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A stochastic feedback system model of a stock exchange

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***Abstract*— Stock exchanges are modelled as nonlinear feedback systems where the plant dynamics is defined by known stock market regulations but the actions of agents are unknown. It is assumed though that each agent submits transaction requests according to his/her beliefs on the price dynamics and his/her behavior. The action of the agents may contain a random element, thus we get a non-linear stochastic feedback system. The market is in equilibrium when the actions of the agents reinforce their beliefs on the price dynamics. Assuming that an $AR(k)$ predictor is used for prediction of the price process, a stochastic approximation procedure for finding market equilibrium is described. The proposed procedure is analyzed using the theory of Benveniste, Métivier and Priouret, [1].**

I. INTRODUCTION

In this paper we present a closed-loop model for a stock exchange. The plant itself is defined by a market clearance mechanism, by which the agent's transaction requests are collected, transactions are carried out, and new stock prices are determined.

The agents action are based on predictions of the observed price process. Using these predictions the agent will make a buy or sell decision according to his/her behavior. A variety of behaviors of economic players, such as loss aversion, conservatism, or risk seeking was identified by experimental psychologists, see for example Kahneman [5].

A key factor in the above model is the agent's belief of the price dynamics, and his/her predictive capability. In this paper we assume that all agents use the same standardized predictor, which is fitted to actual data. In this paper the standard predictor will be obtained by fitting an $AR(k)$ model to the observed price process, and thus the predictor, denoted by M, will be an FIR filter.

For any fixed predictor M the closed loop dynamics will define a price process, the dynamics of which depends on M. The standard predictor for this price process will be typically different from M, thus the agents will adjust their standard predictor. This adjustment will be continued until an equilibrium is reached: until the actions of the agents reinforce their beliefs. An on-line, data-driven version of the above adjustment procedure will be presented and analyzed using the theory of Benveniste, Métivier and Priouret, [1].

II. A BEHAVIORAL STOCK MARKET MODEL

Market dynamics. The price of a given stock at time $n \in \mathbb{N}$ is denoted by p_n. The prices are given by the market: based on the transaction requests of the agents, the next period's price is calculated by an automated trading system that uses an equilibrium-price transaction matching algorithm. The transaction request of the i^{th} agent, $1 \leq i \leq N$, consists of a demand quantity d^i and a bid price b^i. The demand can take any values in $\mathbb{R}$, a negative value meaning that the agent would like to sell. The bid prices show the maximum price the buyer is willing to pay when buying stocks or the minimum the seller will accept when selling.

A basic building block of the transaction matching algorithm is the following. For a given offer/bid price p calculate the total number of transactions that would take place at this price, and choose the price at which trading volume is maximized. If no transactions are realized, the price remains unchanged. The algorithms are complemented by minor refinements that handle non-uniqueness problems. For details see e.g. the opening rules of the New York Stock Exchange [9]. The aggregated demand at price p, say $D(p)$, is given by

$$D(p) = \sum_i (d^i)^+ H(b^i - p),$$

where H is the Heaviside function or step-function and x^+ denotes the positive part of x. Similarly, the aggregated supply is calculated by

$$S(p) = \sum_i (d^i)^- H(p - b^i),$$

where x^- denotes the negative part of x. The volume maximizing price is obtained by solving the non-linear optimization problem

$$\max \ \delta \text{ subject to } D(p) \geq \delta, \ S(p) \geq \delta.$$

This mechanism can be formalized as

$$P \begin{pmatrix} d \\ b \end{pmatrix} = p, \tag{1}$$

where P is a known static nonlinear function of the transaction requests. Obviously $p \leq \max_i b^i$ thus P has linear growth rate. To ensure Lipschitz-continuity in the variables (d, b), we assume that the Heaviside function is replaced by a smooth so-called sigmoid function σ, a standard element in neural networks.

The control action. The transaction requests serving as the input of P are determined by a strictly non-anticipating operator C:

$$C \begin{pmatrix} p \\ v \end{pmatrix} = \begin{pmatrix} d \\ b \end{pmatrix}, \tag{2}$$

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where v denotes external investments or withdrawal. Throughout the paper v is taken to be constantly 0, i.e we consider a self-financing scenario. Another simplification in our model is that unmatched demands and supplies are not recorded in an order book, but simply cancelled. A more detailed picture of the controller C will be given below. Combining equations (1) and (2) we get the closed-loop system shown in Fig. 1.

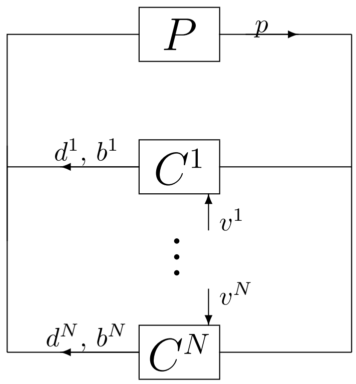

Fig. 1 A stock market model.

The predictor. The financial market is made up of a heterogenous mixture of traders. The only information the agents get from the stock market is the observed stock price process p, i.e. the bid prices of the other traders are unseen. The agents operate by trying to predict price movements: based on their beliefs they construct a price predictor M

$$\hat{p} = Mp. \tag{3}$$

Here M is assumed to be a strictly causal linear predictor. In this paper we will consider only FIR predictors. It is also assumed that all agents use the same standardized predictor.

Behaviors. The agents use the predicted price to determine their own demand d and bid price b. Researchers of behavioral finance argue that various psychological factors prevent decision makers from acting in a fully rational manner, see for example Greenfinch [3], Kostolany [6] and Shefrin [10]. Critics of this theory, see the works of Lucas [8] and Simon [11], claim that the behavior of the agents is always rational from a particular perspective.

The prediction combined with the decision made on assumed future prices will lead to a controller structure, where a dynamic linear element is followed by a static non-linearity as follows:

$$\begin{pmatrix} d \\ b \end{pmatrix} = f(Lp),$$

see Fig. 2. Here f is a Lipschitz-continuous static nonlinearity satisfying a linear growth condition, and L is a strictly causal linear filter that contains the price predictor M.

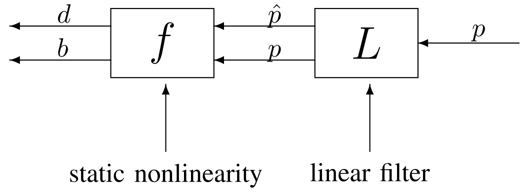

Fig. 2. The control action.

III. More on behaviors

A psychological phenomenon extensively studied by behaviorists is the so-called *loss aversion*. Nobel prize winner psychologists Kahneman and Tversky [5] find that even simple risk aversion can be biased: empirical evidence shows that a loss has about two and a half times the impact of a gain of the same magnitude. This behavior can be formalized by the equation

$$d_n = (\hat{p}_n - p_{n-1})^+ - 0.4(\hat{p}_n - p_{n-1})^-. \tag{4}$$

Now turning to *bid prices*, note that bid prices are usually less than the expected price if buying, and the other way round if selling. One such strategy is given by the rule

$$b_n = \frac{1}{2}(\hat{p}_n + p_{n-1}).$$

Introducing randomness. In case the predicted prices and the current stock prices differ by much, say by at least a value δ, the decision of the agents is straightforward: they buy according to their behavior if $\hat{p}_n - p_{n-1} > \delta$ and sell if $\hat{p}_n - p_{n-1} < -\delta$. However if this is not the case then the agents do not have a clear conception about future price movements, hence they may act randomly: they make buy/sell/hold decisions with prescribed probabilities. These probabilities and the parameter δ are characteristics of the agent.

Another source of randomness may be the uncertainty of the agent in the correctness of his/her strategy. In this case the agent makes a random buy/sell decision with a small probability.

The diversity of bid prices is crucial for a well-functioning market. This fact is supported by the abundant literature on bid-ask spreads (see for example the seminal paper of Glosten and Milgrom [4]). Thus it makes sense to have random elements in bid prices as well.

The noise model. From now on, the source of randomness is taken to be a random adjustment of the model-based pre-computed bid prices, say

$$\begin{pmatrix} d \\ b \end{pmatrix} = f(Lp) + e$$

where (e_n) is a strictly stationary sequence of random variables with zero mean.

IV. UPDATING THE PREDICTOR

Now imagine how an agent having a fixed behavior B would determine the demand for a particular stock. Fixing $M = M_0$ the controller $C_0 = C(M_0)$ is determined, and assuming that the closed loop system is well-defined, and the mapping from e to p is stable in an appropriate sense, we get a strictly stationary price process $p = p(M_0)$, with a spectrum depending on M_0. The agent observes this process and calculates its least squares predictor M^+, which will also depend on C_0, say $M^+ = M^+(C_0)$. Next the agent compares the two predictors. Now if $M^+(C_0) \neq M_0$ then it is reasonable to switch to the new predictor $M_1 := M^+(C_0)$. Let us define the mapping

$$f(M) = M^+(C(M)).$$

Assume that there exists a predictor M^* for which the market is in equilibrium, i.e. M^* is a fixed point of the operator equation

$$M^* = f(M^*).$$

Then we may ask if the iterative procedure

$$M_{i+1} = f(M_i)$$

converges.

The updating of M can be easily calculated if the mapping from e to p is rational and minimum phase. In our model this can not be guaranteed. Also in real life financial markets the mapping from exogenous noise to price is often non-rational and non-minimum phase. It is therefore more reasonable to identify M^*, or its approximation directly from the data.

$AR(k)$-approximation. Write the innovation representation of p in the form

$$p = H\nu,$$

where ν is the innovation process of p. Since we have no prior information on p we fit an $AR(k)$ model to our data, and use use a low order predictor based on this model. Let A be a polynomial of the shift operator of degree k and let A_0 be its leading coefficient. Then we consider the model class

$$\mathcal{A}_k = \{A \mid \deg A \leq k,\ A_0 = I,\ A \text{ stable}\}.$$

If the $AR(k)$-model is parametrized by η then we have for the corresponding least squares predictor $M(\eta) = I - A(\eta)$.

Let us now fix a predictor, or equivalently fix an $\eta \in \mathcal{A}_k$. Closing the loop we get the price process $p(\eta)$. To adjust the initial predictor find the best k-th-order AR-model by a least squares fit: minimize in θ

$$E|A(\theta)p(\eta)|^2$$

subject to $A(\theta) \in \mathcal{A}_k$. Using the notation

$$\nu(\theta, \eta) := A(\theta)p(\eta)$$

and

$$W(\theta, \eta) := \frac{1}{2}E|\nu(\theta, \eta)|^2$$

we have to solve the linear equation

$$W_\theta(\theta, \eta) = 0.$$

The solution will be denoted by $\varphi(\eta)$. It is then reasonable to adjust our predictor using this best fit, and redefine M as

$$M^+ = I - A(\varphi(\eta)).$$

Thus the mapping $M^+ = f(M)$ defined above in terms of transfer functions will be reduced to a mapping

$$\eta^+ = \varphi(\eta).$$

Market equilibrium is achieved if $\eta = \varphi(\eta)$, or equivalently

$$W_\theta(\eta, \eta) = 0. \tag{5}$$

Let the solution of (5) be denoted by η^*.

V. A DATA-DRIVEN PROCEDURE

The above conceptual procedure translates to the following data-driven procedure when working with real data. First, for any fixed stationary price process $p = p(\eta)$ observed over a time horizon 1 to N estimate the best $AR(k)$ coefficients by solving the minimization problem

$$\min_{A \in \mathcal{A}_k} \sum_{n=1}^{N} (Ap)^2 \ .$$

This is quadratic in the coefficients of A and thus can easily be computed. Then update your a priori price model η.

Introduce

$$G(\eta) := W_\theta(\eta, \eta) = E\nu_\theta(\eta, \eta)\nu(\eta, \eta).$$

Then η^* is simply the solution of

$$G(\eta) = 0. \tag{6}$$

Since $W_\theta(\theta, \eta)$ is computable experimentally for each θ and η, the general stochastic approximation procedure developed by Benveniste, Métivier and Priouret, [1], or Ljung and Söderström [7], the latter being extensively analyzed in [2], is applicable.

Adjusting the price model. Thus we arrive at the following data-driven procedure to solve (6):

$$\eta_{n+1} = \eta_n - \frac{c}{n}\nu_{\theta n}\nu_n, \tag{7}$$

where ν_n, $\nu_{\theta n}$ are on-line estimates of $\nu(\eta_n, \eta_n)$ and $\nu_\theta(\eta_n, \eta_n)$, respectively, and $c > 0$ is a step size. Taking into consideration the definition of $\nu(\theta, \eta)$ we have

$$\nu_n = [A(\eta_n)p]_n$$

and

$$\nu_{\theta n} = (p_{n-1}, \ldots, p_{n-k}).$$

The convergence properties of the above procedure can be analyzed via the general theory developed in [1]. Details will be given subsequently. Here we remark only one aspect of potential difficulty. Convergence of general recursive estimators is usually proven using the so-called ODE-method. The associated ODE in the present case is given by

$$\dot{\eta}_s = -c\ G(\eta_s). \tag{8}$$

The Jacobian of G at any η is

$$G_\eta(\eta) = W_{\theta\theta}(\eta,\eta) + W_{\theta\eta}(\eta,\eta).$$

We know that $W_{\theta\theta}(\theta,\eta)$ is at least positive semidefinite for each θ and η, and in fact it is independent of θ for fixed η. However, note that we have no control of the second term $W_{\theta\eta}(\eta,\eta)$ even for $\eta = \eta^*$. Therefore the asymptotic stability of (8) is not a priori guaranteed.

VI. The BMP scheme

In this section we present the basics of the theory of recursive estimation developed by Benveniste, Métivier and Priouret, BMP henceforth (see Chapter 2, Part II. of [1]).

Let a family of transition probabilities $\{\Pi_\theta,\ \theta \in D \subset \mathbb{R}^d\}$ on $\mathbb{R}^k$ be given. Here D is an open set. Assume that for any $\theta \in D$ there exists a unique invariant probability measure, say μ_θ. Let $(X_n(\theta))$ be a Markov-chain such that its initial state $X_0(\theta)$ has distribution μ_θ. Let $H(\theta, x)$ be a mapping from $\mathbb{R}^d \times \mathbb{R}^k$ to $\mathbb{R}^d$. Then the basic estimation problem of the BMP-theory is to solve the equation

$$E_{\mu_\theta} H(\theta, X(\theta)) = 0.$$

Assume that a solution $\theta^* \in D$ exists.

The BMP-scheme. The recursive estimation procedure to solve the above equation is then defined as

$$\theta_{n+1} = \theta_n + \frac{1}{n} H(\theta_n, X_n), \tag{9}$$

where X_n is the time-varying process defined by

$$P(X_{n+1} \in A | \mathcal{F}_n) = \Pi_{\theta_n}(X_n, A).$$

Here $\mathcal{F}_n$ is the σ-field of events generated by the random variables $X_0, \ldots, X_n$ and A is any Borel subset of $\mathbb{R}^k$.

To specify the class of functions H for which the theory is developed define for real-valued functions g on $\mathbb{R}^k$ and any $p \geq 0$ the norms

$$||g||_p := \sup_x \frac{|g(x)|}{1 + |x|^p},$$

and

$$||\Delta g||_p = \sup_{x_1 \neq x_2} \frac{|g(x_1) - g(x_2)|}{|x_1 - x_2|(1 + |x_1|^p + |x_2|^p)}.$$

Introduce the class of functions

$$C(p) = \{\ g : g \text{ is continuous and } ||g||_p < \infty\}.$$

and

$$Li(p) = \{\ g\ :\ ||\Delta g||_p < +\infty\}.$$

Note that $Li(p) \subseteq C(p+1)$ for any $p \geq 0$.

Conditions of BMP. All but one condition will be formulated in terms of the Markov chain $\{X_n(\theta) : n \geq 0\}$ for a fixed $\theta \in D$ with an arbitrary non-random initial value $X_0(\theta) = x$. Throughout the section we will illustrate these conditions on the *benchmark* example of linear dynamical processes:

$$Y_{n+1} = A(\theta) Y_n + B(\theta) W_{n+1},$$

where $A(\theta)$ and $B(\theta)$ are $k \times k$ real-valued matrices and $(W_n)_{n \geq 0}$ is an i.i.d. sequence of random variables. The conditions are as follows. The real number $p \geq 0$ is fixed all over the conditions A1.-A3. below.

A1. For any compact subset $Q \subset D$ there exists a constant $K = K(Q)$ such that for all $\theta \in Q$, $n \geq 0$ and $X_0(\theta) = x \in \mathbb{R}^k$:

$$\int \Pi_\theta^n(x, dy)(1 + |y|^{p+1}) \leq K(1 + |x|^{p+1}).$$

In the case of linear processes this condition is satisfied if (W_n) has bounded moments up to order $p+1$, and there exist constants K and $0 < \alpha < 1$ such that for all $n \geq 0$ and any $\theta \in Q$ we have $|A^n(\theta)| \leq K\alpha^n$ and $|B(\theta)| \leq K$.

A2. For any compact subset Q of D there exist constants $K = K(Q)$ and $0 < \rho < 1$ such that for all $g \in Li(p)$, any $\theta \in Q, n \geq 0$ and $x, x' \in \mathbb{R}^k$:

$$\begin{aligned}|\Pi_\theta^n g(x) - \Pi_\theta^n g(x')| &\leq \\ &\leq K ||\Delta g||_p\ \rho^n |x - x'|(1 + |x|^p + |x'|^p).\end{aligned}$$

In the linear case assumption A2 easily follows from the conditions that we formulated for A1.

Conditions A1 and A2 imply geometric ergodicity of the Markov chains in the following sense: for any $\theta \in D, x \in \mathbb{R}^k$ and any $g \in C(p+1)$ there exists a $\Gamma_\theta g$ such that

$$|\Pi_\theta^n g(x) - \Gamma_\theta g| \leq ||g||_{p+1} \rho^n (1 + |x|^{p+1}).$$

A key contribution of the BMP theory is that the above geometric ergodicity is derived by verifying conditions on a much more convenient class of test functions, namely $Li(p)$. It follows that that there exists a unique invariant measure μ_θ such that

$$\Gamma_\theta g = \int g(x) d\mu_\theta(dx)$$

for $g \in C(p+1)$.

A3. For any compact subset Q of D there exists a constant $K = K(Q)$ such that for all $g \in Li(p)$, any $\theta, \theta' \in Q$ and $n \geq 0, x \in \mathbb{R}^k$:

$$|\Pi_\theta^n g(x) - \Pi_{\theta'}^n g(x)| \leq K ||\Delta g||_p\ |\theta - \theta'|(1 + |x|^{p+1}).$$

In other words the kernels Π_θ^n are supposed to be Lipschitz-continuous, uniformly in n, with respect to the parameter θ when applied to a small set of test functions $Li(p)$. In the linear case A3 is satisfied if, in addition to what we required above, the matrices $A(\theta)$ and $B(\theta)$ are Lipschitz-continuous in θ.

Let $D_0 \subset D$ be a fixed compact truncation domain such that $\theta^* \in \mathrm{int} D_0$. Define the stopping time

$$\tau = \inf\{n : \theta_{n+1} \notin D_0\}.$$

In addition let ε be a fixed small positive number, and define

$$\sigma = \inf\{n : |\theta_n - \theta_{n-1}| > \varepsilon\}.$$

The stability of the time-varying process X_n is enforced by stopping it at $\tau \wedge \sigma$.

A4. For any compact subset Q of D there exists a constant $K = K(Q)$ such that for any $n \geq 0$ and arbitrary starting values $a \in Q$, $x \in \mathbb{R}^k$

$$E_{a,x}\{I(n < \tau \wedge \sigma)(1 + |X_{n+1}|^{p+1}\} \leq K(1 + |x|^{p+1})$$

Regularity of the function H is required in the next condition:

A5. For any compact subset Q of D there exists a constant $K = K(Q)$ such that for all $\theta, \theta' \in Q$

$$|H(\theta, x)| \leq K(1 + |x|^{p+1})$$
$$|H(\theta, x) - H(\theta', x)| \leq K|\theta - \theta'|(1 + |x|^{p+1})$$
$$||\Delta H(\theta, \cdot)||_p \leq K.$$

Remark: In fact it is sufficient to require the above condition for $\Pi_\theta H_\theta$, thus H may be discontinuous.

Since $H(\theta, \cdot) \in Li(p)$ we may set as above

$$h(\theta) = \lim_{n\to\infty} \Pi_\theta^n H(\theta, X_n(\theta)) = E_{\mu_\theta} H(\theta, X(\theta)).$$

The associated ODE is then given by

$$\dot{\theta}_s = h(\theta_s). \qquad (10)$$

To ensure the convergence of the SA-procedure we require global asymptotic stability of the associated ODE by assuming the existence of a Lyapunov function:

A6. There exists a real-valued C^2-function U on D such that

(i) $U(\theta^*) = 0,\ U(\theta) > 0$ for all $\theta \in D\backslash\{\theta^*\}$

(ii) $U'(\theta)h(\theta) < 0$ for all $\theta \in D\backslash\{\theta^*\}$

(iii) $U(\theta) \to \infty$ if $\theta \to \partial D$ or $|\theta| \to \infty$.

Theorem 13, p. 236 of [1] yields the following convergence result.

Theorem VI.1 Assume that Conditions A1 - A6 are satisfied, and ε is sufficiently small. Let $a \in \operatorname{int} D_0$, $X_m = x \in \mathbb{R}^k$, and consider the stopped process $\theta_n^\circ = \theta_{n\wedge\tau\wedge\sigma}$. Then for any $0 < \lambda < 1$ there exist constants B and s such that such that for all $m \geq 0$ we have $\lim \theta_n^\circ = \theta^*$ with probability at least

$$1 - B(1 + |x|^s) \sum_{n=m+1}^{+\infty} n^{-1-\lambda}.$$

VII. Convergence of the SA-procedure

In this section we outline the proof of the validity of Conditions A1-A3 for algorithm (7) for a stock market where the demand function of the agent satisfies a sector condition similar to that of Zames [12].

The proposed stochastic approximation procedure fits into the general BMP scheme by choosing the state vector

$$X_n(\theta) = (p_n(\theta), p_{n-1}(\theta), \ldots, p_{n-k}(\theta))^T.$$

The correction term will be obtained via the function $H : \mathbb{R}^k \times \mathbb{R}^{k+1} \mapsto \mathbb{R}^k$ defined by

$$H(\eta, z) = -(z^1, \ldots, z^k)^T(z^0 + \eta^1 z^1 + \ldots + \eta^k z^k).$$

Next we impose conditions on the stock-exchange model itself. For the *market dynamics* let us define the average absolute demand

$$\overline{d} = \sum_{n=1}^{N} |d^i|/N.$$

Similarly the average absolute deviation of demands will be

$$\overline{\delta d} = \sum_{n=1}^{N} |d^i - d'^i|/N.$$

For the vector of bid prices we use the sup-norm. The price matching function g is assumed to satisfy the following condition.

G. The price forming function satisfies a linear growth condition and a Lipschitz condition in the following sense: there exists an $L_g \in \mathbb{R}$ such that for any $d, d' \in \mathbb{R}^N$

$$|g(d, b)| \leq L_g(\overline{d} + |b|),$$
$$|g(d, b) - g(d', b)| \leq L_g \overline{\delta d}.$$

For the *price predictors* we consider a feasible set

$$D_0 \subset D = \{\ \eta \in \mathbb{R}^k :\ A(\eta) \text{ is stable}\}.$$

Empirical evidence on real data shows that the coefficient of the one-period lag is typically close to 1 while the other coefficients are close to 0. Thus a typical price predictor is 'close' to an $AR(1)$ predictor. Thus if our model is properly describing the mechanism of a stock-exchange then the set of feasible parameters can be chosen so that $|\eta^1 - 1|$ and $|\eta^i|$ are all small. Let $\delta_1 = \sup_{\eta\in D_0} |\eta^1 - 1|$ and for $i = 2, \ldots, k$ define

$$\delta_i = \sup_{\eta\in D_0} |\eta^i|.$$

Set

$$\Delta = \sum_{i=1}^{k} \delta_i.$$

Then we expect that Δ is small.

The *behavior* of the agent is characterized by the static non-linear element f_i. For a current price predictor η_n the i-th agent determines his/her transaction request according to

$$d_n^i = f_i(\hat{p}_n(\eta_n) - p_{n-1})$$

The demand functions f_i are assumed to satisfy the following sector condition.

F. There exists $L_f \in \mathbb{R}$ such that for any $x, y \in \mathbb{R}$ and any $1 \leq i \leq N$

$$|f_i(x)| \leq L_f |x| \text{ and } |f_i(x) - f_i(y)| \leq L_f |x - y|.$$

Theorem VII.1 Assume that $E|e_n|^q < \infty$ for any $q \geq 0$ and Conditions F and G hold. Then if

$$L_f L_g \Delta < 1,$$

then assumptions A1.-A3. are satisfied.

Remark: Note that if all agents have the same behavior and have approximately the same demand, then merging them into a single agent has the effect that L_f is multiplied while L_g is divided by approximately the same scalar.

Proof: For the proof we use a Lyapunov function argument. A new norm in $\mathbb{R}^{k+1}$ will be introduced to ensure the contraction of the norms of the state vector $X_n(\theta)$. Let $\varepsilon_1, \ldots, \varepsilon_k$ be positive constants and for $z \in \mathbb{R}^{k+1}$ define the new vector norm as

$$|z| := |z^0| + \sum_{i=1}^{k} \varepsilon_i |z^i|.$$

The key observation is that the input for the behavior is $\hat{p}_n - p_{n-1}$ which is small if Δ is small. Using the sector constants L_f and L_g we conclude that there exist positive constants $\varepsilon_1, \ldots, \varepsilon_k$ and $0 < \beta < 1$ such that using the norm defined above we have for any $\theta \in D_0$

$$|X_n(\theta)| \leq \beta |X_{n-1}(\theta)| + L_1 |e_n|.$$

The verification of A1 is now trivial. To verify A2. let $X_n(\theta)$ and $X'_n(\theta)$ denote the frozen-parameter processes with initial values $X_0 = x$ and $X'_0 = x'$, respectively. Then

$$|\Pi^n_\theta g(x) - \Pi^n_\theta g(x')| = |E[g(X_n(\theta)) - g(X'_n(\theta))]|.$$

Using $g \in Li(p)$ the right hand side can be bounded by

$$||\Delta g||_p \; E\left\{|X_n(\theta) - X'_n(\theta)|(1 + |X_n(\theta)|^p + |X'_n(\theta)|^p)\right\}.$$

Straightforward calculations yield

$$\begin{aligned} |X_n(\theta) - X'_n(\theta)| &= |p_n - p'_n| + \sum_{i=1}^{k} \varepsilon_i |p_{n-i} - p'_{n-i}| \leq \\ &\leq \beta |X_{n-1}(\theta) - X'_{n-1}(\theta)| \leq \ldots \leq \\ &\leq \beta^n |x - x'|. \end{aligned}$$

Thus the Markov chain $X_n(\theta)$ forgets its initial condition exponentially fast. Using A1. the validity of assumption A2. follows immediately.

Now let $X_n(\theta)$ and $X_n(\theta')$ denote the frozen-parameter processes with the same initial state $X_0 = X'_0 = x$. Exploiting the fact that $g \in Li(p)$, the difference $|\Pi^n_\theta g(x) - \Pi^n_{\theta'} g(x)|$ is easily seen to be majorated by

$$||\Delta g||_p \; E\left\{|X_n(\theta) - X_n(\theta')|(1 + |X_n(\theta)|^p + |X_n(\theta')|^p)\right\}.$$

From here easy calculations lead to

$$\begin{aligned} |X_n(\theta) - X_n(\theta')| \leq \beta |X_{n-1}(\theta) - X_{n-1}(\theta')| + \\ + K|\theta - \theta'||X_{n-1}(\theta)|. \end{aligned}$$

Iterating this inequality backwards we are led to

$$|X_n(\theta) - X_n(\theta')| \leq K|\theta - \theta'| \sum_{i=1}^{n} \beta^{i-1} |X_{n-i}(\theta)|.$$

Since $X_n(\theta)$ is L_q-bounded, so is $|X_n(\theta) - X_n(\theta')|/|\theta - \theta'|$. Using a Cauchy-Schwartz inequality and taking into account A1., we get the validity of assumption A3. ∎

Example. Assume the agent uses geometrically decreasing weights for prediction: $\delta_i = \alpha^i$ for some $0 < \alpha < 1$. Then assumptions A1.-A3. hold if the loop gain satisfies $L_f L_g < (1 - \alpha)/\alpha$.

VIII. Conclusion

A behavioral stock market model has been developed. The proposed model is a nonlinear stochastic feedback system. The control action of the agents is composed by choosing a behavior and a prediction step applied to the observed price process. The predictor is automatically adjusted in an iterative manner. Conditions for the convergence to a market equilibrium have been given using the BMP theory.

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