherein the same controller as above, acts on a *weakly* coupled 3D pendulum with moment of inertia

$$J = \begin{bmatrix} 30 & 1 & 7 \\ 1 & 40 & 2 \\ 7 & 2 & 50 \end{bmatrix}.$$

The moment of inertia matrix shows that the nonlinear coupling is not large, since the off-diagonal terms are small. It can be easily verified that the previous controller satisfies the conditions of Theorem 2. The path of the center of mass in Figure 3 clearly shows oscillations that are lightly damped. This may be compared to the case in Figure 2, where there are almost no oscillations.

References

[1] J. Shen, A. K. Sanyal, N.A. Chaturvedi, D.S. Bernstein and N.H. McClamroch, "Dynamics and Control of a 3D Pendulum," *Proceedings of the 43rd IEEE Conference on Decision & Control*, Bahamas, 2004, 323-328.

[2] N. A. Chaturvedi, F. Bacconi, A. K. Sanyal, D. S. Bernstein and N. H. McClamroch, "Stabilization of a 3D Rigid Pendulum," *Proceedings of the American Control Conference*, Portland, Oregon, 2005, 3030-3035.

[3] F. Bullo and R. M. Murray, "Tracking for Fully Actuated Mechanical Systems: A Geometric Framework," *Automatica*, 35(1), 1999, 17-34.

[4] D. E. Chang and J. E. Marsden, "Asymptotic Stability of Heavy Top using Controlled Lagrangian," *Proceedings of the 39th IEEE-Conference on Decision & Control*, Sydney, Australia, 2000, 269-273.

[5] J. Shen, A. K. Sanyal, and N. H. McClamroch, "Asymptotic Stability of Rigid Body Attitude Systems," *Proceedings of the 42nd IEEE Conference on Decision & Control*, Maui, Hawaii, 2003, 544-549.

[6] J. Shen, A. K. Sanyal, and N. H. McClamroch, "Asymptotic Stability of Multi-Body Attitude Systems," *Stability and Control of Dynamical Systems with Applications*, Chapter 3, Ed. Liu and Antsakalis, Birkhauser, Boston, 2003, 47-70.

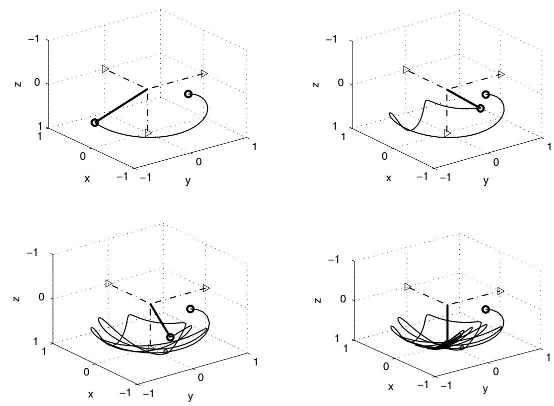

Fig. 1. Motion of the vector between the pivot and the center of mass of the 3D pendulum in the inertial frame.

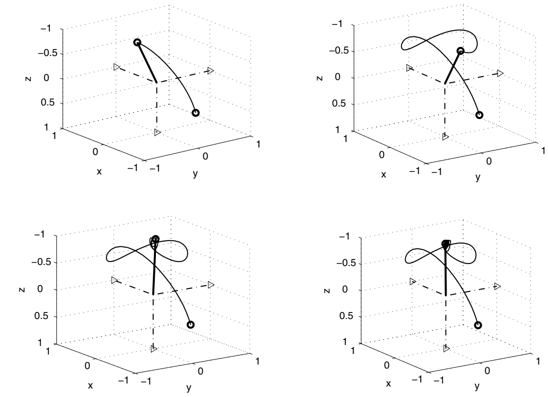

Fig. 2. Swing-up motion of the vector between the pivot and the center of mass of the 3D pendulum in the inertial frame.

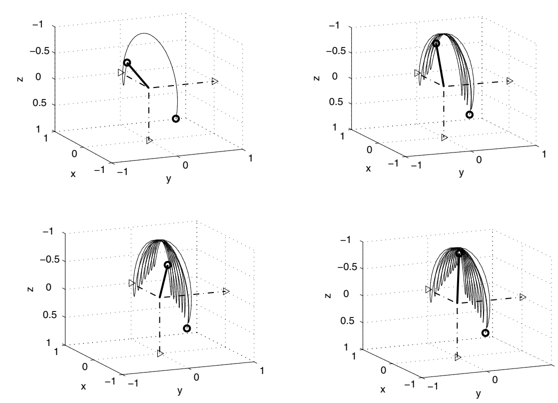

Fig. 3. Swing-up motion of the vector between the pivot and the center of mass of a weakly coupled 3D pendulum in the inertial frame.