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The Nyquist Stability Criterion For A Class Of Spatially Periodic Systems

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Abstract— **The Nyquist stability criterion is extended to a class of spatially periodic systems with spatially distributed inputs and outputs. It is demonstrated that the exponential stability of this class of systems can be guaranteed by checking the Nyquist stability criterion for a family of finite-dimensional systems. In order to show this result, a new version of the argument principle is derived that is applicable to systems with infinite-dimensional input/output spaces and unbounded system operators.**

I. Introduction

Of the frequency-domain methods of stability analysis, the Nyquist criterion is of particular interest as it offers a simple visual test to determine the stability of a closed-loop system for a family of feedback gains [1].

Extensions of the Nyquist stability criterion exist for certain classes of time periodic [2] and distributed systems [1], where by distributed it is meant that the state-space of the system is infinite-dimensional. Time-delay (retarded) systems and systems governed by partial differential equations (spatially extended systems) are examples of distributed systems. But to the authors' best knowledge, the existing literature on the Nyquist stability criterion for distributed systems deals only with those systms that have finite-dimensional input and output spaces. An example of such a system would be one described by a partial differential equation with *pointwise* sensing and *pointwise* actuation.

In contrast, in the present work we aim to extend the Nyquist stability criterion to a class of spatially periodic systems that possess *spatio-temporal* (i.e. infinite-dimensional) input and output spaces, and possibly *unbounded* system operators.

Some of the difficulties in the application of the Nyquist criterion to the class of systems described above include (a) the concept of the *characteristic function* and its zeros are not immediately extendable to unbounded system operators, and (b) due to the infinite dimensionality of the input/output spaces one has to locate an *infinite* number of eigenloci.

Hence the main contributions of this paper are to (a) derive a version of the argument principle that lends itself to unbounded system operators, and (b) show that one can always truncate the infinite-dimensional system operators so that the problem can be reduced to checking the Nyquist stability criterion for (a family of) finite-dimensional systems.

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Our presentation is organized as follows: Section II gives a short overview of the frequency-domain representation of periodic operators. We lay out the problem setup in Section III and describe the general conditions for stability of spatially-periodic systems in Section IV. Section V contains the main contributions of the paper, where the Nyquist stability criterion is developed for spatially-periodic systems. We apply the theory to a simple example in Section VI, and finish with some conclusions and directions for future research in Section VII. All proofs and technical details have been placed in the Appendix.

Notation: $k \in \mathbb{R}$ denotes the spatial-frequency variable, also known as the wave-number. $\Sigma(T)$ is the spectrum of T, $\Sigma_p(T)$ its point spectrum, and $\rho(T)$ its resolvent set. $\sigma_n(T)$ is the n^{th} singular-value of T. $\mathscr{B}(\ell^2)$ denotes the bounded operators on ℓ^2, $\mathscr{B}_0(\ell^2)$ the compact operators on ℓ^2, and $\mathscr{B}_1(\ell^2)$ the nuclear operators on ℓ^2, i.e. operators T that have the property $\sum_{n=1}^{\infty} \sigma_n(T) < \infty$; $\mathscr{B}_1(\ell^2) \subset \mathscr{B}_0(\ell^2) \subset \mathscr{B}(\ell^2)$. $\mathrm{tr}[T]$ denotes the trace of T and $\det[T]$ its determinant. $\mathbb{C}^+$ and $\mathbb{C}^-$ denote the closed right-half and the open left-half of the complex plane, respectively, and $j := \sqrt{-1}$. $C(z_0; \mathfrak{P})$ is the number of counter-clockwise encirclements of the point $z_0 \in \mathbb{C}$ by the closed path $\mathfrak{P}$.

II. Preliminaries

Let us consider a linear time invariant (LTI) spatially distributed system of the form $\partial_t \psi = A\psi$, where A is a spatial operator on $L^2(\mathbb{R})$ and $\psi(t,x)$ is a spatio-temporal function. An example of such a system would be the heat equation on the real line $\partial_t \psi = \partial_x^2 \psi$, $x \in \mathbb{R}$.

Now for those systems where A is a spatially invariant operator (i.e., the action of A on ψ can be represented by a convolution), Fourier methods can significantly simplify the problem. For example in the heat equation introduced above, ∂_x^2 is a spatially invariant operator on $L^2(\mathbb{R})$, and thus the system can be rewritten in the spatial-frequency domain as $\partial_t \hat{\psi} = -k^2 \hat{\psi}$. Here $\hat{\psi}(t,k)$ is the spatial Fourier transform of $\psi(t,x)$, $\hat{A}(k) = (jk)^2 = -k^2$ is the Fourier symbol of the spatial operator ∂_x^2, and $k \in \mathbb{R}$ is the spatial-frequency variable.

In general if an LTI system is composed of only spatially invariant operators then as illustrated above, the spatial Fourier transform 'diagonalizes' all such operators (i.e., turns them into multiplication by a function of k). Therefore the Fourier-transformed system collapses to a 'continuum' of finite-dimensional LTI systems parameterized by k. This means that the original infinite-dimensional problem has been effectively 'decoupled' in the spatial-frequency domain.

But in this paper we will be dealing with the broader class of *spatially periodic* systems, i.e., systems of the form $\partial_t \psi = A\psi$ where A is a spatially periodic operator on $L^2(\mathbb{R})$. An example of such a system would be $\partial_t \psi = \partial_x^2 \psi + \cos(x)\psi$.

In light of the previous discussion, the difficulty faced in the analysis of spatially periodic systems becomes clear. For the Fourier methods described above to simplify the representation of a spatially periodic system, the Fourier transform has to *simultaneously* diagonalize the spatially invariant *and* spatially periodic operators present in the system. But this is not possible since these two classes of operators, in general, do not commute.[1] For example take the two spatial operators ∂_x and $\cos(x)$. Clearly $\cos(x)\ \partial_x\ \psi(x) \neq \partial_x\ \cos(x)\ \psi(x) = -\sin(x)\ \psi(x) + \cos(x)\ \partial_x\ \psi(x)$, for a general nonzero differentiable function $\psi(x)$. Thus $\cos(x)$, viewed as a spatially periodic pure multiplication operator, does not commute with the spatially invariant differentiation operator ∂_x.

Yet this does not mean that Fourier methods are not applicable or are not useful in the analysis of periodic systems. Quite to the contrary, analysis of periodic systems in the Fourier (frequency) domain often leads to numerical tractability and valuable insight.

In [3] and [4] it is demonstrated that a general spatially periodic operator A on $L^2(\mathbb{R})$ with spatial-period $\mathrm{X} = 2\pi/\Omega$ can be represented in the Fourier domain by a *family* of operators on ℓ^2. These operators can be written as bi-infinite matrices $\mathcal{A}_\theta$ parametrized by a variable $\theta \in [0, \Omega]$. In this paper we only give the $\mathcal{A}_\theta$ representation of certain special subclasses of spatially periodic operators, namely spatially invariant operators and spatially periodic pure multiplication operators, with the additional note that any spatially periodic operator can be written as the summation and/or multiplication of a countable number of such operators.

1) A *spatially-invariant* operator A has the *diagonal* representation

$$\mathcal{A}_\theta = \begin{bmatrix} \ddots & & \\ & \hat{A}(\theta + \Omega n) & \\ & & \ddots \end{bmatrix},$$

where $\hat{A}(\,\cdot\,)$ is the Fourier symbol of A, and for every given $\theta \in [0, \Omega]$ the diagonal elements of $\mathcal{A}_\theta$ are the equally-spaced samples $\{\hat{A}(\theta + n\Omega)\}_{n\in\mathbb{Z}}$ of $\hat{A}(\,\cdot\,)$.

2) A *spatially-periodic pure multiplication* operator F has the *Toeplitz* representation

$$\mathcal{F} = \begin{bmatrix} \ddots & \ddots & \ddots & & \ddots \\ \ddots & f_0 & f_{-1} & f_{-2} & \\ \ddots & f_1 & f_0 & f_{-1} & \ddots \\ & f_2 & f_1 & f_0 & \ddots \\ \ddots & & \ddots & \ddots & \ddots \end{bmatrix},$$

where f_l are the Fourier series coefficients of $F(x)$, i.e. $F(x) = \sum_{l=-\infty}^{\infty} f_l\ e^{jl\Omega x}$. Notice that $\mathcal{F}$ is independent of θ.

Remark 1: If A is a bounded operator on $L^2(\mathbb{R})$, then $\sup_{k\in\mathbb{R}} |\hat{A}(k)| \leq M$ for some $M > 0$, and thus $\mathcal{A}_\theta$ is a bounded operator on ℓ^2 for every $\theta \in [0, \Omega]$. ∎

III. Problem Setup

Consider the spatially-invariant system $\mathbf{S}^{\text{ol}}$

$$\begin{aligned} \big(\partial_t \psi\big)(t, x) &= \big(A\psi\big)(t, x) + \big(Bu\big)(t, x), \\ y(t, x) &= \big(C\psi\big)(t, x), \end{aligned} \tag{1}$$

where $t \in [0, \infty)$, $x \in \mathbb{R}$, A, B, and C, which we call the *system operators*, are spatially invariant and are all defined on a dense domain $\mathscr{D} \subset L^2(\mathbb{R})$. u, y and ψ are the spatio-temporal input, output and state of the system, respectively. Clearly, for any given time t, $\psi(t, \cdot)$ is a spatial function on $L^2(\mathbb{R})$, and thus (1) is an infinite-dimensional linear system.

Next, we place the spatially-invariant system $\mathbf{S}^{\text{ol}}$ in feedback with a spatially-periodic operator $\gamma F(x)$, $\|F(x)\| = 1$, $\gamma \in \mathbb{C}$, to form the closed-loop system $\mathbf{S}^{\text{cl}}$ as in Figure 1. It is our aim here to determine the stability properties of $\mathbf{S}^{\text{cl}}$ as the feedback gain γ varies in $\mathbb{C}$.

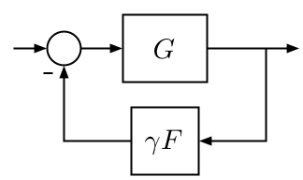

Fig. 1. The closed-loop system $\mathbf{S}^{\text{cl}}$.

For simplicity, in this paper we assume that all operators A, B, C and F are scalar. We also make the following assumptions on $\mathbf{S}^{\text{ol}}$.

Assumption (i): A is such that $\mathrm{Re}\big(\hat{A}(k)\big) \leq \beta$ for $|k| > K_1$, and $|\hat{A}(k)| \geq a\,|k|^{1+\eta}$ for $|k| > K_2$, for some $\beta < 0$, $a > 0$, $\eta > 0$, $K_1 > 0$ and $K_2 > 0$,

Assumption (ii): B and C are bounded operators.

[3] and [4] show how (1) can also be represented as

$$\begin{aligned} \big(\partial_t \psi_\theta\big)(t) &= \big(\mathcal{A}_\theta \psi_\theta\big)(t) + \big(\mathcal{B}_\theta u_\theta\big)(t), \\ y_\theta(t) &= \big(\mathcal{C}_\theta \psi_\theta\big)(t), \end{aligned} \tag{2}$$

with $\theta \in [0, \Omega]$, with $\mathcal{A}_\theta$, $\mathcal{B}_\theta$, and $\mathcal{C}_\theta$ being the bi-infinite matrices introduced previously. (2) describes the open-loop system $\mathbf{S}_\theta^{\text{ol}}$ with temporal impulse response $\mathcal{G}_\theta(t) := \mathcal{C}_\theta\, e^{\mathcal{A}_\theta t}\, \mathcal{B}_\theta$, and transfer function

$$\begin{aligned} \mathcal{G}_\theta(s) &:= \mathcal{C}_\theta\,(s\mathcal{I} - \mathcal{A}_\theta)^{-1}\,\mathcal{B}_\theta \\ &= \mathrm{diag}\Big\{\cdots, \frac{\hat{C}(\theta + \Omega n)\hat{B}(\theta + \Omega n)}{s - \hat{A}(\theta + \Omega n)}, \cdots\Big\}. \end{aligned} \tag{3}$$

Finally, using the same bi-infinite representation for the periodic operator $\gamma F(x)$ to get $\gamma\mathcal{F}$, and placing it in feedback with $\mathbf{S}_\theta^{\text{ol}}$, we obtain the close-loop system $\mathbf{S}_\theta^{\text{cl}}$ in Figure 2 with A-operator $\mathcal{A}_\theta^{\text{cl}} := \mathcal{A}_\theta - \mathcal{B}_\theta\, \gamma\mathcal{F}\, \mathcal{C}_\theta$.

[1] $TS = ST$ is a necessary and sufficient condition for T and S to be simultaneously diagonalizable by a transformation U.

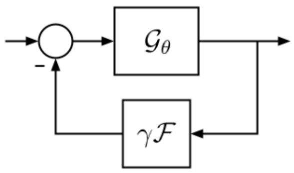

Fig. 2. The closed-loop system $\mathbf{S}_\theta^{\text{cl}}$.

IV. Stability of Linear Spatially-Periodic Systems

A semigroup e^{At} on a Hilbert space is called exponentially stable if there exist constants $M \geq 1$ and $\alpha > 0$ such that $\|e^{At}\| \leq Me^{-\alpha t}$ for $t \geq 0$. It is well-known [5] [6] that if A is an ifinite-dimensional operator, then in general $\Sigma(A) \subset \mathbb{C}^-$ is not sufficient for the exponential decay of $\|e^{At}\|$. In this paper we focus on systems which *do* satisfy the so-called *spectrum-determined growth condition*, namely systems for which $\Sigma(A) \subset \mathbb{C}^-$ implies exponential decay of the semigroup. Examples of such semigroups are numerous and include analytic semigroups [7] [8].

In [3] it is shown that for a general spatially periodic operator A we have

$$\Sigma(A) \;=\; \bigcup_{\theta\in[0,\Omega]} \Sigma(\mathcal{A}_\theta). \tag{4}$$

Thus to prove $\Sigma(A^{\text{cl}}) \subset \mathbb{C}^-$, as needed to guarantee the exponential stability of $\mathbf{S}^{\text{cl}}$, it is necessary and sufficient to show that $\mathcal{A}_\theta^{\text{cl}} = \mathcal{A}_\theta - \mathcal{B}_\theta\,\gamma\mathcal{F}\,\mathcal{C}_\theta$ has spectrum inside $\mathbb{C}^-$ for all $\theta \in [0,\Omega]$. In the next section we aim to develop a graphical method of checking whether or not $\Sigma(\mathcal{A}_\theta^{\text{cl}}) \subset \mathbb{C}^-$. Also, henceforth in this paper wherever we use the term stability we mean exponential stability.

V. The Nyquist Stability Criterion for Spatially Periodic Systems

A. The Determinant Method

To motivate the development in this section, let us first consider a finite-dimensional (multi-input multi-output) LTI system $G(s)$ placed in feedback with a constant gain γI. In analyzing the closed-loop stability of such a system, we are concerned with the eigenvalues in $\mathbb{C}^+$ of the closed-loop A-matrix A^{cl}. If s is an eigenvalue of A^{cl}, then it satisfies $\det[sI - A^{\text{cl}}] = 0$. Now to check whether the equation $\det[sI - A^{\text{cl}}] = 0$ has solutions inside $\mathbb{C}^+$, one can apply the argument principle to $\det[I + \gamma G(s)]$ as s traverses some curve $\mathfrak{D}$ enclosing $\mathbb{C}^+$. More precisely, since

$$\det[I + \gamma G(s)] = \frac{\det[sI - A^{\text{cl}}]}{\det[sI - A]},$$

if one knows the number of unstable open-loop poles, then one can determine the number of unstable closed-loop poles by looking at the plot of $\det[I + \gamma G(s)]\big|_{s\in\mathfrak{D}}$.

But in the case of spatially-distributed systems the open-loop and closed-loop A-operators $\mathcal{A}_\theta$ and $\mathcal{A}_\theta^{\text{cl}}$ are, in general, unbounded. Hence it does not make sense to talk about the characteristic functions $\det[s\mathcal{I} - \mathcal{A}_\theta]$ and $\det[s\mathcal{I} - \mathcal{A}_\theta^{\text{cl}}]$, and one has to resort to operator theoretic arguments to relate the plot of $\det[\mathcal{I} + \gamma\mathcal{F}\mathcal{G}_\theta(s)]\big|_{s\in\mathfrak{D}}$ to the unstable modes of the open-loop and closed-loop systems. But first it has to be clarified what is meant by $\det[\mathcal{I} + \gamma\mathcal{F}\mathcal{G}_\theta(s)]$ for the infinite-dimensional operator $\mathcal{I} + \gamma\mathcal{F}\mathcal{G}_\theta(s)$. We need the following lemma.

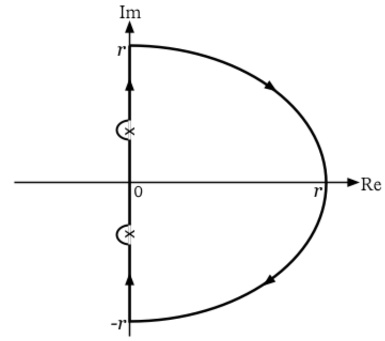

Fig. 3. The closed contour $\mathfrak{D}^\theta$ traversed in the clockwise direction taken as the Nyquist path as $r \to \infty$. The indentations are arbitrarily made to avoid open-loop modes on the imaginary axis.

Lemma 1: $\mathcal{F}\mathcal{G}_\theta(s) \in \mathscr{B}_1(\ell^2)$ for all $s \in \rho(\mathcal{A}_\theta)$.

Proof: See Appendix. ■

Since $\mathcal{F}\mathcal{G}_\theta(s) \in \mathscr{B}_1(\ell^2)$, one can now define [9] [10]

$$\det[\mathcal{I} + \gamma\mathcal{F}\mathcal{G}_\theta(s)] := \prod_{n=-\infty}^{\infty} \big(1 + \gamma\lambda_n^\theta(s)\big),$$

where $\lambda_n^\theta(s)$, $n \in \mathbb{Z}$, are the eigenvalues of $\mathcal{G}_\theta(s)$. We are now ready to state a generalized form of the argument principle applicable to systems with unbounded A-operators.

Theorem 2: If $\det[\mathcal{I} + \gamma\mathcal{F}\mathcal{G}_\theta(s)] \neq 0$ for all $s \in \mathfrak{D}^\theta$,

$$C\Big(0; \det[\mathcal{I} + \gamma\mathcal{F}\mathcal{G}_\theta(s)]\big|_{s\in\mathfrak{D}^\theta}\Big) =$$

$$\operatorname{tr}\bigg[\frac{1}{2\pi j}\int_{\mathfrak{D}^\theta}(s\mathcal{I} - \mathcal{A}_\theta^{\text{cl}})^{-1}ds\bigg] - \operatorname{tr}\bigg[\frac{1}{2\pi j}\int_{\mathfrak{D}^\theta}(s\mathcal{I} - \mathcal{A}_\theta)^{-1}ds\bigg] =$$

$$-\,(\text{number of eigenvalues of } \mathcal{A}_\theta^{\text{cl}} \text{ in } \mathbb{C}^+) + (\text{number of eigenvalues of } \mathcal{A}_\theta \text{ in } \mathbb{C}^+),$$

where $\mathfrak{D}^\theta$ is the Nyquist path shown in Figure 3 that does not pass through any eigenvalues of $\mathcal{A}_\theta$, and encloses a finite number of them.

Proof: See Appendix. ■

Remark 2: $\mathcal{P} = -\frac{1}{2\pi j}\int_{\mathfrak{D}^\theta}(s\mathcal{I} - \mathcal{A}_\theta)^{-1}ds$ is the group-projection [11] [10] corresponding to the eigenvalues of $\mathcal{A}_\theta$ inside $\mathfrak{D}^\theta$, and $\operatorname{tr}[\mathcal{P}]$ gives the total number of such eigenvalues [12]. Similarly $\operatorname{tr}[-\frac{1}{2\pi j}\int_{\mathfrak{D}^\theta}(s\mathcal{I} - \mathcal{A}_\theta^{\text{cl}})^{-1}ds]$ gives the total number of eigenvalues of $\mathcal{A}_\theta^{\text{cl}}$ in $\mathfrak{D}^\theta$. Thus Theorem 2 allows us to determine the number of $\mathbb{C}^+$ eigenvalues of $\mathcal{A}_\theta^{\text{cl}}$ from knowledge of the number of $\mathbb{C}^+$ eigenvalues of $\mathcal{A}_\theta$ and the number of encirclements of the origin by the plot of $\det[\mathcal{I} + \gamma\mathcal{F}\mathcal{G}_\theta(s)]$ as s traverses $\mathfrak{D}^\theta$. Notice that since $\mathcal{A}_\theta = \operatorname{diag}\big\{\cdots, \hat{A}(\theta + \Omega n), \cdots\big\}$, the eigenvalues of $\mathcal{A}_\theta$ are known and we have $\Sigma_{\text{p}}(\mathcal{A}_\theta) = \big\{\hat{A}(\theta + \Omega n), n \in \mathbb{Z}\big\}$. ■

Remark 3: It now becomes clear why we have used the $\mathcal{A}_\theta$ representation of the operator A; in this representation and under the assumptions of Section III, the open- and closed-loop A-operators enjoy the property of having discrete (pure point) spectrum. This allows for an extension of the argument principle to be invoked. ∎

Remark 4: Since $\mathcal{A}_\theta$ has discrete spectrum, it has no finite accumulation points in the complex plane [12]. In particular, the eigenvalues of $\mathcal{A}_\theta$ can not converge to any finite point $j\omega_0$ of the imaginary axis. Yet this does not rule out the possibility of the eigenvalues accumulating at $\pm j\infty$. But Assumption (i) guards against this by requiring that $\hat{A}(k)$ be bounded away from the imaginary axis as $|k| \to \infty$. Thus the Nyquist path can be taken to run to infinity along the imaginary axis without any technical difficulties. ∎

As a direct consequence of Theorem 2 we have

Theorem 3: Assume p_+^θ denotes the number of eigenvalues of $\mathcal{A}_\theta$ inside $\mathbb{C}^+$. For $\mathfrak{D}^\theta$ taken as above, $\Sigma_{\mathrm{p}}(\mathcal{A}_\theta^{\mathrm{cl}}) \subset \mathbb{C}^-$ iff

(a) $\det[\mathcal{I} + \gamma \mathcal{F}\mathcal{G}_\theta(s)] \neq 0, \ \forall\, s \in \mathfrak{D}^\theta$,

and

(b) $C\Big(0; \det[\mathcal{I} + \gamma \mathcal{F}\mathcal{G}_\theta(s)]\big|_{s\in\mathfrak{D}^\theta}\Big) = p_+^\theta$.

Finally, the closed-loop system G^{cl} is exponentially stable iff $\Sigma_{\mathrm{p}}(\mathcal{A}_\theta^{\mathrm{cl}}) \subset \mathbb{C}^-$ for all $\theta \in [0, \Omega]$. ∎

B. The Eigenloci Method

The setback with the method described in the previous paragraph is that to show $\Sigma_{\mathrm{p}}(\mathcal{A}_\theta^{\mathrm{cl}}) \subset \mathbb{C}^-$, $\mathcal{A}_\theta^{\mathrm{cl}} = \mathcal{A}_\theta - \mathcal{B}_\theta \gamma \mathcal{F} \mathcal{C}_\theta$, for different values of γ, one has to plot $\det[\mathcal{I} + \gamma \mathcal{F}\mathcal{G}_\theta(s)]\big|_{s\in\mathfrak{D}^\theta}$ for each γ. This also includes having to calculate the determinant of an infinite dimensional matrix. This motivates the following eigenloci approach to Nyquist stability analysis, which is very similar to that performed in [2] for the case of time-periodic systems.

Let $\lambda_n^\theta(s)$, $n \in \mathbb{Z}$, constitute the eigenvalues of $\mathcal{F}\mathcal{G}_\theta(s)$. Then

$$\angle \ \det[\mathcal{I} + \gamma \mathcal{F}\mathcal{G}_\theta(s)] \ = \ \angle \prod_{n=-\infty}^{\infty} \big(1 + \gamma \lambda_n^\theta(s)\big). \tag{5}$$

But recall from Lemma 1 that $\mathcal{F}\mathcal{G}_\theta(s) \in \mathscr{B}_1(\ell^2)$ for every $s \in \rho(\mathcal{A}_\theta)$. This, in particular, means that $\mathcal{F}\mathcal{G}_\theta(s)$ is a compact operator and thus its eigenvalues $\lambda_n^\theta(s)$ accumulate at the origin as $|n| \to \infty$ [13]. As a matter of fact one can make a much stronger statement.

Lemma 4: The eigenvalues $\lambda_n^\theta(s)$ converge to the origin uniformly on $\mathfrak{D}^\theta$.

Proof: See Appendix. ∎

Take the positive integer N_ϵ to be such that $|\lambda_n^\theta(s)| < \epsilon$, $s \in \mathfrak{D}^\theta$, for all $|n| > N_\epsilon$. Let us rewrite (5) as

$$\begin{aligned}
&\angle \ \det[\mathcal{I} + \gamma \mathcal{F}\mathcal{G}_\theta(s)] = \\
&\quad \angle \prod_{|n|\le N_\epsilon} \big(1 + \gamma \lambda_n^\theta(s)\big) \ + \ \angle \prod_{|n| > N_\epsilon} \big(1 + \gamma \lambda_n^\theta(s)\big) = \\
&\quad \sum_{|n|\le N_\epsilon} \angle \, \big(1 + \gamma \lambda_n^\theta(s)\big) \ + \ \sum_{|n| > N_\epsilon} \angle \, \big(1 + \gamma \lambda_n^\theta(s)\big). \qquad (6)
\end{aligned}$$

It is clear that if $|\gamma| < \frac{1}{\epsilon}$ then for $|n| > N_\epsilon$, $|\gamma \lambda_n^\theta(s)| < 1$, and $1 + \gamma \lambda_n^\theta(s)$ can never circle the origin as s travels around $\mathfrak{D}^\theta$. Thus for $|\gamma| < \frac{1}{\epsilon}$ the final sum in (6) will not contribute to the encirclements of the origin, and hence we lose nothing by considering only the first N_ϵ eigenvalues. There still remain some minor technicalities.

First, let D_ϵ denote the disk $|s| < \epsilon$ in the complex plane. Then said truncation may result in some eigenloci (part of which resides inside D_ϵ) not forming closed loops. But notice that these can be arbitrarily closed inside D_ϵ, as this does not affect the encirclements [2].

The second issue is that for some values of $s \in \mathfrak{D}^\theta$, $\mathcal{F}\mathcal{G}_\theta(s)$ may have multiple eigenvalues, and hence there is ambiguity in how the eigenloci of the Nyquist diagram should be indexed. But this poses no problem as far as counting the encirclements is concerned, and it is always possible to find such an indexing; for a detailed treatment see [1].

Let us denote by λ_n^θ the indexed eigenloci that make up the generalized Nyquist diagram. From (6) and the above discussion it follows that

$$C\Big(0; \det[\mathcal{I} + \gamma \mathcal{F}\mathcal{G}_\theta(s)]\big|_{s\in\mathfrak{D}^\theta}\Big) \ = \ \sum_{|n|\le N_\epsilon} C\Big(-\frac{1}{\gamma}; \lambda_n^\theta\Big)$$

which together with Theorem 3 gives

Theorem 5: Assume p_+^θ denotes the number of eigenvalues of $\mathcal{A}_\theta$ inside $\mathbb{C}^+$. For $\mathfrak{D}^\theta$ and N_ϵ as defined previously, $\Sigma_{\mathrm{p}}(\mathcal{A}_\theta^{\mathrm{cl}}) \subset \mathbb{C}^-$ for $|\gamma| < \frac{1}{\epsilon}$ iff

(a) $-\dfrac{1}{\gamma} \notin \big(\lambda_n^\theta\big)_{|n|\le N_\epsilon}$,

and

(b) $\sum_{|n|\le N_\epsilon} C\Big(-\dfrac{1}{\gamma}, \lambda_n^\theta(s)\Big) = p_+^\theta$.

Finally, the closed-loop system G^{cl} is exponentially stable iff $\Sigma_{\mathrm{p}}(\mathcal{A}_\theta^{\mathrm{cl}}) \subset \mathbb{C}^-$ for all $\theta \in [0, \Omega]$. ∎

C. Finite Truncations of System Operators

The above development means that for a given $\epsilon > 0$, the eigenloci that fall within the disk $D_\epsilon = \{s \text{ s.t. } |s| < \epsilon\}$ play no role in the Nyquist stability analysis and can be ignored as long as $|\frac{1}{\gamma}| > \epsilon$.

This suggests that one could *truncate* $\mathcal{F}\mathcal{G}_\theta$, or equivalently truncate $\mathcal{A}_\theta$, $\mathcal{B}_\theta$, $\mathcal{C}_\theta$, and $\mathcal{F}$, and effectively treat the stability problem as one for a family of (finite-dimentional) multivariable systems parameterized by the variable θ.

The complication here is that although a truncation removes the infinite number of eigenloci that shrink to zero, it also affects *all other* eigenloci, no matter how large the truncation is taken to be. Nevertheless, it can be shown that by increasing the size of the truncation, the eigenloci of $\mathcal{F}\mathcal{G}_\theta$ can be recovered to any accuracy.

Let Π_N be the projection on the first $2N+1$ standard basis elements of ℓ^2, $\{e_{-N}, \cdots, e_0, \cdots, e_N\}$. Thus $\Pi_N \mathcal{F}\mathcal{G}_\theta \Pi_N$ is the $(2N+1) \times (2N+1)$ truncation of $\mathcal{F}\mathcal{G}_\theta$. We have

Lemma 6: If $\zeta \in \rho(\Pi_N \mathcal{F}\mathcal{G}_\theta \Pi_N)$ and

$$\|\mathcal{F}\mathcal{G}_\theta - \Pi_N \mathcal{F}\mathcal{G}_\theta \Pi_N\| \ \|(\zeta - \Pi_N \mathcal{F}\mathcal{G}_\theta \Pi_N)^{-1}\| < 1,$$

then $\zeta \in \rho(\mathcal{F}\mathcal{G}_\theta)$.

Proof: This is a direct consequence of Theorem 3.17, Chap IV of [12]. ∎

The decay of the diagonal elements of $\mathcal{G}_\theta$, and the decay with increasing l of the f_l elements of $\mathcal{F}$, can be used to show that $\|\mathcal{F}\mathcal{G}_\theta - \Pi_N \mathcal{F}\mathcal{G}_\theta \Pi_N\|$ can be made arbitrarily small for large enough N.[2] Hence, by Lemma 6 the eigenloci of $\Pi_N \mathcal{F}\mathcal{G}_\theta \Pi_N$ approximate those of $\mathcal{F}\mathcal{G}_\theta$ arbitrarily well as N grows.

D. Regularity in the θ Parameter

Regarding Theorems 3 and 5, one would hope for some kind of 'regularity' with respect to the variable θ. More precisely, in practice one would like to plot the eigenloci and check the Nyquist stability criterion for a finite number of θ_l, say $\theta_1, \cdots, \theta_L$, and be able to conclude stability for all $\theta \in [0, \Omega]$ if the θ_l are chosen close enough to each other.

It is possible to show that under certain mild conditions on A, B and C, all points of the plot $\det[\mathcal{I} + \gamma \mathcal{F}\mathcal{G}_\theta(s)]\big|_{s \in \mathfrak{D}^\theta}$ change continuously with θ. Moreover we can show that the eigenloci λ_n^θ of $\mathcal{F}\mathcal{G}_\theta(s)$ change continuously with θ. We do not present the details here.

VI. An Illustrative Example

In this section, we will consider the example of an open-loop system governed by

$$\begin{aligned} \partial_t \psi(t,x) &= \partial_x^2 \psi(t,x) + \psi(t,x) + u(t,x), \\ y(t,x) &= \psi(t,x), \end{aligned}$$

put in feedback with $\gamma F(x) = \gamma \cos(x)$. Here $\Omega = 1$, and B, C are the identity operator. The representation of the system in the frequency domain is $\partial_t \psi_\theta(t) = (\mathcal{A}_\theta - \gamma \mathcal{F})\psi_\theta(t)$, where $\mathcal{A}_\theta = \mathrm{diag}\{\cdots, -(\theta+n)^2 + 1, \cdots\}$ for every $\theta \in [0,1]$, and $\mathcal{F}$ has the form shown at the end of Section II with $f_1 = f_{-1} = 1$ and $f_i = 0$, $i \neq \pm 1$. Notice that the open-loop system is unstable.

Recall that to test the stability of the closed-loop system, one has to apply the Nyquist criterion for every $\theta \in [0,1]$. We take Nyquist paths $\mathfrak{D}^\theta$ of the form shown in Figure 4(a) with $r = 20$. The indentation is chosen appropriately as to avoid the eigenvalues of $\mathcal{A}_\theta$ at the origin. Let us take a look at the Nyquist plots for two particular values of θ:

$\underline{\theta = 0}$: $\lambda = 0, 0, 1$ are the eigenvalues of $\mathcal{A}_0$ inside $\mathfrak{D}^0$, hence $p_+^0 = 3$, and we need three counter-clockwise encirclements of $-1/\gamma$ to achieve closed-loop stability. As can be seen in Figure 4(b) and its blown-up version (c), one possible choice would be to take $-1/\gamma$ to be purely imaginary and $-0.2j \leq -1/\gamma \leq 0.2j$. Clearly such $-1/\gamma$ is encircled three times by the eigenloci.

$\underline{\theta = 0.5}$: $\lambda = 0.75, 0.75$ are the eigenvalues of $\mathcal{A}_{0.5}$ inside $\mathfrak{D}^{0.5}$, hence $p_+^{0.5} = 2$, and we need two counter-clockwise encirclements of $-1/\gamma$ to achieve closed-loop stability. Again, from Figure 4(d), if $-1/\gamma$ is taken to be purely imaginary and $-0.2j \leq -1/\gamma \leq 0.2j$, then $-1/\gamma$ is encircled twice by the eigenloci.

As a matter of fact, it can be shown that $-0.2j \leq -1/\gamma \leq 0.2j$ stabilizes the closed-loop system for all values of $\theta \in [0,1]$.

[2] This is also a direct consequence of the fact that $\mathcal{F}\mathcal{G}_\theta \in \mathscr{B}_1(\ell^2)$ is a compact operator and can therefore be approximated arbitrarily well by a sequence of finite-dimensional operators [13].

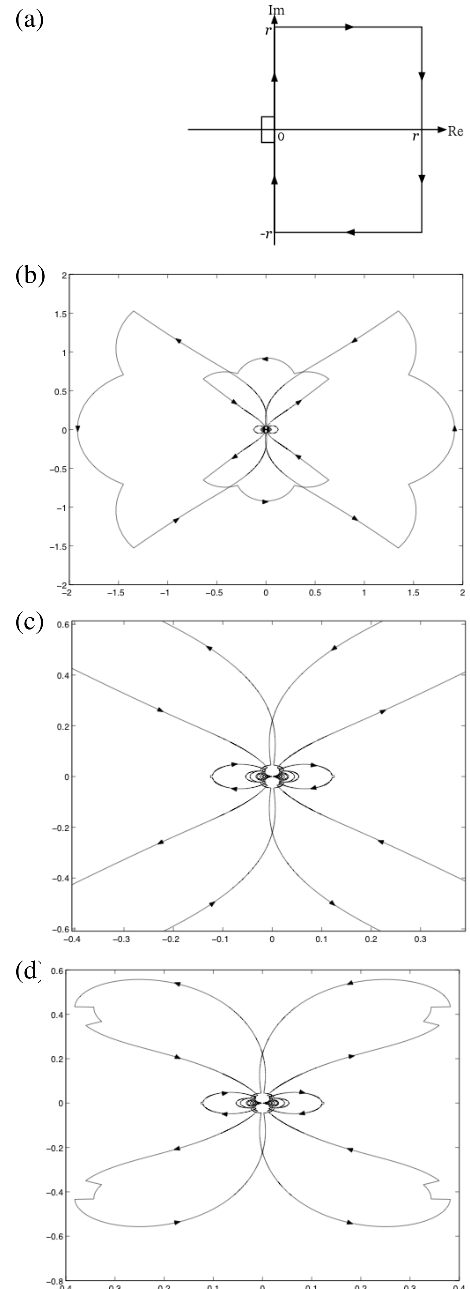

Fig. 4. (a) The Nyquist path $\mathfrak{D}^\theta$; (b) Nyquist plot for $\theta = 0$; (c) Blown-up version of plot in (b); (d) Nyquist plot for $\theta = 0.5$.

VII. Conclusions and Future Work

We develop a generalized Nyquist stability criterion that is applicable to a class of spatially-distributed systems with infinite-dimensional input/output spaces and unbounded system operators.

Future work in this direction would include considering a wider class of spatially-distributed systems, for example those for which the Fourier symbol of A is itself a matrix (i.e., the state has Euclidean dimension larger than one). Also, the affect on stability of different basic frequencies Ω of the periodic feedback, and the occurrence of parametric resonance, can be investigated in this framework. It would be of interest to compare such results to those already obtained using norm and spectral analysis [4] [14].

VIII. Appendix

Proof of Lemma 1: Since $s \in \rho(\mathcal{A}_\theta)$, then $s \neq \hat{A}(\theta + \Omega n)$ for any $n \in \mathbb{Z}$, which together with (3) and the boundedness of B and C implies that all diagonal elements of $\mathcal{G}_\theta(s)$ are finite. Let $\varrho_n(\mathcal{G}_\theta(s))$, $n = 1, 2, \cdots$, be a reordering of these elements such that $|\varrho_1(\mathcal{G}_\theta(s))| \geq |\varrho_2(\mathcal{G}_\theta(s))| \geq \cdots \geq 0$. Then clearly $\sigma_n(\mathcal{G}_\theta(s)) = |\varrho_n(\mathcal{G}_\theta(s))|$. But from Assumption (i) it follows that $\sum_{n=1}^{\infty} \sigma_n(\mathcal{G}_\theta(s)) < \infty$. Hence $\mathcal{G}_\theta(s) \in \mathscr{B}_1(\ell^2)$, and since $\mathcal{F} \in \mathscr{B}(\ell^2)$, $\mathcal{F}\mathcal{G}_\theta(s) \in \mathscr{B}_1(\ell^2)$ [10]. ■

To prove Theorem 2 we need the following lemma.

Lemma A1: For $s \in \rho(\mathcal{A}_\theta)$, $\det[\mathcal{I} + \gamma\mathcal{F}\mathcal{G}_\theta(s)]$ is analytic in both γ and s.

Proof: For $s \in \rho(\mathcal{A}_\theta)$, $\gamma\mathcal{F}\mathcal{G}_\theta(s) \in \mathscr{B}_1(\ell^2)$ by Lemma 1. Also $\gamma\mathcal{F}\mathcal{G}_\theta(s) = \gamma\mathcal{F}\mathcal{C}_\theta(s\mathcal{I} - \mathcal{A}_\theta)^{-1}\mathcal{B}_\theta$ is clearly analytic in both γ and s for $s \in \rho(\mathcal{A}_\theta)$. Then it follows from [10, p163] that $\det[\mathcal{I} + \gamma\mathcal{F}\mathcal{G}_\theta(s)]$ too is analytic in both γ and s for $s \in \rho(\mathcal{A}_\theta)$. ■

Proof of Theorem 2: Since $\mathfrak{D}^\theta$ does not pass through any eigenvalues of $\mathcal{A}_\theta$, $s \in \rho(\mathcal{A}_\theta)$ and thus $\gamma\mathcal{F}\mathcal{C}_\theta(s\mathcal{I} - \mathcal{A}_\theta)^{-1}\mathcal{B}_\theta \in \mathscr{B}_1(\ell^2)$ from Lemma 1. Then from [9], $\big(\mathcal{I} + \gamma\mathcal{F}\mathcal{C}_\theta(s\mathcal{I} - \mathcal{A}_\theta)^{-1}\mathcal{B}_\theta\big)^{-1}$ exists and belongs to $\mathscr{B}(\ell^2)$ iff $\det[\mathcal{I} + \gamma\mathcal{F}\mathcal{C}_\theta(s\mathcal{I} - \mathcal{A}_\theta)^{-1}\mathcal{B}_\theta] \neq 0$, which is satisfied by assumption. Applying the matrix inversion lemma to $\big(\mathcal{I} + \gamma\mathcal{F}\mathcal{C}_\theta(s\mathcal{I} - \mathcal{A}_\theta)^{-1}\mathcal{B}_\theta\big)^{-1}$, we conclude that $s \in \rho(\mathcal{A}_\theta^{\rm cl})$ and $(s\mathcal{I} - \mathcal{A}_\theta^{\rm cl})^{-1} = (s\mathcal{I} - \mathcal{A}_\theta + \mathcal{B}_\theta\gamma\mathcal{F}\mathcal{C}_\theta)^{-1} \in \mathscr{B}(\ell^2)$.

Now since $(s\mathcal{I} - \mathcal{A}_\theta^{\rm cl})^{-1}$, $\mathcal{B}_\theta$, $\mathcal{C}_\theta$, and $\mathcal{F}$ all belong to $\mathscr{B}(\ell^2)$, and $(s\mathcal{I} - \mathcal{A}_\theta)^{-1} \in \mathscr{B}_1(\ell^2)$, we have $(s\mathcal{I} - \mathcal{A}_\theta^{\rm cl})^{-1}\mathcal{B}_\theta\gamma\mathcal{F}\mathcal{C}_\theta(s\mathcal{I} - \mathcal{A}_\theta)^{-1} \in \mathscr{B}_1(\ell^2)$ [10]. Thus, from the identity

$$(s\mathcal{I} - \mathcal{A}_\theta^{\rm cl})^{-1} - (s\mathcal{I} - \mathcal{A}_\theta)^{-1} = - (s\mathcal{I} - \mathcal{A}_\theta^{\rm cl})^{-1}\mathcal{B}_\theta\gamma\mathcal{F}\mathcal{C}_\theta(s\mathcal{I} - \mathcal{A}_\theta)^{-1},$$

it follows that $(s\mathcal{I} - \mathcal{A}_\theta^{\rm cl})^{-1} \in \mathscr{B}_1(\ell^2)$. In particular $(s\mathcal{I} - \mathcal{A}_\theta)^{-1}$ and $(s\mathcal{I} - \mathcal{A}_\theta^{\rm cl})^{-1}$ are both in $\mathscr{B}_0(\ell^2)$, which means that $\mathcal{A}_\theta$ and $\mathcal{A}_\theta^{\rm cl}$ both have discrete spectrum [12] and in

$$\frac{1}{2\pi j}\int_{\mathfrak{D}^\theta}(s\mathcal{I} - \mathcal{A}_\theta^{\rm cl})^{-1}ds - \frac{1}{2\pi j}\int_{\mathfrak{D}^\theta}(s\mathcal{I} - \mathcal{A}_\theta)^{-1}ds = - \frac{1}{2\pi j}\int_{\mathfrak{D}^\theta}(s\mathcal{I} - \mathcal{A}_\theta^{\rm cl})^{-1}\mathcal{B}_\theta\gamma\mathcal{F}\mathcal{C}_\theta(s\mathcal{I} - \mathcal{A}_\theta)^{-1}ds$$

each term on the left side is a finite-dimensional projection [10, p11, p15]. Taking the trace of both sides and changing the order of integration and trace on the right[3] we have

$$\mathrm{tr}\Big[\frac{1}{2\pi j}\int_{\mathfrak{D}^\theta}(s\mathcal{I} - \mathcal{A}_\theta^{\rm cl})^{-1}ds\Big] - \mathrm{tr}\Big[\frac{1}{2\pi j}\int_{\mathfrak{D}^\theta}(s\mathcal{I} - \mathcal{A}_\theta)^{-1}ds\Big] = - \frac{1}{2\pi j}\int_{\mathfrak{D}^\theta}\mathrm{tr}[(s\mathcal{I} - \mathcal{A}_\theta^{\rm cl})^{-1}\mathcal{B}_\theta\gamma\mathcal{F}\mathcal{C}_\theta(s\mathcal{I} - \mathcal{A}_\theta)^{-1}]ds. \quad \text{(A1)}$$

[3] Since $(s\mathcal{I} - \mathcal{A}_\theta^{\rm cl})^{-1}\mathcal{B}_\theta\gamma\mathcal{F}\mathcal{C}_\theta(s\mathcal{I} - \mathcal{A}_\theta)^{-1} \in \mathscr{B}_1(\ell^2)$, its trace is well-defined and finite.

From [12], the left side of (A1) is equal to the number of eigenvalues of $\mathcal{A}_\theta$ in $\mathbb{C}^+$ minus the number of eigenvalues of $\mathcal{A}_\theta^{\rm cl}$ in $\mathbb{C}^+$.

On the other hand, let the path $\mathfrak{C}^\theta$ be that traversed by $\det[\mathcal{I} + \gamma\mathcal{F}\mathcal{G}_\theta(s)]$ as s travels once around $\mathfrak{D}^\theta$ (where $\mathfrak{D}^\theta$ lies entirely in $\rho(\mathcal{A}_\theta)$). By Lemma A1, $\det[\mathcal{I} + \gamma\mathcal{F}\mathcal{G}_\theta(s)]$ is analytic in s, and if $\det[\mathcal{I} + \gamma\mathcal{F}\mathcal{G}_\theta(s)] \neq 0$ on $\mathfrak{D}^\theta$ we have

$$C\Big(0; \det[\mathcal{I} + \gamma\mathcal{F}\mathcal{G}_\theta(s)]\big|_{s\in\mathfrak{D}^\theta}\Big) = \frac{1}{2\pi j}\int_{\mathfrak{C}^\theta}\frac{dz}{z} = \frac{1}{2\pi j}\int_{\mathfrak{D}^\theta}\frac{\frac{d}{ds}\det[\mathcal{I} + \gamma\mathcal{F}\mathcal{G}_\theta(s)]}{\det[\mathcal{I} + \gamma\mathcal{F}\mathcal{G}_\theta(s)]}ds. \quad \text{(A2)}$$

Using [10, p163] and the assumption $\det[\mathcal{I} + \gamma\mathcal{F}\mathcal{G}_\theta(s)] \neq 0$, $s \in \mathfrak{D}^\theta$, we arrive at

$$\frac{\frac{d}{ds}\det[\mathcal{I} + \gamma\mathcal{F}\mathcal{G}_\theta(s)]}{\det[\mathcal{I} + \gamma\mathcal{F}\mathcal{G}_\theta(s)]} = \mathrm{tr}\Big[\big(\mathcal{I} + \gamma\mathcal{F}\mathcal{G}_\theta(s)\big)^{-1}\frac{d}{ds}\gamma\mathcal{F}\mathcal{G}_\theta(s)\Big] = -\mathrm{tr}[(s\mathcal{I} - \mathcal{A}_\theta^{\rm cl})^{-1}\mathcal{B}_\theta\gamma\mathcal{F}\mathcal{C}_\theta(s\mathcal{I} - \mathcal{A}_\theta)^{-1}].$$

This, together with (A2) and (A1) gives the required result. ■

Proof of Lemma 4: For $s \in \rho(\mathcal{A}_\theta)$, $\det[\mathcal{I} + \gamma\mathcal{F}\mathcal{G}_\theta(s)]$ is analytic in both γ and s by Lemma A1. The proof now proceeds exactly as in [2, p140] and is omitted. ■

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