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H_∞ Full Information Control of Discrete-time Systems with Multiple Input Delays

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Abstract— **In this paper, we present an explicit solution to H_∞ full-information control for discrete-time systems with multiple input delays. Our solution is given in terms of one standard Riccati difference equation of the same order as the plant under investigation. Thus, it has much computational advantage over methods such as system augmentation. As special cases, solutions to the H_∞ control problem for systems with single input delay and the H_∞ preview control are obtained.**

I. Introduction

H_∞ control for continuous-time systems with single input delay has been analytically solved in [7]. [16] has addressed the H_∞ control for a broader class of systems with delays in disturbance and control inputs, containing the H_∞ control with preview as a special case. Very recently, a complete solution to the H_∞ control with preview for both continuous-time and discrete-time systems have been proposed in [8], [9] whereas [6] is concerned with systems with multiple input/output delays and a nested set of solutions to the so-called adobe delay problems.

In the discrete-time context, the control problem for systems with input delays has also received some renewed interests due to the applications in network congestion control and networked control systems; see, e.g. [1], [10], [21]. For discrete-time systems with delays, one might tend to consider augmenting the system and convert a delay problem into a delay free problem. While it is certainly possible to do so, the augmentation approach, however, generally results in higher state dimension and thus high computational cost, especially when the system under investigation involves multiple delays and the delays are large [9]. Further, in the state feedback case, the augmentation approach generally leads to a static output feedback control problem which is non-convex; see the work of [21].

In this paper, the H_∞ full-information control problem for linear discrete time-varying systems with multiple constant input delays is investigated. We present a simple solution in terms of the solution of one standard Riccati difference equation of the same order as the original plant. Our approach is based on converting the problem into an optimization problem in Krein space and showing that the H_∞ control problem is in fact a dual problem of H_∞ fixed-lag smoothing. As special cases of the full-information control problem, solutions to the H_∞ control of systems with single input delay and H_∞ control with preview are obtained. Note that the H_∞ control with preview has been solved in [9] using a different approach.

The rest of the paper is organized as follows. In Section 2, the system under consideration and the H_∞ control problem is formulated. Our solution to the H_∞ full-information control and the discussions on the special cases of systems with single input delay and the H_∞ control with preview are presented in Section 3. Some conclusions are drawn in Section 4. Due to the page limitation, all proofs shall be omitted. Their details can be found in [24].

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II. Problem Statement

We consider the following discrete linear time-varying system for the H_∞ control problem.

$$\begin{aligned} x(t+1) &= \Phi_t x(t) + \sum_{i=0}^{d} B_{i,t} w_i(t-h_i) \\ &\quad + \sum_{i=0}^{d} C_{i,t} v_i(t-h_i),\ d \geq 1, \quad (1) \\ s(t) &= L_t x(t), \quad (2) \end{aligned}$$

where $x(t) \in \mathcal{R}^n, w_i(t) \in \mathcal{R}^{m_{i,w}}, v_i(t) \in \mathcal{R}^{m_{i,v}}$, and $s(t) \in \mathcal{R}^r$ represent the state, the exogenous input, control input and the controlled signal, respectively. $\Phi_t,\ B_{i,t},\ C_{i,t}$, and L_t are bounded time-varying matrices. It is assumed that the input noises are deterministic signals and are from $\ell_2[0, N]$ where N is the time-horizon of the control problem under investigation. Without loss of generality, we assume that the delays are in an increasing order: $0 = h_0 < h_1 < \cdots < h_d$ and the control inputs $v_i,\ i = 0, 1, \cdots, d$ and the exogenous inputs $w_i,\ i = 0, 1, \cdots, d$ respectively have the same dimension, i.e., $m_{0,v} = m_{1,v} = \cdots = m_{d,v} = m_v$ and $m_{0,w} = m_{1,w} = \cdots = m_{d,w} = m_w$.

The H_∞ full-information control under investigation is stated as follows: *Find a finite-horizon full-information control strategy* $v_i(t) = \mathcal{F}_i(x(0),\ (w_j(\tau), v_j(\tau)) \mid_{0 \leq j \leq d, 0 \leq \tau < t})$, *such that*

$$\sup_{\{x(0), w_j(t) |_{0 \leq j \leq d, 0 \leq t < N-h_j}\}} J(x(0), w_j(t), v_j(t)) < \gamma^2 \quad (3)$$

where

$$J(x(0), w_j(t), v_j(t)) = \frac{x'(N+1)P_{N+1}x(N+1) + \sum_{i=0}^{d}\sum_{t=0}^{N-h_i} v_i(t)'R^v_{i,t}v_i(t) + \sum_{t=1}^{N} s'(t)Q_t s(t)}{x'(0)\Pi_0^{-1}x(0) + \sum_{i=0}^{d}\sum_{t=0}^{N-h_i} w_i'(t)R^w_{i,t}w_i(t)}, \tag{4}$$

and P_{N+1} is a given positive definite matrix which reflects the uncertainty of the initial state relative to the exogenous input.

Remark 1: For linear continuous-time systems with single input delay, Tadmor [7] has presented a complete solution to the H_∞ control problem for both the state and output feedback cases whereas [16] has studied the H_∞ full-information control for systems with multiple input delays and a solution is given in terms of a Riccati equation and a matrix differential equation. The present study can be viewed as the discrete-time counterpart of [16]. We present an explicit solution in terms of a standard Riccati difference equation of the same order as the original plant (ignoring the delays).

III. Solution to H_∞ Full Information Control

Consider the performance index (3) and define

$$J_N^\infty \triangleq x'(0)\Pi_0^{-1}x(0) - \gamma^{-2}J_N, \tag{5}$$

where

$$\begin{aligned} J_N &= x'(N+1)P_{N+1}x(N+1) \\ &+ \sum_{i=0}^{d}\sum_{t=0}^{N-h_i} u_i'(t)R_{i,t}u_i(t) + \sum_{t=1}^{N} s'(t)Q_t s(t), \end{aligned} \tag{6}$$

with

$$R_{i,t} = diag\{R^v_{i,t}, \ -\gamma^2 R^w_{i,t}\}, \tag{7}$$

$$u_i(t) = \begin{bmatrix} v_i(t) \\ w_i(t) \end{bmatrix}. \tag{8}$$

Also, we can rewrite the system (1) as

$$x(t+1) = \Phi_t x(t) + \sum_{i=0}^{d}\Gamma_{i,t}u_i(t-h_i), \ d \ge 1, \tag{9}$$

where

$$\Gamma_{i,t} = [\, B_{i,t} \quad C_{i,t} \,]. \tag{10}$$

It is clear that an H_∞ controller $v_i(t)$ that achieves (3) exists if and only if there exists

$$v_i(t) = \mathcal{F}_{i,t}(x(0), (w_j(\tau), v_j(\tau)) \mid_{0\le j\le d, 0\le\tau<t}), \tag{11}$$

such that J_N^∞ of (5) is positive for all non-zero $\{x(0);\ w_i(t), 0 \le t \le N-h_i, 0 \le i \le d\}$.

For any given $\tau \ge 0$, denote:

$$u^\tau(t) \triangleq \begin{cases} \begin{bmatrix} u_0(t+\tau-h_0) \\ \vdots \\ u_i(t+\tau-h_i) \end{bmatrix}, & h_i \le t < h_{i+1}, \\ \begin{bmatrix} u_0(t+\tau-h_0) \\ \vdots \\ u_d(t+\tau-h_d) \end{bmatrix}, & t \ge h_d \end{cases} \tag{12}$$

$$\bar{u}^\tau(t) \triangleq \begin{cases} \sum_{j=i+1}^{d}\Gamma_{j,t+\tau}u_j(t+\tau-h_j), & h_i \le t < h_{i+1}, \\ 0, & t \ge h_d \end{cases} \tag{13}$$

$$\Gamma_t^\tau \triangleq \begin{cases} [\, \Gamma_{0,t+\tau} \ \cdots \ \Gamma_{i,t+\tau} \,], & h_i \le t < h_{i+1} \\ [\, \Gamma_{0,t+\tau} \ \cdots \ \Gamma_{d,t+\tau} \,], & t \ge h_d, \end{cases} \tag{14}$$

$$R_t^\tau \triangleq \begin{cases} diag\{R_{0,t+\tau-h_0}, \cdots, R_{i,t+\tau-h_i}\}, & h_i \le t < h_{i+1}, \\ diag\{R_{0,t+\tau-h_0}, \cdots, R_{d,t+\tau-h_d}\}, & t \ge h_d. \end{cases} \tag{15}$$

Using the notations of (12)-(15), for a given $\tau \ge 0$, the system (1)-(2) and the cost (3) can be rewritten respectively as

$$x(t+\tau+1) = \begin{cases} \Phi_{t+\tau}x(t+\tau) + \Gamma_t^\tau u^\tau(t) + \bar{u}^\tau(t), & h_i \le t < h_{i+1}, \\ \Phi_{t+\tau}x(t+\tau) + \Gamma_t^\tau u^\tau(t), & t \ge h_d \end{cases} \tag{16}$$

and

$$\begin{aligned} J_N &= J_N^\tau + \sum_{i=0}^{d}\sum_{t=0}^{\tau-1} u_i(t)'R_{i,t}u_i(t) \\ &+ \sum_{t=1}^{\tau} s'(t)Q_t s(t), \end{aligned} \tag{17}$$

where

$$\begin{aligned} J_N^\tau &= x'(N+1)P_{N+1}x(N+1) + \sum_{t=0}^{N-\tau} u^\tau(t)'R_t^\tau u^\tau(t) \\ &+ \sum_{t=1}^{N-\tau} s'(t+\tau)Q_{t+\tau}s(t+\tau). \end{aligned} \tag{18}$$

Define the following stochastic state-space model associated with (18):

$$\mathbf{x}^\tau(t) = \Phi_t^{\tau'}\mathbf{x}^\tau(t+1) + L_t^{\tau'}\mathbf{u}^\tau(t), \tag{19}$$

$$\mathbf{y}^\tau(t) = \Gamma_t^{\tau'}\mathbf{x}(t+\tau+1) + \mathbf{v}^\tau(t), \tag{20}$$

where $\Phi_t^\tau = \Phi_{t+\tau}$, $L_t^\tau = L_{t+\tau}$, $\langle \mathbf{u}^\tau(t), \mathbf{u}^\tau(t)\rangle = Q_t^\tau$ and $\langle \mathbf{v}^\tau(t), \mathbf{v}^\tau(t)\rangle = R_t^\tau$. It is easy to see that $Q_t^\tau = Q_{t+\tau}$.

Let

$$u^\tau = col\{u^\tau(0), \cdots, u^\tau(N-\tau)\}, \tag{21}$$

$$\mathbf{y}^\tau = col\{\mathbf{y}^\tau(0), \cdots, \mathbf{y}^\tau(N-\tau)\}, \tag{22}$$

$$\mathbf{x}_0^\tau = col\{\mathbf{x}^\tau(0), \mathbf{x}^\tau(1), \cdots, \mathbf{x}^\tau(h_d)\}, \tag{23}$$

Then, by completing the square for J_N^τ, we have

Lemma 1:

$$J_N^\tau = \xi^{\tau\prime}\mathcal{P}^\tau\xi^\tau + (u^\tau - u^{\tau*})' R_{\mathbf{y}_a^\tau}(u^\tau - u^{\tau*}), \quad (24)$$

where

$$\begin{aligned} R_{\mathbf{y}^\tau} &= \langle \mathbf{y}^\tau, \mathbf{y}^\tau \rangle, \\ \xi^\tau &= col\{x(\tau), \bar{u}^\tau(0), \cdots, \bar{u}^\tau(h_d-1)\}, && (25)\\ \mathcal{P}^\tau &= \langle \mathbf{x}_0^\tau - \hat{\mathbf{x}}_0^\tau,\ \mathbf{x}_0^\tau - \hat{\mathbf{x}}_0^\tau \rangle, && (26)\\ u^{\tau*} &= col\{u^{\tau*}(0), \cdots, u^{\tau*}(N-\tau)\}, && (27) \end{aligned}$$

and for $t < h_d$

$$\begin{aligned} u^{\tau*}(t) &= -[\mathcal{F}_0^\tau(t)]' x(\tau) - \sum_{l=1}^{t} [\mathcal{F}_l^\tau(t-l)]' \bar{u}^\tau(l-1) \\ &\quad - \sum_{l=t+1}^{h_d} [\mathcal{S}_l^\tau(t)]' \bar{u}^\tau(l-1), \quad (28) \end{aligned}$$

and for $t \ge h_d$,

$$u^{\tau*}(t) = -[\mathcal{F}_0^\tau(t)]' x(\tau) - \sum_{l=1}^{h_d} [\mathcal{F}_l^\tau(t-l)]' \bar{u}^\tau(l-1), \quad (29)$$

while $\mathcal{S}_l^\tau(\cdot)$ and $\mathcal{F}_l^\tau(\cdot)$ are given as

$$\begin{aligned} \mathcal{S}_l^\tau(t) &= P_l^\tau\left[(\bar{\Phi}_{t+1,l}^\tau)'\Gamma_t^\tau(M_t^\tau)^{-1} - (\bar{\Phi}_{t,l}^\tau)'G^\tau(t)K_t^\tau\right], \\ &\quad \bar{\Phi}_{0,l}^\tau = 0, \quad 0 \le t \le l-1, \\ \mathcal{F}_l^\tau(t) &= [I_n - P_l^\tau G^\tau(l)]\,\bar{\Phi}_{l,t}^\tau K_t^\tau, \quad l \le t \le N, \quad (30) \end{aligned}$$

and

$$G^\tau(t) = \sum_{j=1}^{t}(\bar{\Phi}_{j,t}^\tau)'\Gamma_{j-1}^\tau(M_{j-1}^\tau)^{-1}(\Gamma_{j-1}^\tau)'\bar{\Phi}_{j,t}^\tau, \quad (31)$$

while $\bar{\Phi}_{m,m}^\tau = I$ and

$$\begin{aligned} \bar{\Phi}_{j,m}^\tau &= \bar{\Phi}_j^\tau \cdots \bar{\Phi}_{m-1}^\tau, \ m \ge j, && (32)\\ \bar{\Phi}_j^\tau &= \Phi_{\tau+j}' - K_j^\tau(\Gamma_j^\tau)', && (33)\\ K_j^\tau &= \Phi_{\tau+j}'P_{j+1}^\tau\Gamma_j^\tau(M_j^\tau)^{-1}, && (34)\\ M_j^\tau &= R_j^\tau + (\Gamma_j^\tau)'P_{j+1}^\tau\Gamma_j^\tau. && (35) \end{aligned}$$

In the above, the matrix P_{j+1}^τ obeys the following backward RDE

$$\begin{aligned} P_j^\tau &= \Phi_{\tau+j}'P_{j+1}^\tau\Phi_{\tau+j} + L_{\tau+j}'Q_{\tau+j}L_{\tau+j} \\ &\quad - K_j^\tau M_j^\tau(K_j^\tau)', \quad P_{N-\tau+1}^\tau = P_{N+1}, \quad (36) \end{aligned}$$

with terminal condition P_{N+1} as given in (1).

Remark 2: In view of (12), $u_i^{\tau*}(t)$ is given by

$$u_i^{\tau*}(t) = \overbrace{[0\ \cdots\ 0\ I_m]}^{i+1 \text{ blocks}} u^{\tau*}(t+h_i). \quad (37)$$

We further denote

$$\bar{u} \triangleq col\{\bar{u}(0), \cdots, \bar{u}(N)\}, \quad (38)$$

where

$$\bar{u}(t) \triangleq \begin{cases} col\{u_0(t), \cdots, u_d(t)\}, & 0 \le t \le N-h_d, \\ col\{u_0(t), \cdots, u_i(t)\}, & \\ \qquad N-h_{i+1} < t \le N-h_i \end{cases} \quad (39)$$

Theorem 1: With the definition (39) and (38), the linear quadratic form J_N can be rewritten as

$$\begin{aligned} &J_N = \xi^{0\prime}\mathcal{P}\xi^0 + \\ &\sum_{\tau=0}^{N}\left\{\bar{u}(\tau) - \bar{u}^{\tau*}(0)\mid_{u_i(s)=u_i^*(s)(\ 0\le s<\tau;\ 0\le i\le d)}\right\}'\bar{M}_\tau \\ &\times\left\{\bar{u}(\tau) - \bar{u}^{\tau*}(0)\mid_{u_i(s)=u_i^*(s)(\ 0\le s<\tau;\ 0\le i\le d)}\right\}, \end{aligned} \quad (40)$$

where $\bar{u}^{\tau*}(0)$ is obtained from $\bar{u}(\tau)$ with the replacement of $u_i(\tau)$ by $u_i^{\tau*}(0)$, and $u_i^{\tau*}(0)$ is given by (28)-(29).

In addition, for $\tau \le N - h_d$, the covariance matrix $\bar{M}_\tau$ is given by

$$\begin{aligned} \bar{M}_\tau &= diag\{\Gamma_{0,\tau}', \cdots, \Gamma_{d,\tau}'\}\left[\bar{P}_{\tau+1}(i,j)\right]_{(d+1)\times(d+1)} \times \\ &\quad diag\{\Gamma_{0,\tau}, \cdots, \Gamma_{d,\tau}\} + diag\{R_{0,\tau}, \cdots, R_{d,\tau}\}, \end{aligned} \quad (41)$$

where $\bar{P}_\tau(i,j) = \bar{P}_\tau'(j,i)$, and for $i \ge j$, $\bar{P}_\tau(i,j)$ is given by

$$\bar{P}_\tau(i,j) = P_{h_i}^\tau(\bar{\Phi}_{h_j,h_i}^\tau)'\left[I - G^\tau(h_j)P_{h_j}^\tau\right], \quad (42)$$

where $\bar{\Phi}_{h_j,h_i}^\tau$ and $G^\tau(h_j)$ are given respectively in (32) and (31), and P_t^τ satisfies the Riccati equation (36).

For $N - h_{l+1} < \tau \le N - h_l$, the covariance matrix $\bar{M}_\tau$ is given by

$$\begin{aligned} \bar{M}_\tau &= diag\{\Gamma_{0,\tau}', \cdots, \Gamma_{l,\tau}'\}\left[\bar{P}_{\tau+1}(i,j)\right]_{(l+1)\times(l+1)} \\ &\quad \times diag\{\Gamma_{0,\tau}, \cdots, \Gamma_{l,\tau}\} + diag\{R_{0,\tau}, \cdots, R_{l,\tau}\}, \quad (43) \end{aligned}$$

where $\bar{P}_\tau(i,j)$ is calculated by (42).

Lemma 2: Assume that $u_i(t) = 0$ for $t < 0$. Denote

$$\bar{u}_r(t) \triangleq \begin{bmatrix} \bar{v}(t) \\ \bar{w}(t) \end{bmatrix}, \quad (44)$$

with

$$\bar{v}(t) \triangleq \begin{cases} col\{v_0(t), \cdots, v_d(t)\}, & 0 \le t \le N-h_d, \\ col\{v_0(t), \cdots, v_i(t)\}, & \\ \qquad N-h_{i+1} < t \le N-h_i \end{cases} \quad (45)$$

and

$$\bar{w}(t) \triangleq \begin{cases} col\{w_0(t), \cdots, w_d(t)\}, & 0 \le t \le N-h_d, \\ col\{w_0(t), \cdots, w_i(t)\}, & \\ \qquad N-h_{i+1} < t \le N-h_i. \end{cases} \quad (46)$$

Then the linear quadratic form J_N of (18) can be rewritten as

$$\begin{aligned} J_N &= x'(0)P_0x(0) + \\ &\sum_{\tau=0}^{N}\left[\bar{u}_r(\tau) - \bar{u}_r^{\tau*}(0)\mid_{u_i(s)=u_i^*(s)(\ 0\le s<\tau;\ 0\le i\le d)}\right]'\tilde{M}_\tau \\ &\times\left[\bar{u}_r(\tau) - \bar{u}_r^{\tau*}(0)\mid_{u_i(s)=u_i^*(s)(\ 0\le s<\tau;\ 0\le i\le d)}\right], \end{aligned} \quad (47)$$

where

- $\bar{u}_r^{\tau*}(0)$ is obtained from $\bar{u}_r(\tau)$ with $w_i(\tau)$ and $v_i(\tau)$ replaced by $w_i^{\tau*}(0) \triangleq [0, I_{m_w}]u_i^{\tau*}(0)$ and $v_i^{\tau*}(0) \triangleq [I_{m_v}, 0]u_i^{\tau*}(0)$, respectively, while $u_i^{\tau*}(0)$ is given in (28).
- The matrix $\tilde{M}_t$ is calculated by
  - For $t \le N - h_d$
  $$\begin{aligned} \tilde{M}_t &= \Theta'_{d,t}\left[\bar{P}_{t+1}(i,j)\right]_{(d+1)\times(d+1)}\Theta_{d,t} \\ &+diag\{R^v_{0,t}\cdots R^v_{d,t}; -\gamma^2R^w_{0,t}\cdots-\gamma^2R^w_{d,t}\}, \end{aligned} \quad (48)$$
  - For $N - h_{i+1} < t \le N - h_i$
  $$\begin{aligned} \tilde{M}_t &= \Theta'_{i,t}\left[\bar{P}_{t+1}(i,j)\right]_{(i+1)\times(i+1)}\Theta_{i,t} \\ &+diag\{R^v_{0,t}\cdots R^v_{i,t}; -\gamma^2R^w_{0,t}\cdots-\gamma^2R^w_{i,t}\}. \end{aligned} \quad (49)$$

  In the above, $\Theta_{i,t} = diag\{B_{0,t}\cdots B_{i,t}; C_{0,t}\cdots C_{i,t}\}$ and $\bar{P}_{t+1}(i,j)$ is given by (42).

Introduce the following LDU factorization for the covariance matrix $\tilde{M}_t$ as

$$\begin{aligned} \tilde{M}_t &\equiv \begin{bmatrix}\tilde{M}_{1,1}(t) & \tilde{M}_{1,2}(t)\\ \tilde{M}_{2,1}(t) & \tilde{M}_{2,2}(t)\end{bmatrix} = \begin{bmatrix} I & \tilde{M}_{1,2}(t)\tilde{M}_{2,2}^{-1}(t)\\ 0 & I\end{bmatrix} \\ &\times\begin{bmatrix}\Delta(t) & 0\\ 0 & \tilde{M}_{2,2}(t)\end{bmatrix}\begin{bmatrix} I & 0\\ \tilde{M}_{2,2}^{-1}(t)\tilde{M}_{2,1}(t) & I\end{bmatrix}, \end{aligned} \quad (50)$$

where

$$\Delta(t) = \tilde{M}_{1,1}(t) - \tilde{M}_{1,2}(t)\tilde{M}_{2,2}^{-1}(t)\tilde{M}_{2,1}(t). \quad (51)$$

The main result of this section follows

Theorem 2: Consider the system (1)-(2) and the performance (3). If P_j^τ is the bounded solution to the backward recursive Riccati equation to (36). Then an H_∞ controller that achieves performance (3) exists if and only if

$$\begin{aligned} \Pi^{-1} - \gamma^{-2}P_0^0 &> 0, & (52)\\ \tilde{M}_{2,2}(t) &< 0, & (53) \end{aligned}$$

where P_0^0 is the terminal value of P_0^τ of (36), and $\tilde{M}_{2,2}(t)$ is $(2,2)$-block of $\tilde{M}_t$ which is given by (48) for $t \le N - h_d$ or (49) for $N - h_{i+1} < t \le N - h_i$. In this case, a suitable H_∞ controller $v_i^*(\tau)$ is given by

$$v_i^*(\tau) = [I_{m_v}, 0]u_i^*(\tau), \quad (54)$$

where $u_i^*(\tau)$, $i = 0, 1, \cdots, d$, is calculated by

$$\begin{aligned} u_i^*(\tau) &= -\overbrace{[0\ \cdots\ 0\ I_m]}^{i+1\text{ blocks}} \\ &\times\Bigg\{-\left[\mathcal{F}_0^\tau(h_i)\right]'x(\tau) - \sum_{l=1}^{h_i}\left[\mathcal{F}_l^\tau(h_i-l)\right]'\bar{u}^{\tau*}(l-1) \\ &\qquad - \sum_{l=h_i+1}^{h_d}\left[\mathcal{S}_l^\tau(h_i)\right]'\bar{u}^{\tau*}(l-1)\Bigg\}, \end{aligned} \quad (55)$$

while $\mathcal{S}_\cdot^\tau(\cdot)$ and $\mathcal{F}_\cdot^\tau(\cdot)$ are given in (30), and

$$\bar{u}^{\tau*}(t) \triangleq \begin{cases} \sum\limits_{j=i+1}^{d}\Gamma_{j,t+\tau}u_j^*(t+\tau-h_j), \\ \quad h_i \le t < h_{i+1},\ i = 0, 1, \cdots, d-1 \\ 0, \qquad t \ge h_d \end{cases} \quad (56)$$

A. Discussion on special cases

1) H_∞ control for single input delay systems : In this subsection we consider the system (1) with $B_{i,t} = 0$ for $i > 0$ and $C_{0,t} = 0$ and $C_{i,t} = 0$ for $i > 1$, i.e.,

$$x(t+1) = \Phi_t x(t) + B_{0,t}w_0(t) + C_{1,t}v_1(t-h_1). \quad (57)$$

The cost function of (4) is simplified as

$$\begin{aligned} &J(x(0), w_0(t), v_1(t)) = \\ &\frac{x'(N+1)P(N+1)x(N+1) + \sum_{t=0}^{N-h_1}v_1'(t)R^v_{1,t}v_1(t) + \sum_{t=0}^{N}s'(t)Q_ts(t)}{x'(0)\Pi_0^{-1}x(0) + \sum_{t=0}^{N}w_0'(t)R^w_{0,t}w_0(t)} \end{aligned} \quad (58)$$

Associated with the system (57) and cost function (58), we introduce the following notations: $R_{0,t} \triangleq -\gamma^2R^w_{0,t}$, $R_{1,t} \triangleq R^v_{1,t}$, $u_0(t) \triangleq w_0(t)$, $u_1(t) \triangleq v_1(t)$, $\Gamma_{0,t} \triangleq B_{0,t}$, $\Gamma_{1,t} \triangleq C_{1,t}$, and

$$u^\tau(t) \triangleq \begin{cases} u_0(t+\tau), & 0 \le t < h_1, \\ \begin{bmatrix} u_0(t+\tau) \\ u_1(t+\tau-h_1)\end{bmatrix}, & t \ge h_1 \end{cases} \quad (59)$$

$$\bar{u}^\tau(t) \triangleq \begin{cases} C_{1,t+\tau}v_1(t+\tau-h_1), & 0 \le t < h_1, \\ 0, & t \ge h_1 \end{cases} \quad (60)$$

$$\Gamma_t^\tau \triangleq \begin{cases} \Gamma_{0,t+\tau}, & 0 \le t < h_1 \\ \left[\Gamma_{0,t+\tau}\quad \Gamma_{1,t+\tau}\right], & t \ge h_1 \end{cases} \quad (61)$$

$$R_t^\tau \triangleq \begin{cases} R_{0,t+\tau}, & 0 \le t < h_1, \\ diag\{R_{0,t+\tau}, R_{1,t+\tau-h_1}\}, & t \ge h_1. \end{cases} \quad (62)$$

Following a similar discussion as in Theorem 2, we have

Theorem 3: Suppose the following Riccati equation

$$\begin{aligned} P_j^\tau &= \Phi'_{\tau+j}P_{j+1}^\tau\Phi_{\tau+j} + L'_{\tau+j}Q_{\tau+j}L_{\tau+j} \\ &\quad - K_j^\tau M_j^\tau(K_j^\tau)', \quad P_{N-\tau+1}^\tau = P_{N+1} \end{aligned} \quad (63)$$

where

$$\begin{aligned} K_j^\tau &= \Phi'_{\tau+j} P_{j+1}^\tau \Gamma_j^\tau (M_j^\tau)^{-1}, & (64)\\ M_j^\tau &= R_j^\tau + (\Gamma_j^\tau)' P_{j+1}^\tau \Gamma_j^\tau, & (65)\end{aligned}$$

with given boundary condition P_{N+1}, has a bounded solution for any given $\tau \geq 0$. Then there exists a controller that solves the H_∞ control problem if and only if

$$\begin{aligned} \Pi^{-1} - \gamma^{-2} P_0^0 &> 0, & (66)\\ \bar{M}_{1,1}(t) &< 0, & (67)\end{aligned}$$

where $P_0^0 = P_0$ is the terminal value of P_0^τ of (63) for $\tau = 0$, and for $t \leq N - h_1$, $\bar{M}_{1,1}(t)$ is the (1,1)-block of matrix $\bar{M}_t$ which is given by

$$\begin{aligned} \bar{M}_t &= diag\{B'_{0,t}, C'_{1,t}\} \left[\bar{P}_{t+1}(i,j)\right]_{2\times 2} diag\{B_{0,t}, C_{1,t}\}\\ &\quad + diag\{-\gamma^2 R_{0,t}^w, R_{1,t}^v\} \qquad (68)\end{aligned}$$

and for $N - h_1 < t \leq N$,

$$\bar{M}_{1,1}(t) = \bar{M}_t = B'_{0,t} \bar{P}_{t+1}(0,0) B_{0,t} - \gamma^2 R_{0,t}^w, \qquad (69)$$

where $\bar{P}_{t+1}(i,j)$ is as

$$\begin{aligned} \bar{P}_\tau(0,0) &= P_0^\tau,\\ \bar{P}_\tau(1,0) &= \bar{P}'_\tau(0,1) = P_{h_1}^\tau (\bar{\Phi}_{0,h_1}^\tau)',\\ \bar{P}_\tau(1,1) &= P_{h_1}^\tau \left[I - G^\tau(h_1) P_{h_1}^\tau\right].\end{aligned}$$

In this situation, a suitable H_∞ controller is given by

$$\begin{aligned} v_1^*(\tau) &= -[0, I_{m_v}] \times \Big([\mathcal{F}_0^\tau(h_1)]' x(\tau)\\ &\quad + \sum_{l=1}^{h_1} [\mathcal{F}_l^\tau(h_1 - l)]' C_{1,\tau+l-1} v_1^*(\tau + l - h_1 - 1) \Big), \qquad (70)\end{aligned}$$

where $\mathcal{F}_l^\tau(.)$ is as (30).

2) H_∞ preview control: In this subsection we consider the system (1)-(2) with $B_{0,t} = B_{i,t} = 0$ for $i > 1$ and $B_{1,t} \neq 0$, $C_{0,t} \neq 0$ and $C_{i,t} = 0$ for $i > 1$, i.e.,

$$\begin{aligned} x(t+1) &= \Phi_t x(t) + B_{1,t} w_1(t - h_1) + C_{0,t} v_0(t), & (71)\\ s(t) &= L_t x(t). & (72)\end{aligned}$$

The H_∞ preview control for system (71)-(72) is described as: *Find a finite-horizon full-information H_∞ sub-optimal control strategy $v_0(t)$, such that the following is satisfied*

$$\sup_{\{x(0), w_1(\tau)|_{0 \leq \tau < N - h_1}\}} J(x(0), v_0(t), w_1(t)) < \gamma^2 \qquad (73)$$

where

$$J(x(0), v_0(t), w_1(t)) = \frac{x'(N+1) P_{N+1} x(N+1) + \sum_{t=0}^{N} v_0(t)' R_{0,t}^v v_0(t) + \sum_{t=1}^{N} s'(t) Q_t s(t)}{x'(0) \Pi_0^{-1} x(0) + \sum_{t=0}^{N-h_1} w_1'(t) R_{1,t}^w w_1(t)} \qquad (74)$$

Associated with the system (71) and the cost function (74), we introduce the following notations, $R_{0,t} \triangleq R_{0,t}^v$, $R_{1,t} \triangleq -\gamma^2 R_{1,t}^w$, $u_0(t) \triangleq v_0(t)$, $u_1(t) \triangleq w_1(t)$, $\Gamma_{0,t} \triangleq C_{0,t}$, $\Gamma_{1,t} \triangleq B_{1,t}$, and

$$u^\tau(t) \triangleq \begin{cases} u_0(t+\tau), & 0 \leq t < h_1\\ \begin{bmatrix} u_0(t+\tau)\\ u_1(t+\tau-h_1)\end{bmatrix}, & t \geq h_1 \end{cases} \qquad (75)$$

$$\bar{u}^\tau(t) \triangleq \begin{cases} B_{1,t+\tau} w_1(t+\tau-h_1), & 0 \leq t < h_1\\ 0, & t \geq h_1 \end{cases} \qquad (76)$$

$$\Gamma_t^\tau \triangleq \begin{cases} \Gamma_{0,t+\tau}, & 0 \leq t < h_1\\ [\Gamma_{0,t+\tau} \quad \Gamma_{1,t+\tau}], & t \geq h_1 \end{cases} \qquad (77)$$

$$R_t^\tau \triangleq \begin{cases} R_{0,t+\tau}, & 0 \leq t < h_1\\ diag\{R_{0,t+\tau}, R_{1,t+\tau-h_1}\}, & t \geq h_1 \end{cases}, \qquad (78)$$

Then, the following result follows.

Theorem 4: Suppose the following Riccati equation

$$\begin{aligned} P_j^\tau &= \Phi'_{\tau+j} P_{j+1}^\tau \Phi_{\tau+j} + L'_{\tau+j} Q_{\tau+j} L_{\tau+j}\\ &\quad - K_j^\tau M_j^\tau (K_j^\tau)', \quad P_{N-\tau+1}^\tau = P_{N+1} \qquad (79)\end{aligned}$$

where

$$\begin{aligned} K_j^\tau &= \Phi'_{\tau+j} P_{j+1}^\tau \Gamma_j^\tau (M_j^\tau)^{-1}, & (80)\\ M_j^\tau &= R_j^\tau + (\Gamma_j^\tau)' P_{j+1}^\tau \Gamma_j^\tau, & (81)\end{aligned}$$

with given boundary condition condition P_{N+1}, has a bounded solution for any given $\tau \geq 0$. Then there exists a controller that solves the H_∞ control with preview if and only if

$$\begin{aligned} \Pi^{-1} - \gamma^{-2} P_0^0 &> 0, & (82)\\ \bar{M}_{2,2}(t) &< 0, \quad t \leq N - h_1 & (83)\end{aligned}$$

where $P_0^0 = P_0$ is the terminal value of P_0^τ of (79) for $\tau = 0$, and $\bar{M}_{2,2}(t)$ is the (2,2)-block of matrix $\bar{M}_t$ which, for $t \leq N - h_1$, is given by

$$\begin{aligned} \bar{M}_t &= diag\{C'_{0,t}, B'_{1,t}\} \left[\bar{P}_{t+1}(i,j)\right]_{2\times 2} diag\{C_{0,t}, B_{1,t}\}\\ &\quad + diag\{R_{0,t}^v, -\gamma^2 R_{1,t}^w\}, \qquad (84)\end{aligned}$$

where $\bar{P}_{t+1}(i,j)$ is as

$$\begin{aligned} \bar{P}_\tau(0,0) &= P_0^\tau,\\ \bar{P}_\tau(1,0) &= \bar{P}'_\tau(0,1) = P_{h_1}^\tau (\bar{\Phi}_{0,h_1}^\tau)',\\ \bar{P}_\tau(1,1) &= P_{h_1}^\tau \left[I - G^\tau(h_1) P_{h_1}^\tau\right].\end{aligned}$$

In this situation, the center controller $v_0^*(\tau)$ is as

$$\begin{aligned} v_0^*(\tau) &= -\Big([\mathcal{F}_0^\tau(0)]' x(\tau)\\ &\quad + \sum_{l=1}^{h_1} [\mathcal{S}_l^\tau(0)]' B_{1,\tau+l-1} w_1^*(\tau + l - h_1 - 1) \Big), \qquad (85)\end{aligned}$$

where $\mathcal{F}_0^\tau(0) = K_0^\tau$, and $\mathcal{S}_l^\tau(\cdot)$ is given as (30).

IV. Conclusion

We have presented a simple solution to the H_∞ control of systems with multiple input delays which has not been studied in existing literature. An explicit controller is given in terms of the solution of a Riccati difference equation of the same order as the original plant, which is advantageous over methods such as system augmentation. Our result assumes the existence of a bounded solution of Riccati difference equation (25) whose necessity deserves further studies.

Appendix A Proof of Lemma 2

Substituting (47) into (5) and using (50) yields

$$\begin{aligned}J_N^\infty &= x'(0)\Pi^{-1}x(0)-\gamma^{-2}J_N\\ &= x'(0)[\Pi^{-1}-\gamma^{-2}P_0]x(0)\\ &-\gamma^{-2}\sum_{\tau=0}^{N}[\bar v(\tau)-\bar v^{\tau*}(0)]^T\Delta(\tau)[\bar v(\tau)-\bar v^{\tau*}(0)]\\ &-\gamma^{-2}\sum_{t=0}^{N}[\bar w(\tau)-\tilde w^{\tau*}(0)]^T\tilde M_{2,2}(t)[\bar w(\tau)-\tilde w^{\tau*}(0)],\end{aligned} \tag{A.1}$$

where

$$\begin{aligned}&\begin{bmatrix}\bar v(\tau)-\bar v^{\tau*}(0)\\ \bar w(\tau)-\tilde w^{\tau*}(0)\end{bmatrix}\\ =&\begin{bmatrix}I & 0\\ \tilde M_{2,2}^{-1}(t)\tilde M_{2,1}(t) & I\end{bmatrix}\begin{bmatrix}\bar v(\tau)-\bar v^{\tau*}(0)\\ \bar w(\tau)-\bar w^{\tau*}(0)\end{bmatrix}\end{aligned} \tag{A.2}$$

and $\bar v^{\tau*}(0)$ is obtained from $\bar v(\tau)$ with replacements of $v_i(\tau)$ by $v_i^{\tau*}(0)$ for $i=0,\cdots,d$, and $\bar w^{\tau*}(0)$ is obtained from $\bar w(\tau)$ with replacements of $w_i(\tau)$ by $w_i^{\tau*}(0)$ for $i=0,\cdots,d$. Recall the discussion in [3] (Theorem 9.5.1), an H_∞ control input $v(\tau)$ that achieves $J_N^\infty>0$ exists if and only if

$$\Pi^{-1}-\gamma^{-2}P_0 > 0, \tag{A.3}$$
$$\tilde M_{2,2}(t) < 0. \tag{A.4}$$

In view of (A.1), the suitable controller can be chosen such that

$$\bar v(\tau)-\bar v^{\tau*}(0)=0. \tag{A.5}$$

Therefore, the controller is $v_i^{\tau*}(0)$, which is given by (54) and (55).

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