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Locally Optimal Takagi-Sugeno Fuzzy Controllers

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Abstract— **This paper presents a new suboptimal design for stable fuzzy controller. Using fuzzy plant model and locally optimal gains (based on LQR design), we suggest a sufficient condition for stability of the closed-loop system that can be expressed as Linear Matrix Inequalities. Provided that we have an accurate fuzzy model of the plant, the real system is globally stable too. This new approach facilitates the design process much yielding sub-optimal global design. Simulation results show that even if we have not an exact fuzzy model representation, this method increases attraction domain of the system comparing with conventional linear optimal controllers.**

I. Introduction

Fuzzy controllers have found popularity in many practical situations. Many difficult plants have been controlled very well using fuzzy controllers without any difficult analyses common in classical control design. In designing a controller (e.g. for driving a car), expert's knowledge (driver's ability to drive) can be used easily, which is not the case for classical controllers. Beside that, fuzzy controllers are general nonlinear controller, which their benefits are well-known [1]. In spite of these advantageous properties of fuzzy controllers, the main crisis of them was the absence of a formal method for proving the system's stability. However, after introducing the fuzzy plant modeling in [2], some methods for stable controller design have arisen. The approach is based on fuzzy modelling of a nonlinear plant as a sum of nonlinear-weighted linear subsystems. Following this approach, one can design a linear controller for each subsystem and satisfying some constraints expressible as Linear Matrix Inequalities (LMI), stability of the whole system can be proved [3]-[11]. This is what we call Takagi-Sugeno (T-S) fuzzy controller. The idea is similar to traditional gain scheduling method in which controlling gains change according to the state of the system [12]. The difference is that in T-S controllers, feedback gains changes gradually as the system's state changes and usually this property provides a better performance.

Although the common approach to proving stability of the fuzzy controller is well-defined, incorporating optimality criterion in this framework is difficult. Most researches seek for a stable controller and adding optimality measure such as H_2 or H_∞ make the constrained optimization problem very complex. In this paper, we propose a method to facilitate this process by using locally optimal design instead of trying to prove the stability and optimality of the whole controller. Doing so, we gain suboptimal solution with much less complicated procedure.

After this introduction, we present T-S plant model and propose a locally optimal fuzzy nonlinear controller in section II. In section III, we test the method on a nonlinear plant, and after that in section IV, conclusions are made.

II. Fuzzy Plant Model and Nonlinear State Feedback Controller

We consider a nonlinear controller with all system states accessible. We assume that the plant can be represented by a fuzzy plant model. Our goal is designing a nonlinear state feedback controller.

A. Fuzzy Plant Model

Jacobian Linearization is a common approximation for nonlinear plants. With the linearized plant, we can use linear control theory methods to design controllers. However, the main deficiency of this method is its local validity, i.e. there is no guarantee of being stable, or retaining the performance and robustness in state space places other than linearization point. However, the region of stability can be expanded by linearizing at different points of the state space and interpolating between them. It can be proved that we can approximate nonlinear plant using these nonlinearly combined linear plants with any accuracy, or more precisely, homogenous T-S model (that we will introduce it later) is dense in C^1 and general inhomogeneous T-S model is dense in C^3 [13]. In addition, if we have a physical model of the system, T-S fuzzy model can represent exact nonlinear dynamics of it (globally or semi-globally), although we may need too many rules for that representation ([8] and [5]). We do not follow this exact method in this paper.

Let r be the number of fuzzy rules describing the nonlinear plant. The i^{th} rule is of the following form

$$\begin{aligned} \mathrm{R_i} &: \text{If } x_1(t) \text{ is } \mathrm{M_1^i} \text{ and} \ldots \text{and } x_n(t) \text{ is } \mathrm{M_n^i} \\ &\quad \text{then } \dot{x}(t) = A_i x(t) + B_i u(t) \end{aligned} \tag{1}$$

where $x(t) \in \Re^n$ is system state variables, $u(t) \in \Re^m$ is system input, M_j^i s are fuzzy variables, and $A_i \in \Re^{n\times n}$ and

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$B_i \in \Re^{n\times m}$ are state transition matrix and input matrix, respectively. Let us use product operator as the t-norm operator of the antecedent part of rules and the center of mass method for defuzzification. If we denote $\mu_{M_i^k}(x_i(t))$ as the grade of membership of $x_i(t)$ to M_i^k, we can write

$$\dot{x}(t) = \sum_{i=1}^{r} h_i(x(t))(A_i x(t) + B_i u(t)) \tag{2}$$

in which

$$h_i(x) = \frac{w_i(x)}{\sum_{i=1}^{r} w_i(x)},$$
$$w_k(x) = \prod_{i=1}^{r} \mu_{M_i^k}(x_i(t)), \quad \sum_{i=1}^{r} h_i(x) = 1 \tag{3}$$

B. Quadratic Stability

After defining the model, we want to find the condition under which the system is stable. The following theorem provides a sufficient condition for quadratic stability.

Theorem 1. *The continuous uncontrolled ($u = 0$) fuzzy system of (1) and (2) is globally quadratically stable if there exists a common positive definite matrix $P = P^T$ such that*

$$A_i^T P + PA_i < 0, \quad \mathrm{i} = 1,...,\mathrm{r} \tag{4}$$

Proof. Select $V(x(t)) = x^T Px$ as a candidate for Lyapunov function. The derivative of V along system trajectory is

$$\dot{V}(x(t)) = \dot{x}^T Px + x^T P\dot{x} =$$
$$\sum_{i=1}^{r} \left[x^T A_i^T h_i^T(x) Px + x^T P h_i(x) A_i x\right]$$

noting that $h_i(x)$ is a scalar and rearranging, we have

$$\dot{V} = \sum_{i=}^{r} h_i(x) x^T \left[A_i^T P + PA_i\right]x$$

so if we can find a common Positive Definite (P.D.) matrix P that make $A_i^T P + PA_i$ Negative Definite (N.D.) for all A_i, $\dot{V} < 0$ and stability is guaranteed. ■

Finding a common P can be considered as LMI problem. There is no analytical method for solving it, but as it is a convex optimization problem, we can use numerical methods and be sure that if there is any solution, it can be found. We used Matlab LMI toolbox for solving this problem [14].

It is important to note that this theorem provides only sufficient conditions. Beside that, it has a conservative nature. The system may be asymptotically stable but not quadratic stable, and if it is the case, we cannot find common P though the system is stable. Moreover, we do not investigate the effect of $h_i(x)$ at all. We may design $h_i(x)$ in order to make the whole summation N.D. but $A_i^T P + PA_i$ s are not. It seems that it is useful to investigate this effect explicitly (Reference [4] used the effect of membership in his analysis but it was not a general treatment of the problem).

C. Parallel Distributed Compensation

Having T-S plant model, we can use Parallel Distributed Compensation (PDC) control defined as follows

$$\text{Control rule}_\text{i} : \text{If } x_1(t) \text{ is } \mathrm{M}_1^\mathrm{i} \text{ and ... and } x_n(t) \text{ is } \mathrm{M}_\mathrm{n}^\mathrm{i}$$
$$\text{then } \mathrm{u(t) = -K_i x(t)} \tag{5}$$

With similar assumption as T-S plant model, we can write control signal as

$$u = -\sum_{i=1}^{r} h_i(x) K_i x(t) \tag{6}$$

in which K_i s are state feedback gains. We can see it as local gains of gain scheduling design which overall control signal is made from combining each local control signal with different weights according to the closeness to the each rule's region.

We write closed-loop system by combining (2) and (6) as

$$\dot{x}(t) = \sum_{i=1}^{r} \sum_{j=1}^{r} h_i(x) h_j(x) \left[A_i - B_i K_j\right]x(t)$$
$$= \sum_{i=1}^{r} h_i^2(x) G_{ii} x(t) + \tag{7}$$
$$\sum_{i=1}^{r} \sum_{j=1}^{r} h_i(x) h_j(x). \left[\frac{G_{ij} + G_{ji}}{2}\right] x(t)$$

where $G_{ij} = A_i = B_i K_j$. It is easy to obtain the following result using Theorem 1.

Corollary 1: The fuzzy system (2) with fuzzy control of (6) is globally stable if there exists $P = P^T$ such that

$$\left(\frac{G_{ij} + G_{ji}}{2}\right)^T P + P\left(\frac{G_{ij} + G_{ji}}{2}\right) < 0,\ \mathrm{i, j} = 1,\ldots,\mathrm{r} \quad (8)$$

D. Locally Optimal Design

We proposed a sufficient condition for stability of T-S controller but how should we choose K_i s? There are some trends in literature: someone chose it using pole placement method and some other considered it as a variable of design. We suggest using locally optimal LQR design to obtain the gains and then look for some common P in order to prove the stability of the system. Although locally optimal design is not globally optimal too, but it may have a rather good sub-optimal performance, which is desirable, specially considering that designing optimal nonlinear controller is not a trivial task. We should mention that [15] used similar method of fuzzy gain scheduling but they did not use fuzzy plant model and no stability proof was given. In addition, [16] devised an evolutionary-based method for optimal fuzzy controller design, but it was different with our approach as it did not use T-S model and no stability proof was given too.

Steady state linear quadratic regulator (LQR) problem can be formulated as

$$u(t) = \arg\min_{u(t)} \left\{ \begin{array}{l} J\big(x(t),u(t)\big) = \\ \int_0^\infty \left[x^T Q x + u^T R u\right] dt \end{array} \right\} \quad (9)$$

It is not difficult to show that optimal gain is (e.g. [17])

$$K = R^{-1} B^T P \quad (10)$$

in which P is a solution of the following algebraic Ricatti equality:

$$PA + A^T P + Q - PBR^{-1}B^T P = 0 \quad (11)$$

For designing suboptimal T-S controller, we select some membership functions and linearize following system at centers of membership functions

$$\dot{x} = f(x,u) \quad (12)$$

where $x \in \Re^n$ and $u \in \Re^m$, in order to obtain A_i and B_i s of fuzzy plant model (1). After that, we calculate K_i of each subsystem using LQR design and then test for stability of the whole system. If it is not stable, we may change linearization points or feedback gains a little in order to attain stability.

III. Experiments

Designed closed-loop system was proved stable. What can we say about stability of real plant (not T-S model) and proposed controller? If we use an exact fuzzy model (such as [8] and [5]), the controlled system is stable too. On the other hand, if we approximate the model with a T-S model, we may have some modelling error which depends on the number of the rules, places of the rules, and the shape of membership functions. Because of modeling error, T-S plant model is not the same as the real one (12) and therefore we cannot be certain of the global stability of overall system. However, it is supposed that it has larger region of attraction than what we can get from a linear controller. As an example, we test our method on a nonlinear mass-spring-damper system (similar to [6]). The plant is defined as

$$M\ddot{x}(t) + g\big(x(t),\dot{x}(t)\big) + f\big(x(t)\big) = \phi\big(\dot{x}(t)\big)u(t) \quad (13)$$

where M is mass, $g(\cdot)$ is damper nonlinearity, $f(\cdot)$ is spring nonlinearity, $\phi(\cdot)$ is input nonlinearity and u is external force. Choosing nonlinearity as

$$\begin{cases} g\big(x(t),\dot{x}(t)\big) = \dot{x}(t) \\ f\big(x(t)\big) = 0.01x(t) + 0.1x(t)^3 \\ \phi\big(\dot{x}(t)\big) = 1.4387 - 0.13\dot{x}(t)^2 \end{cases} \quad (14)$$

and selecting $x_1 = x$ and $x_2 = \dot{x}_1$, we can write in state space form

$$\dot{\underline{x}}(t) = \begin{pmatrix} x_2 \\ -x_2 - 0.01x_1 - 0.1x_1^3 + \left(1.4387 - 0.13x_2^2\right)u \end{pmatrix} \quad (15)$$

Doing Jacobian linearization, we have

$$A = \left.\frac{\partial f(x,u)}{\partial x}\right|_{\substack{x=x_0\\u=0}} = \left.\begin{pmatrix} 0 & 1 \\ -0.01 - 0.3x_1^2 & -1 \end{pmatrix}\right|_{x_0} \quad (16)$$

$$B = \left.\frac{\partial f(x,u)}{\partial u}\right|_{x=x_0} = \left.\begin{pmatrix} 0 \\ 1.4387 - 0.13x_2^2 \end{pmatrix}\right|_{x_0} \quad (17)$$

where x_0 s are center points of membership functions. Here

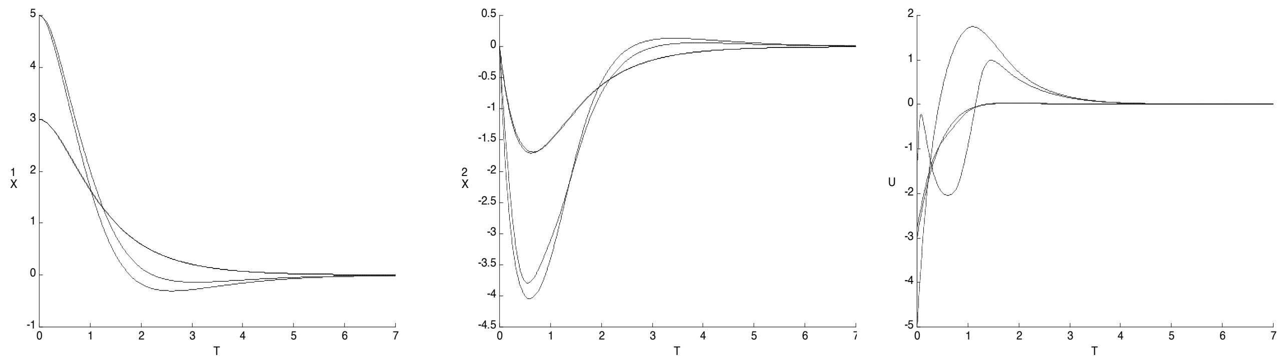

Fig. 1. Comparison of our method (bold) and linear (dotted) controllers with near the equilibrium initial conditions.

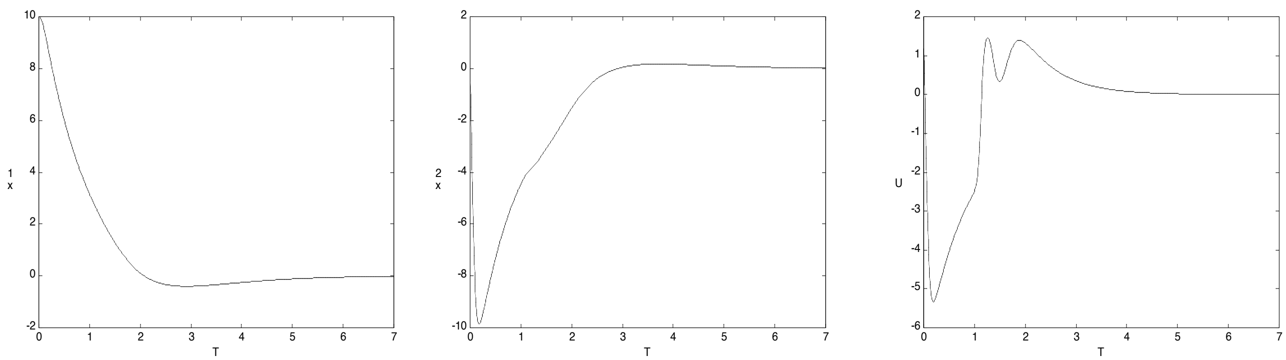

Fig. 2. System trajectory using our method in a region far from the equilibrium. Linear controller is unstable in this region.

we design locally optimal controllers and blend them using aforementioned method. In this experiment, we use five different linearization points placed at $(0 \quad 0)^T$, $(5 \quad 5)^T$, $(5 \quad -5)^T$, $(-5 \quad 5)^T$, and $(-5 \quad -5)^T$ (and correspondingly five rules) with exponential decaying membership function centers around those points as

$$\mu_{M^k}(x(t)) = \exp\left(-\frac{\|(x - x_c)\|^2}{2\sigma^2}\right). \tag{18}$$

In these experiments, we have selected $\sigma^2 = 3$. Using $Q = I_{2\times 2}$ and $R = 1$, we compare optimal linear design with this new method. You can see the comparison of LQR controller with T-S controller in Fig. 1. It is seen that when initial point is near to equilibrium, both of them is almost the same but as the distance increases, the difference becomes apparent. T-S controller has lower overshoot. Beside that linear controller is not stable whenever the initial condition is too far from equilibrium point (e.g. $\underline{x}_0 \in \left\{(x_{01} \quad 0)^T \mid x_{01} > 5.25\right\}$, but this nonlinear T-S controller is. As an example of the behavior of our fuzzy controller in far from equilibrium point, its response to $x_0 = (10 \quad 0)^T$ is depicted in Fig. 2. It is evident that the new controller brings a larger region of attraction.

In order to compare performance of linear controller with fuzzy controller, we have averaged performance measure (J of (9)) of these two methods over a region. The region is a rectangle ranged from $(-4 \quad -4)^T$ to $(4 \quad 4)^T$ which is divided into 1600 points (40×40 uniform grid divisions). In this region, both controllers are stable. We have done the test with three different performance measures: one with $\{Q = I_{2\times 2}, R = 1\}$ another with $\{Q = 10 I_{2\times 2}, R = 1\}$ and the last one with $\{Q = I_{2\times 2}, R = 10\}$. You can see the results in Table 1. Further investigations shows that T-S controller acts a little worse than linear controller when initial conditions are near zero point but is better when it is farther and its overall performance is somehow better except in the last case (Fig. 3). This is due to the effect of other not-

TABLE I
PERFORMANCE COMPARISON BETWEEN LINEAR AND TS CONTROLLERS

Performance measure	LQR	Locally optimal TS
$Q = I_{2\times 2}, R = 1$	5.8023	5.4715
$Q = 10 I_{2\times 2}, R = 1$	9.0696	8.4492
$Q = I_{2\times 2}, R = 10$	5.5805	5.6195

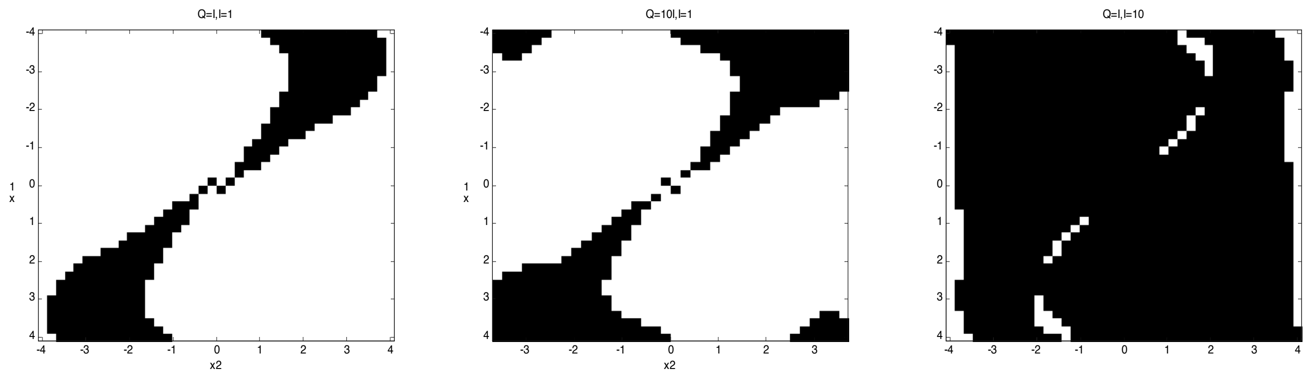

Fig. 3. Performance region comparison for different performance indices: (Q=1, R=1), (Q=10, R=1), and (Q=1, R=10), from left to right, respectively (dark region means linear one has better performance).

around-center controllers that change the control signal. This effect can be reduced by increasing the number of fuzzy rules and decreasing the width of membership functions.

IV. CONCLUSIONS

We have proposed a stable nonlinear fuzzy controller based on parallel distributed fuzzy controllers. Each sub-controller is LQR designed and provides local optimal solutions. If we have the exact fuzzy plant model, the system will be globally stable. Even if we do not have the exact model, the result is satisfactory and has a large attraction region. More investigations are needed in order to clarify the relation between model accuracy and stability region. Beside that, we use locally optimal linear controllers in our design. This is one way for designing suboptimal fuzzy controllers. There exists another method in which an upper bound of performance index that is expressed using LMI framework is minimized ([18] and [19]). This other method results in suboptimal controllers too. Comparison of these two methods will be done in our further researches.

REFERENCES

[1] J.-J. E. Slotine and W. Li. *Applied Nonlinear Control*. Englewood Cliffs, New Jersey: Prentice-Hall, 1991.

[2] T. Takagi and M. Sugeno, "Fuzzy identification of systems and its applications to modeling and control," *IEEE Trans. Syst., Man, Cybern.*, vol. 15, pp. 116-132, 1985.

[3] E. Kim and S. Kim, "Stability analysis and synthesis for an affine fuzzy control system via LMI and ILMI: continuous case," *IEEE Trans. Fuzzy Syst.*, vol. 10. no. 3, pp.391-400, 2002.

[4] H. K. Lam, F. H. F. Leung, and P. K. S. Tam, "Nonlinear state feedback controller for nonlinear systems: stability analysis and design based on fuzzy plant model," *IEEE Trans. Fuzzy Syst.*, vol. 9., no. 4, pp. 657-661, 2001.

[5] K.-Y. Lian, C.-S. Chiu, T.-S. Chiang, and P. Liu, "LMI-based fuzzy chaotic synchronization and communication," *IEEE Trans. Fuzzy Syst.*, vol. 9., no. 4, pp. 539-553, 2001.

[6] K. Tanaka, T. Ikeda, and H. O. Wang, "Robust stabilization of a class of uncertain nonlinear systems via fuzzy control: quadratic stability, H^{∞} control theory, and linear matrix inequalities," *IEEE Trans. Fuzzy Syst.*, vol. 4, pp. 1-13, 1996.

[7] H.O. Wang, K. Tanaka, and M. F. Griffin, "An approach to fuzzy control of nonlinear systems: stability and design issues," *IEEE Trans. Fuzzy Syst.*, vol. 4, pp. 14-23, 1996.

[8] T. Taniguchi, K. Tanaka, h. Ohtake, and H.Wang, "Model construction, rule reduction, and robust compensation for generalized form of takagi-sugeno fuzzy systems," *IEEE Trans. Fuzzy Syst.*, vol. 9, no. 4, pp. 525-538, 2001.

[9] C.-S. Tseng, B.-S. Chen, and H.-J. Uang, "Fuzzy tracking control design for nonlinear dynamics systems via T-S fuzzy model," *IEEE Trans. Fuzzy Syst.*, vol. 9, no. 3, pp. 381-292, 2001.

[10] H. D. Tuan, P. Apkarian, T. Narikiyo, and Y. Yamamoto, "Parameterized linear matrix inequality techniques in fuzzy control system design," *IEEE Trans. Fuzzy Syst.*, vol. 9, no. 2, 2001.

[11] J. Concha and A. Cipriano, "A design method for stable fuzzy lqr controllers," *IEEE Int. Conf. on Fuzzy Systems*, 1997.

[12] A. Packard, "Gain scheduling via linear fractional transformations," *Syst. Contr. Lett.*, vol. 22, pp. 79-92, 1994.

[13] C. Fantuzzi and R. Rovatti, "On the approximation capabilities of the homogeneous takagi-sugeno model," *5^{th} IEEE Int. Conf. Fuzzy Syst.*, 1996.

[14] P. Gahinet, A. Nemirovski, A. Laub, and M. Chilali, *LMI Control Toolbox,* The Math. Works Inc., 1994.

[15] M. Alata and K. Demirli, "Fuzzy control and gain scheduling-case study: robust stabilization of an inverted pendulum," *Joint 9^{th} IFSA World Cong. and 20th NAFIPS Int. Conf*, 2001.

[16] D. Kim and S. Rhee, "Design of an optimal fuzzy logic controller using response surface methodology," *IEEE Trans. Fuzzy Syst.*, vol. 9. no. 3, 2001.

[17] J. B. Burl, *Linear Optimal Control: H_2 and H_∞ methods*, Addison Wesley Longman, Inc., 1999.

[18] A. Jadbabaie, "Robust, non-fragile controller synthesis using model-based fuzzy systems: a linear matrix inequality approach," M.S. Thesis, The university of New Mexico, 1997.

[19] H.O. Wang, J. Li, D. Niemann, and K. Tanaka, "Fuzzy optimal control and its application to nonlinear benchmark problem," Duke Tech. Report, 1998.