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Identification and Control: Joint Input Design and $\mathcal{H}_\infty$ State Feedback with Ellipsoidal Parametric Uncertainty

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Abstract— **One obstacle in connecting robust control with models generated from prediction error identification is that very few control design methods are able to directly cope with the ellipsoidal parametric uncertainty regions that are generated by such identification methods. In this contribution we present a sufficient condition for the existence of a $\mathcal{H}_\infty$ state feedback controller for the multi-input/single-output case which accomodates for ellipsoidal parametric uncertainty. The condition takes the form of a linear matrix inequality whose solution also provides a set of valid feedback gains. The model class considered corresponds to systems with known poles but uncertain zero locations. A second important contribution of the paper is to integrate the input design problem in system identification with this control synthesis method. This means that given $\mathcal{H}_\infty$ specifications on the closed loop transfer function are translated into the requirements on the input signal spectrum used to identify the process so that the ellipsoidal model uncertainty resulting from model identification using this input spectrum will be shaped such that the control specifications are satisfied for all models in the uncertainty set and hence guaranteed for the true system. The procedures are illustrated on a numerical example.**

Index Terms— **Robust control, system identification, control design, ellipsoidal uncertainty set, input design**

I. INTRODUCTION

The statistical theory on which prediction error identification and many other identification methods are founded has proven very fruitful in practical applications, see, e.g., the proceedings from the IFAC symposium series in System Identification. This theory leads to a characterization of the noise induced model uncertainty as an ellipsoidal set in the model parameter space. Recently, it has been pointed out that also in the case of restricted complexity modelling such uncertainty descriptions are relevant [1], [2].

In robust control, results addressing the *synthesis* of controller design with parametric uncertainty are scarce. However there are some important contributions in this area. In [3] a convex parametrization of all controllers that simultaneously stabilize a system for all norm-bounded parameters is given. An alternative procedure is presented in [4], which addresses the same problem formulated in [3]. The article [5] presents a procedure where the first step is to determine the set of controllers for which the nominal performance is somewhat better than the desired robust performance. It is then tested whether all controllers in the set stabilize all systems in the model set. For the performance criterion, a similar test is presented. The stability test is equivalent to computing the structured singular value of a certain matrix. In [6] the design specification is that the closed loop poles should be inside a disc with a given radius $\rho < 1$. The objective is to find the controller that maximizes the volume of an ellipsoidal model set such that the closed loop poles satisfy the design objective for all models in the set.

This paper contributes to taking one step further in closing the gap between system identification and robust control. We consider model structures which are linear in the uncertain parameters. Such model structures include systems with uncertain zero locations but known pole locations, so called fixed denominator structures, or equivalently fixed basis function expansions [7] such as Laguerre and Kautz models. We will assume that the parametric uncertainty is ellipsoidal. One of the main contributions is to show that the existence of a solution to a certain linear matrix inequality (LMI) implies the existence of a state feedback controller that guarantee *both* robust stability and performance for this uncertainty structure. We also show that the solution to this LMI can be used to find a set of robust controllers. The result extends the sufficiency part of the state feedback $\mathcal{H}_\infty$ problem for discrete time systems (see, e.g., [8]) to systems with ellipsoidal parametric uncertainty.

In our second main contribution in this paper, the above control synthesis method is linked to system identification. The problem considered is to design a suitable input signal for the identification experiment such that the the above state feedback synthesis method is able to guarantee given performance specifications for all models that are inside the confidence region resulting from the identification. This means that with, e.g., 99% probability the performance specifications will be satisfied for the true system. We show that for linearly parametrized input spectra this input design problem is equivalent to solving a convex program. Input design has a long history and recently, an interesting approach to input design has been opened up. It has been shown that a wide range of input design problems are equivalent to convex programs [9], [10], [11] and our work is a continuation in this direction.

The material is organized as follows. Section II-A presents the the robust control problem. Section II-B summarizes the $\mathcal{H}_\infty$-control theory that we will use. In Section II-C a sufficient condition for the existence of a robust state feedback controller is presented. In Section II-D this result is linked to a set of controllers satisfying the performance specifications. A numerical example is provided in Section II-E. In Section III-A the problem setup for the system identification is defined. In Section III-B the fundamental properties of system identification in the prediction error framework are reviewed and the input design problem is stated in Section III-C. In Section III-D the special parametrization of the input spectrum is introduced. In Sections III-E and III-F the parametrization of quality constraints and input constraints is given. Section IV concludes the paper.

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II. ROBUST STATE FEEDBACK CONTROL WITH ELLIPSOIDAL UNCERTAINTY

A. Problem Definition

Consider the system

$$\mathbf{x}(t+1) = \mathbf{A}\mathbf{x}(t)+\mathbf{B}\mathbf{u}(t)+\mathbf{D_p}\mathbf{w}(t) \qquad (1)$$
$$\mathbf{y}(t) = \mathbf{C}(\theta)\mathbf{x}(t)+\mathbf{B_y}\mathbf{u}(t) \qquad (2)$$

where $\mathbf{x}\in\mathbb{R}^{n}$, $\mathbf{y}\in\mathbb{R}^{n_y}$, $\mathbf{u}\in\mathbb{R}^{n_u}$, $\mathbf{w}\in\mathbb{R}^{n_w}$, $\mathbf{A}\in\mathbb{R}^{n\times n}$, $\mathbf{C}\in\mathbb{R}^{n_y\times n}$, $\mathbf{B}\in\mathbb{R}^{n\times n_u}$, $\mathbf{D_p}\in\mathbb{R}^{n\times n_w}$ and $\mathbf{B_y}\in\mathbb{R}^{n_y\times n_u}$. All matrices are known except $\mathbf{C}(\theta)$, which is linearly parametrized by a vector $\theta\in\mathbb{R}^{n}$ which, in turn, is known to lie in the ellipsoidal set

$$U=\left\{\theta:(\theta-\theta_0)^T\mathbf{R}(\theta-\theta_0)\le 1\right\}. \qquad (3)$$

The quantities θ_0 and $\mathbf{R}\in\mathbb{R}^{n\times n}$ which define the ellipsoid can, e.g., originate from prediction error identification of θ [12], a fact which will come back to in Section III. We will consider the case where the system (1)-(2) is controlled by a state feedback controller $\mathbf{u}=\mathbf{G}\mathbf{x}$. Denote the closed loop transfer function from the disturbance $\mathbf{w}$ to the output $\mathbf{y}$ by

$$\mathbf{T}(q,\theta)=(\mathbf{C}(\theta)+\mathbf{B_y}\mathbf{G})(q\mathbf{I}-\mathbf{A}-\mathbf{B}\mathbf{G})^{-1}\mathbf{D_p} \qquad (4)$$

We use the notation $||\mathbf{T}||_\infty\equiv\sup_{\omega\in[0,2\pi]}||\mathbf{T}(e^{j\omega})||$, where $||\cdot||$ is the maximum singular value. The ultimate objective is to characterize all feedback gains $\mathbf{G}$ for which

$$\|\mathbf{T}(\theta)\|_\infty<\gamma,\quad\forall\theta\in U \qquad (5)$$

We will not reach this far in this contribution but a partial answer will be provided in that an LMI which provides a sufficient condition for the existence of such gains will be derived and which can be used for control design.

B. State Feedback $\mathcal{H}_\infty$ Control

Our result is based on the following LMI characterization of the state feedback $\mathcal{H}_\infty$ problem when θ is known[1].

Theorem 1 (Theorem 7.3.1 in [8]): Assume that

$$\mathbf{B_y}^T\left[\mathbf{C}\quad\mathbf{B_y}\right]=\left[\mathbf{0}\quad\mathbf{I}\right]. \qquad (6)$$

and that the state variables $\mathbf{x}$ are measurable without noise. Then the following statements (i), (ii) and (iii) are equivalent.

(i) There exists a (static) feedback controller $\mathbf{u}=\mathbf{G}\mathbf{x}$ which stabilizes the system and yields $||\mathbf{T}||_\infty<\gamma$, where $\gamma>0$ is a scalar.

(ii) There exists a matrix $\mathbf{X}\in\mathbb{R}^{n\times n}$, $\mathbf{X}>0$ such that

$$\mathbf{X}>\mathbf{D_p}\mathbf{D_p^T}$$
$$\begin{pmatrix}\mathbf{X}-\mathbf{A}\mathbf{X}\mathbf{A^T}-\mathbf{D_p}\mathbf{D_p^T}+\gamma^2\mathbf{B}\mathbf{B^T} & \mathbf{A}\mathbf{X}\mathbf{C^T}\\ \mathbf{C}\mathbf{X}\mathbf{A^T} & \gamma^2\mathbf{I}-\mathbf{C}\mathbf{X}\mathbf{C^T}\end{pmatrix}>0 \qquad (7)$$

Now define $\mathbf{Y}\equiv\gamma^2\mathbf{X}^{-1}$.

(iii) There exists a matrix $\mathbf{Q}>0$ such that the Riccati equation

$$\mathbf{Y}=\mathbf{A^T}\mathbf{Y}\mathbf{A}-\mathbf{A^T}\mathbf{Y}\mathbf{E}(\mathbf{E^T}\mathbf{Y}\mathbf{E}+\mathbf{J})^{-1}\mathbf{E^T}\mathbf{Y}\mathbf{A}+\mathbf{C^T}\mathbf{C}+\mathbf{Q} \qquad (8)$$

has a solution $\mathbf{Y}>0$ satisfying

$$||\mathbf{D_p^T}\mathbf{Y}\mathbf{D_p}||<\gamma^2, \qquad (9)$$

where

$$\mathbf{E}\equiv\left[\mathbf{B}\quad\mathbf{D_p}\right],\quad \mathbf{J}\equiv\begin{bmatrix}\mathbf{I}_{n_u} & \mathbf{0}\\ \mathbf{0} & -\gamma^2\mathbf{I}_{n_w}\end{bmatrix}$$

All such controllers are given by

$$\mathbf{G}=-(\mathbf{B^T}\mathbf{P}\mathbf{B}+\mathbf{I})^{-1}\mathbf{B^T}\mathbf{P}\mathbf{A}+(\mathbf{B^T}\mathbf{P}\mathbf{B}+\mathbf{I})^{-\frac{1}{2}}\mathbf{L}\mathbf{Q}^{\frac{1}{2}} \qquad (10)$$

where $\mathbf{L}$ is an arbitrary matrix such that

$$||\mathbf{L}||<1 \qquad (11)$$

and $\mathbf{P}\equiv(\mathbf{Y^{-1}}-\frac{1}{\gamma^2}\mathbf{D_p}\mathbf{D_p^T})^{-1}>0$.

Proof: See [8]. ■

C. Existence of Stabilizing Controller

In this section we will extend the equivalence between (i) and (ii) in Theorem 1 to the case with parametric uncertainty in θ given by the ellipsoidal set (3). Then we will show how to minimize the upper performance bound γ using convex optimization.

We will restrict attention to the case when

$$\mathbf{C}=\mathbf{C}(\theta)=\begin{pmatrix}\theta^T\\ \mathbf{0}\end{pmatrix}\in\mathbb{R}^{n_y\times n} \qquad (12)$$

This assumption effectively means that only one of the system outputs is included in the objective. Any number of inputs can be included (through $\mathbf{B_y}$).

Theorem 2: Consider the system (1)-(2) where $\mathbf{C}$ is given by (12). Further, assume that conditions in Theorem 1 hold. Then (ii) below implies (i).

(i) There exists a (static) feedback controller $\mathbf{u}=\mathbf{G}\mathbf{x}$ which stabilizes the system and yields $||\mathbf{T}(\theta)||_\infty<\gamma$ for all $\theta\in U$.

(ii) There exists a matrix $\mathbf{X}\in\mathbb{R}^{n\times n}$, $\mathbf{X}>0$ and $\tau\in\mathbb{R}$, $\tau>0$ such that $\mathbf{X}>\mathbf{D_p}\mathbf{D_p^T}$ and

$$\begin{pmatrix}-\mathbf{X}+\tau\mathbf{R} & \mathbf{X}\mathbf{A^T} & \tau\mathbf{R}\theta_0\\ \mathbf{A}\mathbf{X} & \mathbf{M} & \mathbf{0}\\ \tau\theta_0^T\mathbf{R} & \mathbf{0} & \gamma^2-\tau(1-\theta_0^T\mathbf{R}\theta_0)\end{pmatrix}>0 \qquad (13)$$

Proof: To begin with, suppose that we are able to prove that a fix matrix $\mathbf{X}$ satisfies the conditions in (ii) in Theorem 1 for all $\theta\in U$. This implies that for all $\theta\in U$, $\mathbf{Y}=\gamma^2\mathbf{X}^{-1}$ is a valid positive definite solution to the Riccati equation (8) which satisfies the constraint (9) in (iii) in Theorem 1. From (iii) in Theorem 1 it then follows that, for each $\theta\in U$, the set of controllers satisfying the performance requirement is given by (10) with $\mathbf{Y}=\gamma^2\mathbf{X}^{-1}$. By taking $\mathbf{L}=0$ we get a state feedback $\mathbf{G}$ which depends only on $\mathbf{Y}$ which is independent of θ, i.e. we have found a θ-independent state feedback $\mathbf{G}$ which satisfies (i) in our theorem.

We will now prove that the existence of a matrix $\mathbf{X}$ and scalar τ satisfying the conditions in (ii) in the theorem is equivalent to the existence of a matrix which satisfies the conditions in (ii) in Theorem 1 for all $\theta\in U$. The proof will also show that $\mathbf{X}$ is such a matrix.

We first reformulate the inequality (7). From (12), we have

$$\gamma^2 I-\mathbf{C}(\theta)\mathbf{X}\mathbf{C}(\theta)^T=\begin{pmatrix}\gamma^2-\theta^T\mathbf{X}\theta & 0\\ 0 & \gamma^2\end{pmatrix}. \qquad (14)$$

[1] We will omit the θ dependence below as θ in this case plays no role.

as well as

$$\mathbf{AXC}(\theta)^T = (\mathbf{AX}\theta \quad \mathbf{0}). \tag{15}$$

Define $\mathbf{M} \equiv \mathbf{X} - \mathbf{AXA^T} - \mathbf{D_p D_p^T} + \gamma^2 \mathbf{BB^T}$. Equations (14)-(15) then give that (7) can be expressed as

$$\begin{pmatrix} \mathbf{M} & \mathbf{AX}\theta & \mathbf{0} \\ \theta^T \mathbf{XA^T} & \gamma^2 - \theta^T \mathbf{X}\theta & 0 \\ \mathbf{0} & 0 & \gamma^2 \end{pmatrix} > 0$$

which is equivalent to

$$\begin{pmatrix} \mathbf{M} & \mathbf{AX}\theta \\ \theta^T \mathbf{XA^T} & \gamma^2 - \theta^T \mathbf{X}\theta \end{pmatrix} > 0. \tag{16}$$

The Schur complement formula gives that (16) is equivalent to

$$\mathbf{M} > 0 \tag{17}$$

and

$$\gamma^2 - \theta^T \mathbf{X}\theta - \theta^T \mathbf{XA^T M^{-1} AX}\theta > 0. \tag{18}$$

Note that (18) can be written as

$$\begin{pmatrix} \theta \\ 1 \end{pmatrix}^T \begin{pmatrix} -\mathbf{X} - \mathbf{XA^T M^{-1} AX} & \mathbf{0} \\ \mathbf{0} & \gamma^2 \end{pmatrix} \begin{pmatrix} \theta \\ 1 \end{pmatrix} > 0 \tag{19}$$

We thus have shown that (7) is equivalent to (17) and (19).

Now we consider the uncertainty region (3) and note that it can be written

$$\begin{pmatrix} \theta \\ 1 \end{pmatrix}^T \begin{pmatrix} -\mathbf{R} & \mathbf{R}\theta_0 \\ \theta_0^T \mathbf{R} & 1 - \theta_0^T \mathbf{R}\theta_0 \end{pmatrix} \begin{pmatrix} \theta \\ 1 \end{pmatrix} \geq 0. \tag{20}$$

This means that we would like (19) to hold for all θ for which (20) holds. We will therefore combine (19) and (20) using the S-procedure (see [13]) which provides the following equivalence. Let $T_0, T_1 \in \mathbb{R}^{n \times n}$ be symmetric matrices. Then the following two conditions are equivalent:

- $\zeta^T T_0 \zeta > 0 \ \forall \zeta \neq 0$ such that $\zeta^T T_1 \zeta \geq 0$
- $\exists \tau \in \mathbb{R}$, $\tau \geq 0$ such that $T_0 - \tau T_1 > 0$, provided that there is some ζ_0 such that $\zeta_0^T T_1 \zeta_0 > 0$.

For (20) we have that $\theta = \theta_0$ results in a strictly positive left-hand side. Hence the S-procedure implies that that conditions (19) and (20) are equivalent to the conditions

$$\tau \geq 0 \tag{21}$$

and

$$\begin{pmatrix} -\mathbf{X} - \mathbf{XA^T M^{-1} AX} & \mathbf{0} \\ \mathbf{0} & \gamma^2 \end{pmatrix} - \tau \begin{pmatrix} -\mathbf{R} & \mathbf{R}\theta_0 \\ \theta_0^T \mathbf{R} & 1 - \theta_0^T \mathbf{R}\theta_0 \end{pmatrix} > 0. \tag{22}$$

Of course (22) can be rewritten as

$$\begin{pmatrix} -\mathbf{X} - \mathbf{XA^T M^{-1} AX} + \tau \mathbf{R} & -\tau \mathbf{R}\theta_0 \\ -\tau \theta_0^T \mathbf{R} & \gamma^2 - \tau(1 - \theta_0^T \mathbf{R}\theta_0) \end{pmatrix} > 0. \tag{23}$$

The Schur complement formula gives that (23) is equivalent to

$$\gamma^2 - \tau(1 - \theta_0^T \mathbf{R}\theta_0) > 0 \tag{24}$$

and

$$-\mathbf{X} - \mathbf{XA^T M^{-1} AX} + \tau \mathbf{R} - \frac{\tau^2 \mathbf{R}\theta_0 \theta_0^T \mathbf{R}}{\gamma^2 - \tau(1 - \theta_0^T \mathbf{R}\theta_0)} > 0. \tag{25}$$

Another use of the Schur complement gives that (25) and (17) are equivalent to

$$\begin{pmatrix} -\mathbf{X} + \tau \mathbf{R} - \frac{\tau^2 \mathbf{R}\theta_0 \theta_0^T \mathbf{R}}{\gamma^2 - \tau(1 - \theta_0^T \mathbf{R}\theta_0)} & \mathbf{XA^T} \\ \mathbf{AX} & \mathbf{M} \end{pmatrix} > 0. \tag{26}$$

Note that (26) is linear in $\mathbf{X}$ but not in τ. We would like it to be linear in τ as well. Therefore we start by rewriting (26) as

$$\begin{pmatrix} -\mathbf{X} + \tau \mathbf{R} & \mathbf{XA^T} \\ \mathbf{AX} & \mathbf{M} \end{pmatrix} - \begin{pmatrix} \frac{\tau^2 \mathbf{R}\theta_0 \theta_0^T \mathbf{R}}{\gamma^2 - \tau(1 - \theta_0^T \mathbf{R}\theta_0)} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} > 0 \tag{27}$$

which can be rewritten as

$$\begin{pmatrix} -\mathbf{X} + \tau \mathbf{R} & \mathbf{XA^T} \\ \mathbf{AX} & \mathbf{M} \end{pmatrix} - \begin{pmatrix} \tau \mathbf{R}\theta_0 \\ 0 \\ 0 \end{pmatrix} \frac{1}{\gamma^2 - \tau(1 - \theta_0^T \mathbf{R}\theta_0)} \begin{pmatrix} \tau(\mathbf{R}\theta_0)^T & 0 & 0 \end{pmatrix} > 0. \tag{28}$$

The Schur complement formula gives that (28) and (24) are equivalent to

$$\begin{pmatrix} -\mathbf{X} + \tau \mathbf{R} & \mathbf{XA^T} & \tau \mathbf{R}\theta_0 \\ \mathbf{AX} & \mathbf{M} & \mathbf{0} \\ \tau \theta_0^T \mathbf{R} & \mathbf{0} & \gamma^2 - \tau(1 - \theta_0^T \mathbf{R}\theta_0) \end{pmatrix} > 0 \tag{29}$$

which is linear in both $\mathbf{X}$ and τ. Note that for the condition (29) to hold we cannot have $\tau = 0$, since that would contradict the condition $\mathbf{X} > 0$. Therefore condition (21) changes to $\tau > 0$. This together with (29) are the conditions in (ii) and this therefore concludes the proof. ■

The minimum performance bound γ for which the sufficient condition in Theorem 2 provides a positive answer to the question of an existence of a robust controller can be obtained from the following convex program.

$$\begin{array}{ll} \underset{\gamma^2, \tau, \mathbf{X}}{\text{minimize}} & \gamma^2 \\ \text{subject to} & \gamma^2 > 0, \ \mathbf{X} - \mathbf{D_p D_p^T} > 0, \\ & \mathbf{X} > 0, \ \tau > 0, \text{ and } (13). \end{array} \tag{30}$$

D. Control Design

Theorem 2 provides a sufficient condition for the existence of a robust state feedback controller for ellipsoidal uncertainty. In this section we will elaborate on how to compute such a state feedback. The key is (iii) in Theorem 1. Starting with $\mathbf{X}$ and τ satisfying the conditions in (ii) in Theorem 2 define $\mathbf{Y} \equiv \gamma^2 \mathbf{X}^{-1}$ and

$$\mathbf{Q}(\theta) = \mathbf{Y} - \mathbf{A^T YA} + \mathbf{A^T YE}(\mathbf{E^T YE} + \mathbf{J})^{-\mathbf{1}} \mathbf{E^T YA} - \theta\theta^T, \tag{31}$$

Then for a particular model corresponding to the parameter vector θ, all controllers satisfying $||\mathbf{T}(\theta)||_\infty < \gamma$ are given by (10):

$$\mathbf{G}(\theta) = -(\mathbf{B^T PB} + \mathbf{I})^{-\mathbf{1}} \mathbf{B^T PA} + (\mathbf{B^T PB} + \mathbf{I})^{-\frac{1}{2}} \mathbf{L}(\theta) \mathbf{Q}^{\frac{1}{2}}(\theta) \tag{32}$$

where we have indicated all quantities that are θ-dependent. We see that the choice $\mathbf{L}(\theta) = 0$ results in

$$\mathbf{G}(\theta) = -(\mathbf{B^T PB} + \mathbf{I})^{-\mathbf{1}} \mathbf{B^T PA} \tag{33}$$

which is a gain matrix independent of θ and hence valid for all $\theta \in U$. This is thus one solution to the robust state feedback $\mathcal{H}_\infty$ problem. However, any valid $\mathbf{G}$ for which $\tilde{\mathbf{L}} \equiv \mathbf{L}(\theta)\mathbf{Q}^{\frac{1}{2}}(\theta)$ is θ-independent also has the desired properties. Since for each θ, $\mathbf{Q}(\theta)$ is given by (31), $\mathbf{L}(\theta)$ has to satisfy $\mathbf{L}(\theta) = \tilde{\mathbf{L}}\mathbf{Q}^{-\frac{1}{2}}(\theta)$. But the norm constraint (11) on $\mathbf{L}(\theta)$ has to be satisfied. This leads to the following set of controllers

$$\Big\{\mathbf{G} = -(\mathbf{B^T P B} + \mathbf{I})^{-1}\mathbf{B^T P A} + (\mathbf{B^T P B} + \mathbf{I})^{-\frac{1}{2}}\tilde{\mathbf{L}} : \sup_{\theta \in U} \|\tilde{\mathbf{L}}\mathbf{Q}^{-\frac{1}{2}}(\theta)\| < 1\Big\} \quad (34)$$

which all result in (5), i.e. they solve the robust state feedback $\mathcal{H}_\infty$ problem with ellipsoidal uncertainty.

E. Numerical Illustration

We will in this section illustrate our results with a numerical example. We will consider a single-input/single-output system with

$$\mathbf{A} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\ \mathbf{B} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ \mathbf{D_p} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ \mathbf{B_y} = 0$$

and $\mathbf{C} = \theta^T = [\ \theta_1 \quad \theta_2\ ]$. The sampling time is 1 second. The objective is to control the impact of the load disturbance w on both the output y and the input u. Therefore we augment the output with the input, resulting in new system matrices $\mathbf{B_y} = (0\ 1)^T$ and $\mathbf{C}$ according to (12). This configuration satisfies (6). The uncertainty ellipsoid, plotted in Figure 1, is given by

$$\theta_0 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \text{ and } \mathbf{R} = \begin{pmatrix} 2273.1 & -428.8 \\ -428.8 & 2273.1 \end{pmatrix}.$$

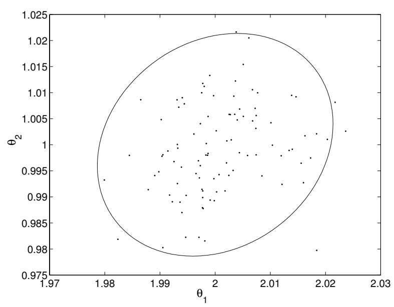

Fig. 1. Solid line: uncertainty ellipsoid. Dots: estimated model parameters from 100 Monte-Carlo runs, see Section III-G.

The solution to (30) is $\gamma = 2.1$. Solving (30) guarantees the existence of a controller $\mathbf{G}$ that fulfils statement (i) in Theorem 2. In Section II-D a procedure to find such controllers was derived. In Figure 2 the maximum singular value $||\mathbf{T}(e^{j\omega})||$ is plotted versus ω for 100 points on U for one such controller (the one corresponding to $\tilde{\mathbf{L}} = 0$). We see that $||\mathbf{T}||_\infty$ is below γ for all points on U. Also, for the center of U, i.e. $\theta = \theta_0$, we have that $||\mathbf{T}||_\infty$ is below γ. Figure 3 shows the Bode plot for the closed loop transfer function $\mathbf{T}$. We conclude that at high frequencies the disturbance w is attenuated in the output y and amplified in the input u and vice versa.

III. INPUT DESIGN FOR ROBUST STATE FEEDBACK CONTROL

A. Problem Definition

Consider the system (1)-(2) and let $u \in \mathbb{R}$. Assume that no information about the parameters θ is available. So we would

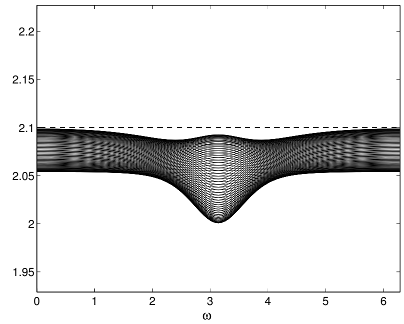

Fig. 2. Solid lines: $||\mathbf{T}(e^{j\omega})||$ versus $\omega \in [0, 2\pi]$ for 100 points on the uncertainty ellipsoid. Dashed line: the performance bound γ.

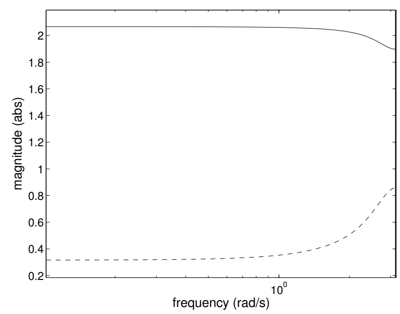

Fig. 3. Solid line: Bode plot for $\mathbf{T}_1$. Dashed line: Bode plot for $\mathbf{T}_2$.

like to design a system identification experiment in open loop using input output measurements to obtain an estimate for θ and characterize the uncertainty region of the estimate. We will assume that the input load disturbance $\mathbf{w}$ is absent in the identification experiment but that white measurement noise is present in the measured output (which is the first element of $\mathbf{y}(t)$ in (2)), resulting in the following input/output model

$$y(t) = G(q,\theta)u(t) + e(t) \quad (35)$$

where the transfer function G is parametrized as

$$G(q,\theta) = \theta^T(q\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} \quad (36)$$

and the white noise e has variance λ_0. We remark that the absence of input load disturbances is not unrealistic. For rolling mills, e.g., this assumptions corresponds to running the experiment without material entering the rolls.

B. System Identification in the Prediction Error Framework

For the identification of the plant (35), we will use prediction error identification, meaning that the parameter estimate is defined by $\hat{\theta}_N = \arg\min_\theta \frac{1}{2N}\sum_{t=1}^{N}(y(t) - G(q,\theta)u(t))^2$. It is well known that the estimate $\hat{\theta}_N$ will converge, under mild assumptions, to the parameters of the true system. Furthermore, when the model is flexible enough to capture the true dynamics, is possible to exactly characterize the asymptotic covariance matrix $\mathbf{P}$ of the parameters when N tends to infinity, see e.g. [12]. It can be shown that the only quantity in open loop operation that can be used to shape $\mathbf{P}$, is actually the spectrum of the input, $\Phi_u(\omega)$. This fact has been very important from an input design perspective and it has been widely used, see i.e. [14], [15], [9], [16] and [10].

It holds [10] that

$$\mathbf{P}^{-1}(\theta_0) = \frac{1}{2\pi\lambda_0}\int_{-\pi}^{\pi} F_u(e^{-j\omega})\Phi_u(\omega)F_u^*(e^{-j\omega})d\omega \quad (37)$$

where $F_u^*(e^{-j\omega}) = (e^{j\omega}\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}$. When the model $G(q,\theta)$ is obtained from an identification experiment it will lie in an uncertainty set whose size and shape is determined by the covariance matrix $\mathbf{P}$. An 95% confidence region $\tilde{U}$ of the parameters is approximately given by the set

$$\tilde{U} = \{\theta : N(\theta-\theta_o)^T \frac{\mathbf{P}^{-1}(\Phi_u)}{\chi}(\theta-\theta_o) \leq 1\}. \quad (38)$$

The parameter χ is determined such that $Pr(\chi^2(n) \leq \chi) = 0.95$ where n denotes the number of parameters in the model[2]. For example, $\chi = 5.99$ when $n = 2$.

Notice that in Section II we considered robust control design for uncertain parametric sets of the ellipsoidal type given in (38). In fact $\tilde{U}$ in (38) corresponds to U in (3) with

$$\mathbf{R} = N\frac{\mathbf{P}^{-1}}{\chi}.$$

However, in Section II the matrix $\mathbf{R}$ was assumed to be known. In the sequel we will open up a new degree of freedom by using $\mathbf{P}^{-1}$ as a variable in the input design problem.

C. Input design problem

Assume that an identification experiment for (35) will be designed before closing the loop with the feedback controller $\mathbf{G}$. For all models in the model set defined by (38), we would like the closed loop properties

$$\|\mathbf{T}(\theta)\|_\infty < \gamma, \quad \forall\theta \in \tilde{U} \quad (39)$$

for a given specification $\gamma > 0$ and where $\mathbf{T}(q,\theta)$ is defined by (4). The objective is to find the input signal with least power that enables us to to find a state feedback gain $\mathbf{G}$ (through the procedure given in Section II) such that (39) is fulfilled. The input design problem is formally stated as

$$\begin{array}{ll} \underset{\Phi_u}{\text{minimize}} & \alpha \\ \text{subject to} & ||\mathbf{T}(\theta)||_\infty < \gamma, \quad \forall\theta \in \tilde{U} \\ & \frac{1}{2\pi}\int_{-\pi}^{\pi}\Phi_u(\omega)d\omega \leq \alpha \\ & \Phi_u(\omega) \geq 0, \quad \forall\omega \end{array} \quad (40)$$

D. Parametrization of the Covariance Matrix

The problem (40) is a non-trivial optimization problem. However, by introducing a suitable parametrization of the input spectrum several input design problems can be written as convex program, see [10], [17]. Generally, a spectrum can be written as a linear expansion in a set of proper stable rational basis functions $\{\beta_k\}$:

$$\Phi_u(e^{j\omega}) = \sum_{k=-\infty}^{\infty} c_k\,\beta_k(e^{j\omega}), \quad (41)$$

see [10]. For the coefficients it is assumed that $c_k = c_{-k} \in \mathbb{R}$. A common choice is to let $\beta_k(e^{j\omega}) = e^{-j\omega k}$, which is an FIR spectrum. This parametrization was originally introduced in [18]. The fact that that the input spectrum can be parametrized in this fashion is the key tool when (40) is rewritten as a convex optimization problem.

Suppose that the input spectrum is written as a linear expansion in $\{c_k\}$. Then (37) implies that $\mathbf{P}^{-1}$ is a linear function of $\{c_k\}$. It is however impractical to use an infinite number of parameters $\{c_k\}$. A key tool is therefore to choose suitable basis functions $\{\beta_k(e^{j\omega})\}$, so that the number of coefficients needed to parametrize $\mathbf{P}^{-1}$ is no longer infinite, but will depend on the order of the system, see [19]. For (41) to define a spectrum it must hold $\Phi_u(\omega) \geq 0, \forall\omega$. In the design we will only determine the M first autocorrelation coefficients but we will however assume that there exists additional coefficients $c_M, c_{M+1}, \ldots \neq 0$. A necessary and sufficient condition for the existence of the expansion $c_M, c_{M+1}, \ldots$ such that (41) defines a spectrum is that a Toeplitz matrix with elements c_{j-i} is positive definite. This means that the partial expansion $\sum_{k=-(M-1)}^{M+1} c_k\,\beta_k(e^{j\omega})$ will not necessarily define a spectrum itself, but is constrained such that there exists additional coefficients in order for the expansion (41) to satisfy the positivity constraint on the input spectrum.

E. Parametrization of the Quality Constraint

In this section the quality constraint in (40) will be reformulated as a convex constraint exploiting the input spectrum parametrization (41). We have the following lemma.

Lemma 1: Statement (ii) in Theorem 2 is equivalent to the following statement. There exists matrices $\mathbf{X} > 0$, $\tilde{\mathbf{R}}, \mathbf{X} \in \mathbb{R}^{n\times n}$ and $\tau > 0$, $\tau \in \mathbb{R}$ such that $\mathbf{X} - \mathbf{D_p}\mathbf{D_p^T} > 0$ and

$$\begin{pmatrix} -\mathbf{X}+\tilde{\mathbf{R}} & \mathbf{X}\mathbf{A^T} & \tilde{\mathbf{R}}\theta_0 \\ \mathbf{A}\mathbf{X} & \mathbf{M} & \mathbf{0} \\ \theta_0^T\tilde{\mathbf{R}} & \mathbf{0} & \gamma^2-\tau+\theta_0^T\tilde{\mathbf{R}}\theta_0 \end{pmatrix} > 0. \quad (42)$$

Proof: The proof follows by introducing $\tilde{\mathbf{R}} \equiv \tau\mathbf{R}$ in Theorem 2. ■

The key idea for the input design is the variable transformation from $\mathbf{R} = N\frac{\mathbf{P}^{-1}}{\chi}$ to $\tilde{\mathbf{R}} \equiv \tau\mathbf{R}$. Define

$$\tilde{c}_k \equiv \tau c_k, \;\forall k. \quad (43)$$

Since $\mathbf{R}$ is a linear function of $\{c_k\}$, then $\tilde{\mathbf{R}}$ is a linear function of $\{\tilde{c}_k\}$. This means that the quality constraint in (40) corresponds to an LMI in $\mathbf{X}, \{\tilde{c}_k\}$ and τ. Furthermore, having $\tau > 0$, the positivity constraint on the input spectrum is equivalent to the positivity of a Toeplitz matrix,

$$\begin{pmatrix} \tilde{c}_0 & \tilde{c}_1 & \ldots & \tilde{c}_{M-1} \\ \tilde{c}_1 & \tilde{c}_0 & \ldots & \tilde{c}_{M-2} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{c}_{M-1} & \ldots & \ldots & \tilde{c}_0 \end{pmatrix} > 0, \quad (44)$$

which also is an LMI in $\{\tilde{c}_k\}$. Once $\{\tilde{c}_k\}$ are determined, $\{c_k\}$ are calculated as $c_k = \tilde{c}_k/\tau$, $\forall k$. The input design will therefore be performed in the variables $\{\tilde{c}_k\}$.

F. Parametrization of Power Constraints

By introducing the parametrization of the input spectrum (41) the power constraint becomes a finite-dimensional affine functions of the sequence $\{c_k\}$, see [10].

Example 1: For an FIR spectrum the total input power is $\frac{1}{2\pi}\int_{-\pi}^{\pi}\Phi_u(\omega)d\omega = c_0$. The power constraint thus yields $c_0 \leq \alpha$. For the sequence $\{\tilde{c}_k\}$ defined by (43) the constraint therefore becomes $\alpha\tau - \tilde{c}_0 \geq 0$. ■

[2] $\chi^2(n)$ denotes a χ^2-distributed random variable with n degrees of freedom.

G. Numerical Illustration

We will again consider the system given in Section II-E. Neglecting the input load disturbance, the system is from an identification point of view given by

$$y(t) = (2q^{-1} + q^{-2})u(t) + e(t) \quad (45)$$

where the white noise e has variance 1. The ultimate objective is to design a state feedback controller such that

$$\|\mathbf{T}\|_\infty < \gamma \quad (46)$$

for this system. To better illustrate the parallel between the example in this section and the one in Section II-E, we will use $\gamma = 2.1$, which was the optimal value in Section II-E. Hence, in this section, γ is a fixed value but the uncertainty ellipsoid $\mathbf{R}$ (or actually the transformed uncertainty ellipsoid $\tilde{\mathbf{R}}$) is a variable in the optimization program. In Section II-E, $\mathbf{R}$ was known and γ was a variable.

As the system parameters are unknown an identification experiment of length $N = 1000$ will be performed and the problem is to design the input such that (46) holds for the true system in closed loop. Assume that the system identification experiment is performed in open loop with the no load disturbance ($w = 0$) but with measurement noise. Consider only the first output. The two unknown parameters are estimated in the experiment. Let the input spectrum be parametrized as an FIR spectrum. With this parametrization, $\mathbf{P}^{-1}$ depends only on the first two input spectrum coefficients c_0 and c_1. So even if the input spectrum is infinitely parametrized, c.f. (41), the matrix $\tilde{\mathbf{R}}$ contains a finite number of parameters. The discussion in Sections III-D and III-F together with Lemma 1 implies that solving (40) is equal to solving the following problem.

$$\begin{array}{ll} \underset{\alpha,\tilde{c}_0,\tilde{c}_1,\ldots,\tilde{c}_{M-1},\tau,\mathbf{X}}{\text{minimize}} & \alpha \\ \text{subject to} & \mathbf{X} > 0, \mathbf{X} - \mathbf{D_p}\mathbf{D_p^T} > 0,\ \tau > 0, \\ & \alpha\tau - \tilde{c}_0 \geq 0,\ (42) \text{ and } (44). \end{array}$$

The smallest feasible α, which is obtained by a line search, is the optimal solution to this optimization program. The first $M = 40$ autocorrelations are solved for. Based on the Yule-Walker equations [20] an autoregressive (AR) input filter is realized. In Figure 4 the optimal input spectrum is shown. We see that more energy has to be injected at higher frequencies. In order to illustrate that the designed input fulfills the constraint, 100 Monte-Carlo runs of the identification experiment (45) are shown in Figure 1 together with the designed uncertainty set $\tilde{U}$. We clearly see that the input performs well, since 95 of the 100 runs are inside $\tilde{U}$.

IV. CONCLUSIONS

In this paper we have derived a sufficient robust stability and robust performance condition for a system with uncertain parameters. The condition is expressed as a linear matrix inequality and the solution is used to generate a set of feedback gains for which the closed loop system gain satisfy the performance specifications. This robust control synthesis method is also linked to system identification. We have shown that the problem of finding an input to the identification experiment such that the resulting model uncertainty is such that the given $\mathcal{H}_\infty$ performance specifications are satisfied with a certain probability, e.g. 99%, is equivalent to a convex program. The results have been illustrated on a numerical example.

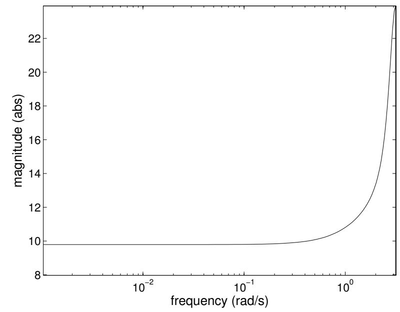

Fig. 4. The optimal input spectrum.

REFERENCES

[1] L. Ljung, "Model validation and model error modeling," in *The Åström Symposium on Control*, B. Wittenmark and A. Rantzer, Eds. Lund, Sweden: Studentlitteratur, Aug 1999, pp. 15 –42.

[2] H. Hjalmarsson, "From experiment design to closed loop control," *Automatica*, vol. 41, pp. 393–438, March 2005.

[3] A. Rantzer and A. Megretski, "A convex parametrization of robustly stabilizing controllers," in *IEEE Transactions on automatic control*, vol. 39, no. 9, September 1994, pp. 1802–1808.

[4] A. Ghulchak and A. Rantzer, "Robustly control under parametric uncertainty via primal-dual convex analysis," in *IEEE Transactions on automatic control*, vol. 47, no. 4, April 2002, pp. 632–636.

[5] X. Bombois, G. Scorletti, B. Anderson, M. Gevers, and P. Van den Hof, "A new robust control design procedure based on a pe identification uncertainty set," in *IFAC World Congress*, Barcelona, Spain, 2002.

[6] H.-F. Raynaud, L. Pronzato, and E. Walter, "Robust identification and control based on ellipsoidal parametric uncertainty descriptions," *European Journal of Control*, vol. 6, pp. 245–255, 2000.

[7] B. Ninness, H. Hjalmarsson, and F. Gustafsson, "The fundamental role of general orthonormal bases in system identification," *IEEE Transactions on Automatic Control*, vol. 44, pp. 1384–1406, July 1999.

[8] R. Skelton, T. Iwasaki, and K. Grigoriadis, *A Unified Algebraic Approach to Linear Control Design, pages 168-170*. London: Taylor and Francis, 1998.

[9] R. Hildebrand and M. Gevers, "Identification for control: Optimal input design with respect to a worst case ν-gap cost function," *SIAM Journal on Control and Optimization*, vol. 41, no. 5, pp. 1586–1608, 2003.

[10] H. Jansson and H. Hjalmarsson, "A general framework for mixed $\mathcal{H}_\infty$ and $\mathcal{H}_2$ input design," *IEEE Transactions on Automatic Control*, 2004, to appear.

[11] X. Bombois, G. Scorletti, M. Gevers, R. Hildebrand, and P. Van den Hof, "Least costly identification experiment for control," *Automatica*, 2004, submitted.

[12] L. Ljung, *System Identification - Theory For the User, 2nd ed.* Upper Saddle River, N.J: PTR Prentice Hall, 1999.

[13] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*. Philadelphia: SIAM Studies in Applied Mathematics, 1994.

[14] G. Goodwin and R. Payne, *Dynamic System Identification: Experiment Design and Data Analysis*. New York: Academic Press, 1977.

[15] B. Cooley and J. H. Lee, "Control-relevant experiment design for multivariable systems described by expansions in orthonormal bases," *Automatica*, vol. 37, pp. 273–281, 2001.

[16] X. Bombois, G. Scorletti, M. Gevers, R. Hildebrand, and P. Van den Hof, "Cheapest open-loop identification for control," in *IEEE Conference on Decision and Control, Bahamas*, December 2004.

[17] M. Barenthin, H. Jansson, and H. Hjalmarsson, "Applications of mixed $\mathcal{H}_\infty$ and $\mathcal{H}_2$ input design in identification," in *IFAC*, Prague, Czech Republic, July 2005.

[18] K. Lindqvist and H. Hjalmarsson, "Identification for control: Adaptive input design using convex optimization," in *IEEE Conference on Decision and Control*, Orlando, US, December 2001.

[19] H. Jansson, "Experiment design with applications in identification for control," Ph.D. dissertation, KTH, December 2004, TRITA-S3-REG-0404.

[20] T. Söderström, *Discrete-time Stochastic Systems*. London: Springer, 2002.