Proceedings of the
44th IEEE Conference on Decision and Control, and
the European Control Conference 2005
Seville, Spain, December 12-15, 2005

Outer Approximations of The Minimal Disturbance Invariant Set

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Abstract— **This paper is concerned with outer approximations of the minimal disturbance invariant set (MDIS) of a discrete-time linear system with an additive set-bounded disturbance. The k-step disturbance reachable sets (Minkowski partial sums) are inner approximations of MDIS that converge to MDIS. Enlarged by a suitable scaling, they lead to outer approximations of MDIS. Two families of approximations, each based on partial sums, are considered: one minimizes the scalings of the partial sums and is not disturbance invariant, the other is generated by maximal disturbance invariant subsets of scaled partial sums. Theoretical properties of the families are proved and interrelated. Algorithmic questions, including error bounds for the approximations, are addressed. The results are illustrated by computational data from several examples.**

I. INTRODUCTION

In this paper we consider sets defined by the Minkowski sum,

$$F_k = EW + AEW + \cdots + A^{k-1}EW, \tag{1}$$

that play a well-known role in the disturbance response of the system

$$\begin{aligned} x(t+1) &= Ax(t) + Ew(t), \\ x(t) &\in R^n, w(t) \in W \subset R^m, m \leq n. \end{aligned} \tag{2}$$

Specifically, the following facts are known [7]. If A is asymptotically stable, W is compact and $0 \in W$, then there exists a compact set $F_\infty \subset R^n$ such that: $0 \in F_k \subset F_\infty$ for $k \in \{1, 2, \cdots, \} = N^+$ and $F_k \to F_\infty$ as $k \to \infty$. Further, F_∞ is: the limit set for all solutions of (2), disturbance-invariant (d-invariant) in the sense that $AF_\infty + EW \subset F_\infty$, minimal over the class of all d-invariant sets. For computational reasons, it is assumed hereafter that W is polytope. Thus, F_k is a polytope. Even so, with the exception of a few trivial cases [8], [7], F_∞ defies exact, concrete characterization. Consequently, efforts have been made to approximate F_∞ in various ways [2], [13], [7], [3], [11]. Since the F_k are inner approximations of F_∞, it is tempting to enlarge them so they then become relatively tight outer approximations. The simplest way of doing this is to use F_k-*approximations*: sets σF_k such that $F_\infty \subset \sigma F_k$ and $\sigma > 1$. See for example, [7] and [6] where algorithmic methods are proposed for approximately minimizing σ. Such 'minimal' F_k approximations do not in general retain the d-invariance property of F_k. This is unfortunate since d-invariance of sets that approximate F_∞ may be useful in a variety of robust control schemes. See, for example, [1], [8], [4] and [9].

Motivated, in part, by the d-invariant F_k approximations in [11], this paper considers two classes of outer approximations: the class of all F_k approximations and the class of d-invariant approximations defined by $O_\infty(\sigma F_k)$, maximal d-invariant subset of σF_k. Contributions include: an algorithm for minimizing σ over the class of F_k approximations that has advantages over the methods in [7] and [6]; results on the convergence of F_k approximations as $k \to \infty$; a comparison of the d-invariant $O_\infty(\sigma F_k)$ approximations and the d-invariant F_k approximation obtained in [11].

Contents of the paper are arranged as follows. This section concludes with assumptions, notations and a review of basic results that are used subsequently without further comment. Section 2 characterizes the class of all F_k approximations and presents the new algorithm for minimizing σ when $F_\infty \subset \sigma F_k$. Section 3 describes the $O_\infty(\sigma F_k)$ and its features. Measures of approximation error and algorithmic issues are addressed in Section 4. Section 5 presents several illustrative examples. Concluding remarks appear in Section 6.

It is assumed hereafter that A is asymptotically stable (spectral radius $\rho(A) < 1$), $E \in R^{n\times m}$ has rank m, the pair A, E is controllable and W is a polyhedral C set [3], i.e., W is compact and contains the origin in its non-empty interior. The assumptions are sufficient to show that there exists $k_{min} \in N^+$ ($k_{min} = 1$ when $m = n$) such that F_k has a non-empty interior for all $k \geq k_{min}$. Thus for $k \geq k_{min}$, F_k is a polyhedral C set. The representation

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for F_k is non-redundant : $F_k = \{x : e^i x \leq 1\, \forall i \in I_k\}$, where $e^i \in R^{1\times n}$, I_k is minimal (removing any i from I_k changes F_k) and has cardinality $|I_k|$. Minimality implies that $e^i \neq 0$ for all $i \in I_k$.

Three operations on subsets of R^n are needed: $\alpha X := \{\alpha x : x \in X\}, \alpha \in R; AX := \{Ax : x \in X\}, A \in R^{n\times n}; X + Y := \{x + y : x \in X, y \in Y\}$, the Minkowski sum. If $\alpha \neq 0$ and A is nonsingular, these operations preserve polyhedral and C set properties of their operands. Thus, for $k \geq k_{min}$ F_k is a polyhedral C set. If $X \subset R^n$, $intX$ and $|X|$ are its interior and cardinality respectively. The p-norm unit ball is $B_p = \{x : ||x||_p \leq 1\}$. For $p = 1$ and $p = \infty$, $B_p \subset R^n$ is a polyhedral C set of the form $B_p = \{x : e^i_p x \leq 1\, \forall i \in I_p\}$.

In this paragraph, $X, Y \subset R^n$ are assumed to be compact. The support function of $X, h_X(\eta) := max_{x\in X}\, \eta x$, is defined for all $\eta \in R^{1\times n}$; $h_{\alpha X}(\eta) = h_X(\alpha\eta)$, $h_{AX}(\eta) = h_X(\eta A)$, $h_{X+Y}(\eta) = h_X(\eta) + h_Y(\eta)$; if $X \subset \gamma B_2, h_X(\eta) \leq \gamma||\eta||_2$; if $0 \in intX, h_X(\eta) > 0$ for all $\eta \neq 0$. Suppose $X = \{x : e^i_X x \leq 1\, \forall i \in I_X\}$. Then $Y \subset X$ if and only if $h_Y(e^i_X) \leq 1$ for all $i \in I_X$; if the representation for X is non-redundant, $h_X(e^i_X) = 1$ for all $i \in I_X$. Suppose $Y \subset X$. Then the Hausdorff distance [12] between X and Y becomes $d(X,Y) = min\{\alpha : X \subset Y + \alpha B_p\}$. Thus, $Y + d(X,Y)B_p$ is the smallest closed p-neighborhood of Y that contains X. It is easy to confirm that $Z \subset Y$ implies $d(X,Y) \leq d(X,Z)$.

II. THE CLASS OF F_k APPROXIMATIONS

Suppose $intF_k \neq \emptyset$. The class of all F_k approximations is given by $\{\sigma F_k : \sigma \geq \underline{\sigma}_k\}$ where

$$\underline{\sigma}_k := min\{\sigma : F_\infty \subset \sigma F_k\}. \tag{3}$$

The indicated minimum exists because F_∞ and F_k are compact and $intF_k \neq 0$. Thus, $\underline{\sigma}_k F_k$ is the tightest F_k approximation of F_∞. While $\underline{\sigma}_k F_k$ is not generally d-invariant, it is useful in determining the size of disturbance induced errors in system (2). It is easy to confirm that $\underline{\sigma}_k \geq 1$. To avoid trivialities involving cases where $\underline{\sigma}_k = 1$, assume hereafter that there exists no $k \in N^+$ such that $F_k = F_\infty$.

Methods for approximately evaluating $\underline{\sigma}_k$ are described in [7] and [6]. Both papers, given $\hat{\sigma} > 1$, test the validity of the inclusion $F_\infty \subset \hat{\sigma} F_k$ by computing $O_\infty(\hat{\sigma} F_k)$, the maximal d-invariant subset of $\hat{\sigma} F_k$. If $O_\infty(\hat{\sigma} F_k) = \emptyset$, the inclusion fails; otherwise, the inclusion holds [7]. While the test is effective, it is computationally expensive. In [7], k is given and approximation of $\underline{\sigma}_k$ is determined by bisection on $\hat{\sigma}$ using the test; in [6], an acceptable value of $\hat{\sigma} > 1$ is specified and k is incrementally increased until $O_\infty(\hat{\sigma} F_k) \neq \emptyset$.

The approach for computing $\underline{\sigma}_k$ given here does not require the computation of $O_\infty(\sigma F_k)$. Using $F_k = \{x : e^i x \leq 1\ \forall i \in I_k\}$ the support-function characterization of inclusion mentioned in Section 1, $F_\infty \subset \sigma F_k$ is equivalent to $h_{F_\infty}(e^i) \leq \sigma h_{F_k}(e^i) = \sigma$ for all $i \in I_k$. Hence,

$$\underline{\sigma}_k = min_{i\in I_k} h_{F_\infty}(e^i) \tag{4}$$

Since F_∞ is compact the values $h_{F_\infty}(e^i)$ exist. Thus, $\underline{\sigma}_k$ exists. While the $h_{F_\infty}(e^i)$ are defined by

$$h_{F_\infty}(e^i) = \sum_{j=0}^{\infty} h_W(e^i A^j), \tag{5}$$

a means for computing them to a specified degree of accuracy is needed. Let $h^J_{F_\infty}(e^i) := \sum_{j=0}^{J-1} h_W(e^i A^j)$. This partial sum is easily evaluated. Its accuracy as an approximation of $h_{F_\infty}(e^i)$ can be determined because there exists a $\nu > 0$ such that $0 < h_W(e^i A^j) \leq \nu(\rho(A))^j$ for all $i \in I_k$. Specifically, $\nu = (max_{i\in I_k}||e^i||_2) \cdot (min_{\gamma>0}\{\gamma : w \in \gamma B_2\}) \cdot M$, where $M(\rho(A))^j \geq ||A^j||_2$ and $||A||_2$ is the induced norm of A. Thus, the series (5) converges geometrically and

$$\underline{\sigma}^J_k := min_{i\in I_k} h^J_{F_\infty}(e^i) + \nu(1-\rho(A))^{-1}(\rho(A))^J \tag{6}$$

approximates $\underline{\sigma}_k$ with error bound,

$$0 \leq \underline{\sigma}^J_k - \underline{\sigma}_k \leq \nu(1-\rho(A))^{-1}(\rho(A))^J. \tag{7}$$

To guarantee $\underline{\sigma}^J_k F_k$ is an outer bound for F_∞ it is necessary that $\underline{\sigma}^J_k \geq \underline{\sigma}_k$. This is the reason for including last term in (6). Without it, it is easy to see that $\underline{\sigma}^J_k < \underline{\sigma}_k$. A specified error bound δ is implemented by choosing J so that $\nu(1-\rho(A))^{-1}(\rho(A))^J < \delta$. With similar tolerances on the accuracy of $\sigma \approx \underline{\sigma}_k$, procedure (6) is considerably faster than the procedures described in [7] and [6]. Computations for the evaluation of the $h^J_{F_\infty}(e^i)$ are much simpler than those for computing $O_\infty(\sigma F_k)$ and do not involve trial and error tests involving the non-emptiness of $O_\infty(\sigma F_k)$.

III. THE d-INVARIANT APPROXIMATION $O_\infty(\sigma F_k)$

While $O_\infty(\sigma F_k)$ has been used in [7] and [6] as a test for inclusion, its potential as a d-invariant approximation of F_k has been neglected. The computation of $O_\infty(\sigma F_k)$ is easily implemented [7] if $O_\infty(\sigma F_k)$ is finitely determined. A sufficient condition for finite determination [7], is $F_\infty \subset int\sigma F_k$. It can be shown that this technical condition holds if $k \geq k_{min}$ and $\sigma > \underline{\sigma}_k$. Since σF_k is a polyhedral C set for $k \geq k_{min}$, finite determination implies $O_\infty(\sigma F_k)$ is also a polyhedral C set. In summary,

$$O_\infty(\sigma F_k) = \{x : e^i_{O_\infty} x \leq 1 \, \forall i \in I_{O_\infty(\sigma F_k)}\} \subset \sigma F_k \quad (8)$$

is a d-invariant approximation of F_∞ for all $k \geq k_{min}$ and $\sigma > \underline{\sigma}_k$.

Some observations are in order. Unlike d-invariant F_k approximations [11], $O_\infty(\sigma F_k)$ exists for all $k \geq k_{min}$. Most importantly, it exists when $m < n$, a case where d-invariant F_k approximations generally fail to exist. The complexity of $O_\infty(\sigma F_k)$, as measured by $|I_{O_\infty(\sigma F_k)}|$, is not known before its computation. It depends on k and σ, and may be less than, equal to or greater than $|I_k|$. The computational availability of $\underline{\sigma}^J_k > \underline{\sigma}_k$ assists the choice of σ. For example, given any $\delta > 0$, it is possible to choose $\sigma = (1+\delta)\underline{\sigma}^J_k$ so that $\sigma > (1+\delta)\underline{\sigma}_k$ and $\sigma \approx (1+\delta)\underline{\sigma}_k$. The ease of specifying such tolerance conditions on σ is valuable in doing computer experiments that relate σ to the complexity of $O_\infty(\sigma F_k)$.

IV. Approximation Errors and Their Algorithmic Implications

The error in F_k approximations of F_∞ is best measured by $d(\sigma F_k, F_\infty)$, the Hausdorff distance between σF_k and F_∞. Since $F_\infty \subset \sigma F_k$, $d(\sigma F_k, F_\infty) = min\{\alpha : \sigma F_k \subset F_\infty + \alpha B_p\}$. Because concrete characterizations of F_∞ are rare, there is no general way of evaluating $d(\sigma F_k, F_\infty)$. Nevertheless, a computationally feasible, fairly tight, upper bound for $d(\sigma F_k, F_\infty)$ is available. Since $F_k \subset F_\infty$, $d(\sigma F_k, F_\infty) \leq d(\sigma F_k, F_k) = min\{\alpha : \sigma F_k \subset F_k + \alpha B_p\} = min\{\alpha : (\sigma-1)F_k \subset \alpha B_p\}$. For $p = 1$ and $p = \infty$, the last set inclusion can be characterized by the support function condition mentioned in Section 1. Specifically, $B_p = \{x : e^i_p x \leq 1 \; \forall i \in I_p\}$ implies

$$d(\sigma F_k, F_\infty) \leq d(\sigma F_k, F_k) = \lambda_k(\sigma - 1), \text{where} \quad (9)$$

$$\lambda_k := max_{i \in I_p} h_{F_k}(e^i_p), \; h_{F_k}(e^i_p) = \sum_{j=0}^{k-1} h_W(e^i_p A^j). \quad (10)$$

Together, (9) and (10) provide a practical means for bounding the approximation errors associated with the sets $\sigma_k F_k$ and $\underline{\sigma}_k F_k$. They also apply to $O_\infty(\sigma F_k)$, since $O_\infty(\sigma F_k) \subset \sigma F_k$ implies $d(O_\infty(\sigma F_k), F_\infty) \leq d(\sigma F_k, F_\infty) = \lambda_k(\sigma - 1)$. It is easy to confirm that for $\sigma > \gamma > 0$, $d(\sigma F_k, \gamma F_k) = \lambda_k(\sigma - \gamma)$. Thus $d(\underline{\sigma}^J_k F_k, \underline{\sigma}_k F_k) \leq \lambda_k(\underline{\sigma}^J_k - \underline{\sigma}_k)$. The following theorem shows that the approximation errors can be made arbitrarily small.

Theorem As $k \to \infty$ the following limits exist: $\underline{\sigma}_k \to 1$, $d(\underline{\sigma}_k F_k, F_\infty) \to 0$, $\underline{\sigma}_k F_k \to F_\infty$

Proof: From (3), $\underline{\sigma}_k = min\{\sigma : F_\infty \subset F_k + (\sigma - 1)F_k\} \leq min\{\sigma : F_\infty \subset F_k + (\sigma-1)\delta B_p\}$ where $k \geq k_{min}$ and $\delta > 0$ is such that $\delta B_p \subset F_k$ (possible because $0 \in intF_k$ for $k \geq k_{min}$). It is shown in [7] (1998) that : given any $\varepsilon > 0$ there exists a $k \in N^+$ such that $F_\infty \subset F_k + \varepsilon B_p$. Thus, given any $\varepsilon > 0$, there exists a $k \in N^+$ such that $\underline{\sigma}_k - 1 < \varepsilon$. Since $\underline{\sigma}_{k+1} \leq \underline{\sigma}_k$ for all $k \in N^+$ (use $F_k \subset F_{k+1}$ in (3)), this implies $\underline{\sigma}_k \to 1$. Clearly, $F_k \subset F_\infty$ and the compactness of F_∞ imply $h_{F_k}(e^i_p) \leq h_{F_\infty}(e^i_p)$ for all $k \in N^+$. Hence, for all k, $\lambda_k \leq \lambda_{max} := max_{i \in I_p} h_{F_\infty}(e^i_p)$ and $d(\underline{\sigma}_k F_k, F_\infty) \leq d(\underline{\sigma}_k F_k, F_k) \leq \lambda(\underline{\sigma}_k - 1)$. This proves $d(\underline{\sigma}_k F_k, F_\infty) \to 0$. Since the Hausdorff distance is a metric on the family of compact set in R^n [12], $\underline{\sigma}_k F_k \to F_\infty$. □

Remark 1. Results similar to those of Theorem 1 are not possible for the d-invariant approximation $O_\infty(\sigma F_k)$. The requirement $\sigma > \underline{\sigma}_k$ does not, in itself, furnish an obvious choice for σ. One possibility is $\sigma = \hat{\sigma}_k := 0.5(\sigma_k + \underline{\sigma}_k)$ where $\sigma_k \to 1$ as $k \to \infty$. Then $F_\infty \subset O_\infty(\hat{\sigma}_k F_k) \subset \hat{\sigma}_k F_k$ and the results of Theorem 1 imply $\hat{\sigma}_k \to 1$, $d(\hat{\sigma}_k F_k, F_\infty) \to 0$ and $O_\infty(\hat{\sigma}_k F_k) \to F_\infty$.

V. Numerical Examples

The numerical results presented in this section involve three examples that represent experiments on many more. Details of the examples are given in Table 1. Results of the experiments are shown in Tables 2-3. They include a comparison with $\sigma_k F_k$, the d-invariant F_k approximations of [11]. The values of σ_k are obtained from their expression for $\alpha^o(k)$ by setting $\sigma_k = (1-\alpha^o(k))^{-1}$. In general, $\sigma_k > \underline{\sigma}_k$. The error measures of $d_k := d(\sigma_k F_k, F_\infty)$, based on the expressions of (9) and (10), are also included in the Tables. For some cases, a d-invariant F_k approximation does not exist. This situation is indicated by $\sigma_k = \infty$. For sufficiently large k, $\sigma_k < \infty$ exists and $\sigma_k \to 1$ as $k \to \infty$.

Each table shows $\sigma_k, \underline{\sigma}_k, d_k$ and $\underline{d}_k$ for values of k selected from a full range of k values. Other quantities are also included: $|I_k|$, the number of inequalities that define F_k; $\hat{\sigma}_k$, the value of σ in $O_\infty(\sigma F_k)$; $|I_{O_\infty}|$, the number of inequalities that define $O_\infty(\sigma F_k)$. The choice of $\hat{\sigma}_k = min\{1.05\underline{\sigma}_k, 0.5(\underline{\sigma}_k + \sigma_k)\} > \underline{\sigma}_k$. It is motivated by the desire to use $\sigma = 1.05\underline{\sigma}_k$ unless $\sigma_k - \underline{\sigma}_k$ is small (see Section 4). This choice guarantees $\underline{\sigma}_k < \hat{\sigma}_k < \sigma_k$. The error bound for $O_\infty(\hat{\sigma}_k F_k)$, $d(O_\infty(\hat{\sigma}_k F_k), F_k)$, is not shown because it can be made to be as close to $\underline{d}_k$ as desired by choosing $\sigma - 1$ to be sufficiently small.

Table 2 shows the results of Examples I and II. Both examples have the same A matrix but different disturbance sets W and matrices E. For Example I, where EW is a slender 2-dimensional set, the d-invariant F_k approximations exist only when $k \geq 4$. As expected, the sequences $\{\sigma_k\}, \{\underline{\sigma}_k\}, \{d_k\}$ and $\{\underline{d}_k\}$ tend toward their expected limits (σ_k, $\underline{\sigma}_k \to 1$ and $d_k, \underline{d}_k \to 0$) with increasing k with $\sigma_k > \underline{\sigma}_k$ and $d_k > \underline{d}_k$. Example II corresponds to the case where EW is a line segment. Here, d-invariant F_k approximations do not exist *for all* values of k. The d-invariant sets $O_\infty(\hat{\sigma}_k F_k)$ do exist for the choice of $\hat{\sigma}_k (= 1.05\underline{\sigma}_k) > \underline{\sigma}_k$. The sequences $\{\underline{d}_k\}$ and $\{\hat{\sigma}_k\}$ tend toward their respective limits of 0 and $\underline{\sigma}_k$ following the results of Theorem 1. The values of $|I_{O_\infty}|$ and $|I_k|$ for all k also do not differ greatly.

Figure 1 shows $\underline{\sigma}_k F_k \subset O_\infty(1.05\underline{\sigma}_k F_k) \subset \sigma_k F_k$ for Example I with $k = 5$. As expected from the differing values of σ_k and $\underline{\sigma}_k$, $\underline{\sigma}_k F_k$ and $O_\infty(1.05\underline{\sigma}_k F_k)$ are much smaller than $\sigma_k F_k$. The set $O_\infty(1.05\underline{\sigma}_k F_k)$ closely approximates $1.05\underline{\sigma}_k F_k$ (not shown), the situation that occurs most frequently in other examples. Figure 2 shows $\underline{\sigma}_k F_k$, $1.05\underline{\sigma}_k F_k$ and $O_\infty(1.05\underline{\sigma}_k F_k)$ for Example II with $k = 3$. The d-invariant F_k approximation is not shown as it does not exist. The set $O_\infty(1.05\underline{\sigma}_k F_k)$ is noticeably smaller than $1.05\underline{\sigma}_k F_k$ at its upper-left and lower-right corners, an improvement in the approximation of F_∞ that cannot be ascertained from the available error bounds.

Table 3 shows the results of Example III, a 3-dimensional example. In this case, the complexity of F_k increases very rapidly with k, a manifestation of the known [5] complexity result (with respect to both k and n) of the Minkowski summation. As in the previous two examples, the complexity of $O_\infty(\hat{\sigma}_k F_k)$ is about the same as F_k.

Several observations emerge from Tables 2-3 and other unreported examples: (a) a tradeoff exists between the accuracy and the complexity of the approximations. Tabulating $\underline{d}_k$ and $|I_k|$ at each k can help resolve this tradeoff. (b) d-invariant F_k approximations may not exist when k is small. (c) d-invariant F_k approximations will most likely not exist when EW is less than full dimension. However, d-invariant sets $O_\infty(\hat{\sigma}_k F_k), \hat{\sigma}_k > \underline{\sigma}_k$, $k \geq k_{min}$ do exist. (d) $O_\infty(\hat{\sigma}_k F_k)$ and F_k have similar complexity even when $\hat{\sigma}_k$ is close to $\underline{\sigma}_k$. (e) The complexity of F_k grows rapidly with k when $n > 2$.

TABLE I

DESCRIPTION OF A, $\rho(A)$ AND EW OF THE THREE EXAMPLES, E_n IS THE n-DIMENSIONAL IDENTITY MATRIX

Sets	A	$\rho(A)$	EW
I	$\begin{matrix} 0.44 & -0.24 \\ -0.56 & -0.24 \end{matrix}$	0.6	$E_2W, W = \{w \in R^2 : \|w_1\| \leq 1.0, \|w_2\| \leq 0.1\}$
II	$\begin{matrix} 0.44 & -0.24 \\ -0.56 & -0.24 \end{matrix}$	0.6	$[1\,0]^T W$, $W = \{w \in R : \|w\| \leq 1\}$
III	$\begin{matrix} 0.44 & -0.24 & 0.1 \\ -0.56 & -0.24 & 0.2 \\ 0.2 & 0.2 & 0.5 \end{matrix}$	0.63	$E_3W, W = \{w \in R^3 : \|\|w\|\|_\infty \leq 0.1\}$

TABLE II

RESULTS FOR SELECTED VALUES OF k FOR EXAMPLES I AND II.

Sets		$k = 2$	$k = 3$	$k = 5$	$k = 10$
I	$\|I_k\|$	8	12	20	40
	$\sigma_k/\underline{\sigma}_k$	∞/1.684	∞/1.413	1.986/1.111	1.035/1.007
	$d_k/\underline{d}_k$	∞/1.001	∞/0.742	2.062/0.232	0.079/0.017
	$\hat{\sigma}_k/\|I_{O_\infty}\|$	1.768/6	1.483/8	1.167/18	1.021/40
II	$\|I_k\|$	4	8	10	20
	$\sigma_k/\underline{\sigma}_k$	∞/1.785	∞/1.488	∞/1.127	∞/1.008
	$d_k/\underline{d}_k$	∞/1.1314	∞/0.863	∞/0.261	∞/0.018
	$\hat{\sigma}_k/\|I_{O_\infty}\|$	1.875/6	1.562/8	1.183/12	1.058/22

VI. CONCLUSIONS

This paper addresses outer approximations of F_∞ that are based on its partial sums F_k. Two classes of polyhedral approximations are investigated: F_k-approximations ($F_\infty \subset \sigma F_k$, $\sigma \geq \underline{\sigma}_k$) and σF_k-constrained maximal d-invariant approximations ($F_\infty \subset O_\infty(\sigma F_k) \subset \sigma F_k$, $\sigma > \underline{\sigma}_k$). Approximations within each class are algorithmically implementable and bounds on the resulting approximation error are available. While any level of approximation accuracy can in principle be achieved, high accuracy generally demands large k and complex approximating polyhedra. In applications, high accuracy is not likely to be needed. Thus, in evaluating error/complexity tradeoff, it is useful to have approximation data at each k.

The $O_\infty(\sigma F_k)$, d-invariant approximations are promising and deserve further attention. Compared to the d-invariant F_k approximations: they exist for all $k \geq k_{min}$; when $\underline{\sigma}_k$ is significantly smaller than σ_k, they permit significantly smaller approximation errors; they apply with essentially no change when EW is less than full dimension.

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TABLE III

RESULTS FOR SELECTED VALUES OF k FOR EXAMPLE III.

	$k=2$	$k=3$	$k=4$	$k=5$
$\|I_k\|$	36	86	151	238
σ_k / $\underline{\sigma}_k$	1.730/1.675	1.427/1.353	1.209/1.193	1.131/1.115
$d/\underline{d}$	0.146/0.135	0.099/0.082	0.054/0.050	0.035/0.032
$\hat{\sigma}_k/\|I_{O_\infty}\|$	1.702/36	1.391/88	1.201/145	1.123/229

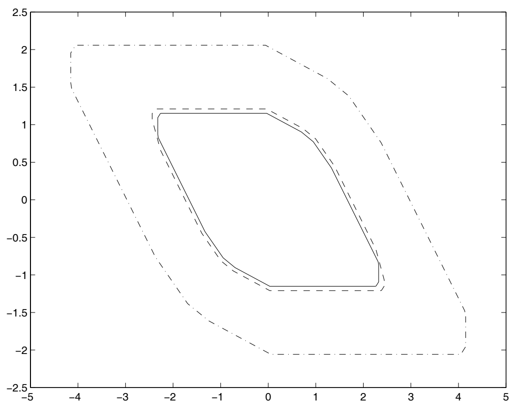

Fig. 1. The sets $\sigma_k F_k$, $O_\infty(1.05\underline{\sigma}_k F_k)$ and $\underline{\sigma}_k F_k$ of Example I, k=5

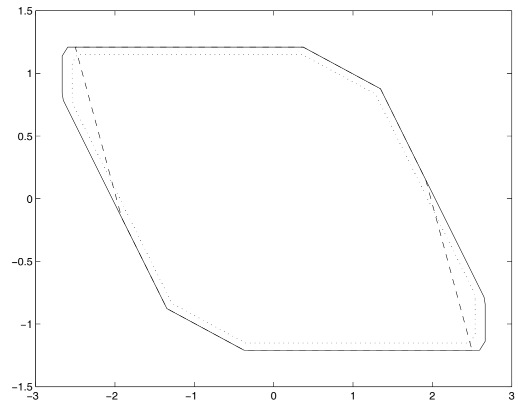

Fig. 2. The sets $1.05\underline{\sigma}_k F_k$, $O_\infty(1.05\underline{\sigma}_k F_k)$ and $\underline{\sigma}_k F_k$ of Example I, k=3

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