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Dissipativity Theory for Switched Systems

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Abstract— **A frame work of dissipativity theory for switched systems using multiple storage functions and multiple supply rates is set up. The exchange of "energy" between the activated subsystem and an inactivated subsystem is characterized by cross supply rates. Stability is reached when all supply rates are non-positive, as long as the total exchanged energy between the activated subsystem and any inactivated subsystems is finite. Passivity and L_2-gain are addressed. For both cases, asymptotic stability is guaranteed under certain "negative" output feedback. Feedback invariance of passivity and a small-gain theorem are also given.**

I. Introduction

Dissipativity theory of nonlinear systems, which was initiated by Willems [20] and developed further by Hill and Moylan [8], has become one of the major approaches to the study of complex systems. Storage functions induced by dissipativity usually provide natural candidates for Lyapunov functions. Therefore, in many cases, stability and stabilization problems can be solved once the dissipativity property is assured.

On the other hand, switched systems as an important class of hybrid systems have drawn considerable attention in recent years. Stability issues are the main concern when studying switched systems [1], [2], [11]. A common *Lyapunov* function method provides stability under an arbitrary switching law but is usually very restrictive [11]. The multiple Lypunov function technique, proposed by Peleties and DeCarlo [16], and further extended by Branicky [1] is a powerful tool for finding stabilizing switching laws [2], [11].

Dissipativity concepts are useful not only for smooth systems, but also for hybrid and switched systems. This has not received much attention until now with few results appearing on the topic. Dissipativity and stability analysis for impulsive systems of a single continuous vector field with impulsive terms were given in [4]. Passivity of nonlinear control systems based on completeness and using controller switching was discussed in [18]. There are many applications of passivity-based control of electrical systems with hybrid nature [3], [17]. However, all of the results mentioned above adopt a unified "storage function" to characterize dissipativity or passivity. This classical notion of dissipativity is obviously too restrictive in a hybrid and switching setting. To overcome this restriction, [21] proposed a notion of passivity by using multiple storage functions. Stability and feedback invariance were proven. However, this passivity concept requires each storage function to be non-increasing on the switching sequence of consecutive "switched on" times as a prerequisite to meet the Branicky's non-increasing condition of multiple Lyapunov functions and then to guarantee stability. Besides, no asymptotic stability is induced by this passivity property.

This paper presents a dissipativity theory for switched systems using multiple storage functions and multiple supply rates. Unlike continuous systems, a switched system has an unusual phenomenon that must taken into consideration when dealing with change of energy. A storage function of a subsystem is still "changing" or even grows on the time intervals when the subsystem is inactivated. This is simply because that all subsystems share the same state variables. This "changing" energy of any inactivated subsystem, though not necessarily real energy, is viewed as "exported energy" from the activated subsystem, and is characterized by cross supply rates.

II. Preliminaries

In this paper, we consider a switched system of the form:

$$\begin{aligned} \dot{x} &= f_\sigma(x, u_\sigma), \\ y &= h_\sigma(x), \end{aligned} \tag{1}$$

where $\sigma : R_+ = [0, \infty) \to M = \{1, 2, \cdots, m\}$ is the switching signal of any form, which may depend on time or state, or both, or even generated by higher level hybrid feedback in the loop, $x \in R^n$ is the state, u_i and $h_i(x)$ are the input vector and output vector of the i-th subsystem respectively. Further, $f_i(\cdot, \cdot) \in R^n$, $f_i(0, 0) = 0$, $h_i(0) = 0$. Here, we adopt the standard notations as in [1], [11]. The switching signal σ can be characterized by the switching sequence:

$$\Sigma = \{x_0; (i_0, t_0), \cdots, (i_n, t_n), \cdots, | i_n \in M, n \in N\}, \tag{2}$$

in which t_0 is the initial time, x_0 is the initial state and N is the set of nonnegative integers. When $t \in [t_k, t_{k+1})$, $\sigma(t) = i_k$, that is, the i_k-th subsystem is activated.

For any $j \in M$, let

$$\Sigma \mid j = \{t_{j_1}, t_{j_2}, \cdots, t_{j_n}, \cdots, i_{j_q} = j, q \in N\}$$

be the sequence of switching times when the j-th subsystem is switched on, and thus

$$\{t_{j_1+1}, t_{j_2+1}, \cdots, t_{j_n+1}, \cdots, i_{j_q} = j, q \in N\}$$

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is the sequence of switching times when the j-th subsystem is switched off.

Assumption 2.1. For any finite $T > t_0$, there exist a positive integer $K = K_T$, such that during the time interval $[t_0,\ T]$ the system (1) switches no more than K times, independently of the initial states in a vicinity of the origin.

This assumption is adopted to rule out arbitrarily fast switching. We also assume the existence and uniqueness of solutions of the system (1). More discussion can be found in [11].

By $L_1[0,\ \infty)$ we denote the usual L_1 function space over $[0,\ \infty)$, that is, $\mu = \mu(t) \in L_1[0,\ \infty)$ if $\int_0^\infty \mid \mu(t) \mid dt < \infty$. Let $L_1^+[0,\ \infty)$ denote the subset of $L_1[0,\ \infty)$ consisting of all no-negative functions. For simplicity, sometime we use $h_j(t)$ to denote $h_j(x(t))$.

III. Dissipativity

This section gives the description of dissipativity for switched systems by using multiple storage functions and multiple supply rates.

A. Definition of Dissipativity

The classical form of dissipativity is still applicable to switched system (1) with a supply rate function $\omega(\cdot, \cdot)$ and a storage function $S(x)$ as

$$S(x(t)) - S(x(t_0)) \le \int_{t_0}^{t} \omega(u_{\sigma(t)}, h_{\sigma(t)})dt. \tag{3}$$

However, this property that is standard for general nonlinear systems is much too restrictive for switched systems because each subsystem usually has its individual supply rate ω_i and thus a storage function $S_i(x)$ when this subsystem is activated. A common supply rate ω and thus a common storage function $S(x)$ for all subsystems may be difficult to find or may not exist at all. Therefore, it is reasonable and necessary to adopt multiple supply rates and multiple storage functions to characterize the dissipativity property for switched systems. However, a simple adoption of multiple storage functions and supply rates of each subsystem may result in the loss of some desirable properties that are expected to be induced by dissipativity, namely, stability. This may happen mainly because the negative impact of inactivated subsystems on the behavior of the whole switched systems. The major difficulty here is that unlike Branicky' multiple Lyapunov functions where a non-increasing condition on a "switched on" time sequence is a basic assumption, storage functions are allowed to grow when corresponding subsystems are inactivated. Therefore, the change of storage functions must be taken into account when the corresponding subsystems are inactivated.

Definition 3.1. System (1) is said to be dissipative under the switching law Σ if there exist positive definite continuous functions $S_1(x), S_2(x), \cdots, S_m(x)$, called storage functions, and functions $\omega_i^i(u_i, h_i), 1 \le i \le m$, called supply rates, and functions $\omega_j^i(x, u_i, h_i, t), 1 \le i, j \le m, i \ne j$, called cross supply rates, such that

$$\text{(i)} \quad \begin{aligned} & S_{i_k}(x(t)) - S_{i_k}(x(s)) \\ \le & \int_s^t \omega_{i_k}^{i_k}(u_{i_k}(\tau), h_{i_k}(\tau))d\tau, \\ & t_k \le s \le t < t_{k+1}, k = 0, 1, 2, \cdots, \end{aligned} \tag{4}$$

$$\text{(ii)} \quad \begin{aligned} & S_j(x(t)) - S_j(x(s)) \\ \le & \int_s^t \omega_j^{i_k}(x(\tau), u_{i_k}(\tau), h_{i_k}(\tau), \tau)d\tau, \\ & j \ne i_k, t_k \le s \le t < t_{k+1}, k = 0, 1, 2, \cdots, \end{aligned} \tag{5}$$

(iii) For any i, j, there exist $u_i(t)$ under which the origin is the equilibrium of all subsystems of the system (1), and $\phi_j^i(t) \in L_1^+[0, \infty)$, which may depend on u_i and the switching sequence Σ, such that

$$\omega_i^i(u_i(t), h_i(t)) \le 0, \forall i. \tag{6}$$

and

$$\omega_j^i(x(t), u_i(t), h_i(t), t) - \phi_j^i(t) \le 0, \forall j \ne i. \tag{7}$$

Remark 3.2. In Definition 3.1, S_j and ω_j^j are the usual storage function and supply rate respectively for the j-th subsystem when being activated. It is worthwhile noticing that the j-th subsystem is inactivated on the time interval $[t_k,\ t_{k+1})$ and thus the "energy" $S_j(x)$ may apparently remain unchanged. However, because all subsystems share the same state variable, $S_j(x)$ indeed changes from $S_j(x(t_k))$ to $S_j(x(t_{k+1}))$. This can be viewed as the result of "imported energy" from the activated i_k-th subsystem into the inactivated j-th subsystem. This "energy" is characterized by the "cross supply rate" ω_j^i from the i-th subsystem to the j-th subsystem and satisfies the dissipation inequalities (5).

Condition (iii) simply means that for at least one $u_i(t)$, if no external energy is supplied to the i-th subsystem when being activated, then the total "energy" coming from the activated i-th subsystem to the inactivated j-th subsystem is finite. This condition is natural and reasonable because otherwise infinitely large energy would be produced by the i-th subsystem without external energy. When the cross supply rates are of the form

$$\omega_j^i(x(t), u_i(t), h_i(t), t) = \vartheta_j^i(x(t))\omega_i^i(u_i(t), h_i(t)) + \phi_j^i(t)$$

with positive functions $\vartheta_j^i(x(t))$ and $\phi_j^i(t) \in L_1^+[0, \infty)$, Condition (iii) is automatically satisfied. In the sequel, for simplicity, we assume that any $u_i(t)$ satisfying (6) also satisfies (7).

Though all subsystems are assumed to be time invariant, the switched systems will have time variant features because of switching. This is even more so for the case of time-dependent switching laws. Considering this, the cross supply rates are defined to be time dependent to cover more general situations.

Remark 3.3. When all subsystems share a common supply rates and cross supply rates $\omega_j^i(\cdot,\ \ \cdot) = \omega(\cdot,\ \ \cdot)$, and thus share a common storage function $S_j(x) = S(x)$, then, (ii) and (iii) are automatically satisfied. Therefore, this notion of dissipativity is a natural generalization of the classical one.

B. Stability Analysis

Theorem 3.4. Under Assumption 2.1, if the system (1) is dissipative with storage function $S_i(x)$ satisfying $S_i(0) = 0$, then, the origin is stable in the sense of Lyapunov for any control $u_i(t)$ satisfying (6).

Proof. Condition (iii) says $\omega_i^i(u_i(t), h_i(t)) \leq 0$, and $\omega_j^i(x(t), u_i(t), h_i(t), t) \leq \phi_j^i(t)$ for some $\phi_j^i(t) \in L_1^+[0, \infty)$. For any constant $c > 0$, let $B(c) = \{x| \parallel x \parallel \leq c\}$, $r_i(c) = \min_x \{S_i(x)| \parallel x \parallel = c\}$ and $r(c) = \min_i \{r_i(c)\}$.

For any $\epsilon > 0$, since $\phi_j^i(t) \in L_1^+[0, \infty)$, there exists $T > 0$ such that for any T_1, T_2 , $T \leq T_1 \leq T_2 \leq \infty$, it holds that

$$\int_{T_1}^{T_2} \phi_j^i(t) dt < \frac{1}{2m} r(\epsilon), \quad i, j \in M, i \neq j. \tag{8}$$

Assumption 2.1 says that the system (1) switches at most K times on the time interval $[t_0,\ T]$ for some integer K. Thus, $t_K \geq T$, no matter where to start. Note that S_i is positive definite and $S_i(0) = 0$, we can find $\delta_1 > 0$, $\delta_1 < \epsilon$, such that $S_i(x) < \frac{1}{2} r(\epsilon)$ when $x \in B(\delta_1)$. For this $\delta_1 > 0$, we can find $\delta_2 > 0$, $\delta_2 < \delta_1$ such that $S_i(x) < r(\delta_1)$ when $x \in B(\delta_2)$. Continuing this procedure, we finally have a sequence

$$\epsilon = \delta_0 > \delta_1 > \delta_2 > \cdots > \delta_K > 0$$

with the property:

$$\begin{array}{l} S_i(x) < r(\delta_p), \text{when } x \in B(\delta_{p+1}), p = 1, 2, \cdots, K-1, \forall i, \\ S_i(x) < \frac{1}{2} r(\epsilon), \quad \text{when} \quad x \in B(\delta_1), \forall i \end{array} \tag{9}$$

Note that $S_{i_k}(x(t))$ decreases when the i_kth subsystem is activated, if we start within $B(\delta_K)$, we will stay in $B(\delta_1)$ as long as we switch no more than K times and no matter how we switch. This implies $x(t) \in B(\delta_1), t \in [t_0,\ t_K]$ if $x(0) \in B(\delta_K)$. In particular, $S_i(x(t_K)) < \frac{1}{2} r(\epsilon), i \in M$.

Now, for any $j \in M$, let $t_{j_q} \in \Sigma \mid j$ and $t_{j_q} > t_K$. Obviously, $j_q \geq K + 1$. It is easy to deduce from (4), (5) that

$$\begin{aligned} & S_j(x(t_{j_q})) - S_j(x(t_K)) \\ = & \sum_{\lambda=0}^{j_q-K-1} \left(S_j(x(t_{K+\lambda+1})) - S_j(x(t_{K+\lambda})) \right) \\ \leq & \sum_{\lambda=0}^{j_q-K-1} \int_{t_{K+\lambda}}^{t_{K+\lambda+1}} \psi_j^{i_{K+\lambda}}(t) dt, \end{aligned} \tag{10}$$

where

$$\begin{array}{l} \psi_j^{i_{K+\lambda}}(t) = \\ \begin{cases} \omega_j^j(u_j(t), h_j(t)) & \text{if} \quad i_{K+\lambda} = j, \\ \omega_j^{i_{K+\lambda}}(x(t), u_{i_{K+\lambda}}(t), h_{i_{K+\lambda}}(t), t) & \text{if} \quad i_{K+\lambda} \neq j \end{cases} \end{array} \tag{11}$$

Note that $\omega_j^j \leq 0$ by assumption and $\omega_j^{i_{K+\lambda}} \leq \phi_j^{i_{K+\lambda}}$, we know that $\psi_j^{i_{K+\lambda}} \leq \phi_j^{i_{K+\lambda}}$.

Therefore,

$$\begin{aligned} & S_j(x(t_{j_q})) - S_j(x(t_K)) \\ & \leq \sum_{\lambda=0,\ i_{K+\lambda} \neq j}^{j_q-K-1} \int_{t_{K+\lambda}}^{t_{K+\lambda+1}} \phi_j^{i_{K+\lambda}}(t) dt < \frac{1}{2} r(\epsilon) \end{aligned} \tag{12}$$

That is,

$$S_j(x(t_{j_q})) \leq S_j(x(t_K)) + \frac{1}{2} r(\epsilon) < r(\epsilon)$$

Thus, $x(t) \in B(\epsilon)$ for all t and stability follows.

Remark 3.5. Normally, stability is addressed for the system (1) with $u_i = 0$. Here we consider stability with respect to a specific class of inputs satisfying (6).

IV. Passivity

Passivity is one of the most useful forms of dissipativity. In this section we define passivity for system (1) and establish a passivity theorem.

Definition 4.1. System (1) is said to be passive under the switching law Σ if it is dissipative with respect to

$$\omega_j^j(u_j, h_j) = u_j^T h_j - \delta_j u_j^T u_j - \epsilon_j h_j^T h_j, \quad j = 1, 2, \cdots, m.$$

for some $\delta_j \geq 0, \epsilon_j \geq 0$, and strict input (output) passive if $\delta_j > 0$ ($\epsilon_j > 0$).

Remark 4.2. We only need the supply rates $\omega_j^j(u_j, h_j)$ to be quadratic. Other cross supply rates which represent energy exchange between different subsystems are allowed to take arbitrary forms. This makes the passivity concept vary broad.

A. Switched KYP Condition

We analyze conditions for passivity of switched systems in this subsection. We focus on smooth affine switched systems of the form:

$$\begin{array}{rcl} \dot{x} & = & f_\sigma(x) + g_\sigma(x) u_\sigma, \\ y & = & h_\sigma(x). \end{array} \tag{13}$$

and look for smooth storage functions and continuous supply rates and cross supply rates.

Since for the system (13) strict input passivity is never satisfied, we only consider passivity with supply rates

$$\omega_i^i(u_i, h_i) = u_i^T h_i - \epsilon_i h_i^T h_i.$$

We have infinitely many choices of cross supply rates. Here, for simplicity, we limit ourself to cross supply rates of the form

$$\omega_j^i(x, u_i, h_i, t) = \varphi_j^i(x) \omega_i^i(u_i, h_i). \tag{14}$$

for some positive continuous functions $\varphi_j^i(x)$.

Also, we only consider a state-dependent switching law of the form $\sigma(t) = \sigma(x(t)) = i$ if $x(t) \in \Omega_i$, and $\bigcup_{i=1}^m \Omega_i = R^n$, int $\Omega_i \bigcap \Omega_j = \emptyset, i \neq j$.

Note that in this case, Condition (i) and (ii) in Definition 3.1 can be written into a unified form:

$$\begin{aligned} & S_j(x(t)) - S_j(x(s)) \\ & \leq \int_s^t \varphi_j^{i_k}(x(\tau)) \omega_{i_k}^{i_k}(u_{i_k}(\tau), h_{i_k}(\tau)) d\tau, \\ & \forall j, k, t_k \leq s \leq t < t_{k+1} \end{aligned} \tag{15}$$

with $\varphi_i^i(x) = 1$, which yields the localized form of the well-known passivity (or KYP) conditions [9]

$$\begin{aligned} L_{f_i} S_j &\leq -\epsilon_i \varphi_j^i h_i^T h_i, \quad x \in \Omega_i, \\ L_{g_i} S_j &= \varphi_j^i h_i^T, \quad x \in \Omega_i. \end{aligned} \tag{16}$$

Condition (iii) is obviously satisfied due to (14).

B. Stabilization by Output Feedback

In this subsection, we show how passivity induces asymptotic stability via output feedback.

We first introduce the concept of asymptotic zero state detectability for nonlinear systems, which will be used to prove the asymptotic stability.

Definition 4.3. A system

$$\begin{aligned} \dot{x} = f(x), \\ y = h(x) \end{aligned} \tag{17}$$

is called asymptotically zero state detectable if for any $\epsilon > 0$, there exists $\delta > 0$, such that when $\| y(t+s) \| < \delta$ holds for some $t \geq 0$, $\Delta > 0$ and $0 \leq s \leq \Delta$, we have $\| x(t) \| < \epsilon$.

Remark 4.4. This asymptotic zero state detectability is a weaker version of small-time norm observability [5].

Theorem 4.5. If the system (1) is passive, then, the origin is stable in the sense of Lyapunov for any control $u_i(t)$ satisfying $u_i^T(t) h_i(t) \leq 0$. If in addition, all S_i are globally defined radially unbounded, there exist at least one j such that $\lim_{k\to\infty}(t_{j_k+1} - t_{j_k}) \neq 0$, and all subsystems of (1) are asymptotically zero state detectable, then, the origin is globally asymptotically stable by the output feedback $u_i = -h_i$.

Proof. Stability follows from Theorem 3.4. We now show global attractiveness.

Substituting the output feedback $u_i = -h_i$ into the passivity inequality (4) gives rise to

$$\begin{aligned} &\zeta_{i_k} \int_s^t \|h_{i_k}(t)\|^2 dt \\ \leq\ & S_{i_k}(x(s)) - S_{i_k}(x(t)), \quad t_k \leq s \leq t < t_{k+1} \end{aligned} \tag{18}$$

with $\zeta_i = 1 + \delta_i + \epsilon_i$. For the integer j satisfying $\lim_{k\to\infty}(t_{j_k+1} - t_{j_k}) \neq 0$, we can select $\delta > 0$ such that the set $\Lambda = \{k | t_{j_k+1} - t_{j_k} \geq \delta\}$ is infinite. Define a function

$$\tilde{h}_j(t) = \begin{cases} h_j(x(t)), & t \in \bigcup_{k\in\Lambda} [t_{j_k}, \quad t_{j_k+1}) \\ 0, & \text{otherwise} \end{cases} \tag{19}$$

For any $t > 0$, when $t_{j_k} \leq t < t_{j_k+1}$ for some $k \in \Lambda$, (18) gives

$$\begin{aligned} &\zeta_j \int_{t_0}^t \tilde{h}_j^T(t) \tilde{h}_j(t) dt \leq S_j(x(t_{j_1})) - S_j(t_{j_k+1}) \\ &\quad + \sum_{p=1}^{k-1} \left(S_j(x(t_{j_{p+1}})) - S_j(x(t_{j_p+1})) \right) \\ \leq\ & S_j(x(t_{j_1})) - S_j(t_{j_k+1}) \\ &\quad + \sum_{p=1}^{k-1} \sum_{\lambda=1}^{j_{p+1}-j_p-1} \int_{t_{j_p+\lambda}}^{t_{j_p+1+\lambda}} \phi_j^{i_{j_p+\lambda}}(t) dt \\ \leq\ & S_j(x(t_{j_1})) - S_j(t_{j_k+1}) \\ &\quad + \sum_{i=1,\ i\neq j}^{m} \int_{t_0}^{\infty} \phi_j^i(t) dt < \infty. \end{aligned} \tag{20}$$

When $t \notin [t_{j_k}, t_{j_k+1})$ for any $k \in \Lambda$, there exists $k \in \Lambda$ such that $t \geq t_{j_k+1}$ and $t < t_{j_q}$ for any $q \in \Lambda$ and $q > k$. In this case, we have $\tilde{h}_j(s) \equiv 0, s \in [t_{j_k+1}, t]$, and (20) still holds. It follows from (20) that $\int_{t_0}^{\infty} \tilde{h}_j^T(t) \tilde{h}_j(t) dt$ is finite. Now , we show $\tilde{h}_j(t) \to 0$ as $t \to \infty$. Suppose this is false, then there exist $\epsilon > 0$ and a sequence of time t, say, $q_1, q_2, \cdots, q_k \to \infty$, satisfying

$$\tilde{h}_j^T(q_i) \tilde{h}_j(q_i) \geq \epsilon, \forall i.$$

Note that (18) and Condition (ii) in Definition 3.1 guarantee the boundedness of $x(t)$, and $\dot{x} = f_\sigma(x, -h_\sigma(x))$ is also bounded. Hence, $\tilde{h}_j(t)$ is uniformly continuous over $\bigcup_{k\in\Lambda}[t_{j_k}, \quad t_{j_k+1})$. In view of $t_{j_k+1} - t_{j_k} \geq \delta$, we have $\int_{t_0}^{\infty} \tilde{h}_j^T(t) \tilde{h}_j(t) dt = \infty$, which contradicts the fact that $\int_{t_0}^{\infty} \tilde{h}_j^T(t) \tilde{h}_j(t) dt$ is finite. Therefore, $\tilde{h}_j(t) \to 0$. So, $x(t_{j_k}) \to 0$ as $k \to \infty$ and $k \in \Lambda$ follows from the asymptotic zero state detectability of the j-th subsystem, and this in turn implies $x(t) \to 0$ as $t \to \infty$ due to stability of the closed-loop system and continuity of $x(t)$.

Remark 4.6. The control u_i can be chosen as output feedback of the form $u_i = -\xi_i(h_i)$ satisfying $\xi_i^T(y) y > 0$ for any y.

Remark 4.7. If the system (1) is strict output passive, global asymptotic stability follows for $u_i = 0$. The proof is similar.

C. Passivity Theorem

Consider the passive switched systems

$$H_1: \quad \begin{aligned} \dot{x}^1 &= f_{\sigma_1}^1(x^1, u_{\sigma_1}^1), \\ y^1 &= h_{\sigma_1}^1(x^1) \end{aligned} \tag{21}$$

with $x^1 \in R^{n_1}$, and

$$H_2: \quad \begin{aligned} \dot{x}^2 &= f_{\sigma_2}^2(x^2, u_{\sigma_2}^2), \\ y^2 &= h_{\sigma_2}^2(x^2) \end{aligned} \tag{22}$$

with $x^2 \in R^{n_2}$. The feedback interconnection of H_1 and H_2 is depicted in Fig.3 below.

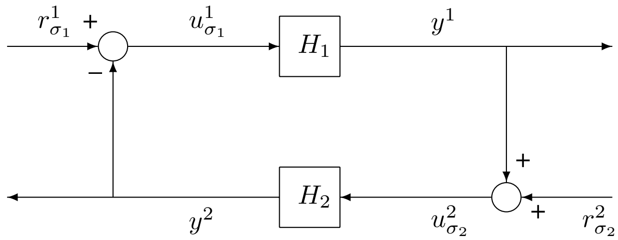

Fig.3. Feedback interconnection

Theorem 4.8. Suppose switched systems H_1 and H_2 are passive. Then, the feedback interconnected system shown in Fig.3 is again a passive switched system as long as the corresponding interconnected subsystems are compatible in the sense of dimensions.

Furthermore, if both H_1 and H_2 are strictly passive, then the interconnected system is also strictly passive.

Proof. Since we do not need the assumption of minimal switching sequence, without loss of generality, we suppose the two switched systems share the same switching times $\{t_0, t_1, \cdots, t_k, \cdots\}$.

Suppose the switching sequences of switched systems H_1 and H_2 are respectively

$$\begin{aligned}\Sigma_1 = \;& \Big\{x_0^1; (i_0^1, t_0), (i_1^1, t_1), \cdots, (i_j^1, t_j), \cdots, \\ & \mid i_j^1 \in M_1 = \{1, 2, \cdots, m_1\}, j \in N\Big\}\end{aligned} \tag{23}$$

and

$$\begin{aligned}\Sigma_2 = \;& \Big\{x_0^2; (i_0^2, t_0), (i_1^2, t_1), \cdots, (i_j^2, t_j), \cdots, \\ & \mid i_j^2 \in M_2 = \{1, 2, \cdots, m_2\}, j \in N\Big\}\end{aligned} \tag{24}$$

and the storage functions are $S_j^1(x^1)$ and $S_j^2(x^2)$. The supply rates and cross supply rates are

$$\omega_{1j}^j = (u_j^1)^T h_j^1 - \epsilon_{1j}(h_j^1)^T h_j^1, \quad \omega_{1j}^i(x^1, u_i^1, h_i^1, t),$$
$$\omega_{2j}^j = (u_j^2)^T h_j^2 - \epsilon_{2j}(h_j^2)^T h_j^2, \quad \omega_{2j}^i(x^2, u_i^2, h_i^2, t),$$

The feedback interconnected system of H_1 and H_2 has the state space representation given by

$$\overline{H}: \begin{aligned} \dot{x} &= \begin{pmatrix} \dot{x}^1 \\ \dot{x}^2 \end{pmatrix} = \begin{pmatrix} f_{\sigma_1}^1(x^1, r_{\sigma_1}^1 - y^2) \\ f_{\sigma_2}^2(x^2, r_{\sigma_2}^2 + y^1) \end{pmatrix} \\ &= f_\sigma(x, \overline{u}_\sigma), \\ y &= \begin{pmatrix} y^1 \\ y^2 \end{pmatrix} = \begin{pmatrix} h_{\sigma_1}^1 \\ h_{\sigma_2}^2 \end{pmatrix} = h_\sigma(x) \end{aligned} \tag{25}$$

where

$$\overline{u}_\sigma = \begin{pmatrix} r_{\sigma_1}^1 \\ r_{\sigma_2}^2 \end{pmatrix}, \sigma = \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} : [0, \ \infty) \to M = M_1 \times M_2.$$

The composite switching law is

$$\Sigma = \{x_0; (i_0, t_0), (i_1, t_1), \cdots, (i_j, t_j), \cdots, |i_j \in M, j \in N\},$$

where $x_0 = ({x_0^1}^T, {x_0^2}^T)^T$, $i_j = \begin{pmatrix} i_j^1 \\ i_j^2 \end{pmatrix} = \begin{pmatrix} \sigma_1(t_j) \\ \sigma_2(t_j) \end{pmatrix}$

Now, define

$$S_{ij}(x) = S_i(x^1) + S_j(x^2), \quad (i, j) \in M_1 \times M_2,$$

Note that

$$\overline{u}_\sigma^T h_\sigma = {u_{\sigma_1}^1}^T h_{\sigma_1}^1 + {u_{\sigma_2}^2}^T h_{\sigma_2}^2,$$

it is easy to check (i) , (ii) and (iii) in Definition 3.1. with cross supply rates

$$\begin{aligned} & \omega_{ij}^{pq}(x, \overline{u}_{(p,q)^T}, h_{(p,q)^T}, t) \\ = \;& \omega_i^p(x^1, r_p^1 - h_q^2, h_p^1, t) + \omega_j^q(x^2, r_q^2 + h_p^1, h_q^2, t). \end{aligned}$$

Therefore, the feedback interconnected system is passive. The proof for strict passivity is trivial.

V. L_2-GAIN

Definition 5.1. The system (1) is said to have L_2-gain $\gamma > 0$ if it is dissipative with respect to $\omega_i^i = \frac{1}{2}\gamma^2 u_i^T u_i - \frac{1}{2} h_i^T h_i, i = 1, 2, \cdots, m.$

Similar to the definition of passivity, we do not specify the form of cross supply rates $\omega_i^j, i \neq j$.

A. Switched Hamilton-Jacobi Inequalities

For affine switched system (13), cross supply rates (14) and state-dependent switching, we can easily have

$$L_{f_i} S_j + \frac{1}{2\varphi_j^i \gamma^2}(L_{g_i} S_j)(L_{g_i} S_j)^T + \frac{1}{2}\varphi_j^i h_i^T h_i \leq 0, \quad x \in \Omega_i. \tag{26}$$

B. Stability

Theorem 5.2. If the system (1) has L_2-gain γ, then, the origin is stable for any control $u_i(t)$ satisfying

$$\| u_i(t) \|^2 \leq \frac{(1 - \zeta_i^2)}{\gamma^2} \| h_i(t) \|^2 \tag{27}$$

for some $\zeta_i, 0 \leq \zeta_i \leq 1$. If in addition, $0 < \zeta_i \leq 1$, all S_i are globally defined radially unbounded, there exist at least one j such that $\lim_{k\to\infty}(t_{j_k+1} - t_{j_k}) \neq 0$, and all subsystems of (1) are asymptotically zero state detectable, then, the origin is globally asymptotically stable.

Proof. Similar to the proof of Theorem 4.5.

C. Small-gain Theorem

Suppose we have two switched systems:

$$H_1: \quad \begin{aligned} \dot{x} &= f_{\sigma_1}(x, u_{\sigma_1}), \\ y &= h_{\sigma_1}(x), \end{aligned} \tag{28}$$

and

$$H_2: \quad \begin{aligned} \dot{z} &= g_{\sigma_2}(z, v_{\sigma_2}), \\ w &= l_{\sigma_2}(z), \end{aligned} \tag{29}$$

where $\sigma_i : R_+ \to M_i = \{1, 2, \cdots, m_i\}, i = 1, 2.$ The meaning of other variables are the same as those in the system (1).

Without loss of generality, we assume that the two switched systems have the same switching time sequence

$\{t_0, t_1, \cdots, t_k, \cdots\}$. When $t \in [t_k, t_{k+1})$, the i_k^1-th and i_k^2-th subsystems of H_1 and H_2 are activated respectively.

Theorem 5.3. Suppose that H_1 has L_2-gain γ_1 with S_{1i}, ω_{1j}^i, and H_2 has L_2-gain γ_2 with S_{2i}, ω_{2j}^i respectively. If $\gamma_1\gamma_2 < 1$, and

$$\begin{aligned} & \omega_{1j}^i(x(t), u_i(t), h_i(t), t) \\ \leq \;& C(t)\left(\omega_{1i}^i(u_i(t), h_i(t))\right) + \varphi_{1j}^i(t), \\ & \omega_{2j}^i(z(t), v_i(t), l_i(t), t) \\ \leq \;& C(t)\left(\omega_{2i}^i(v_i(t), l_i(t))\right) + \varphi_{2j}^i(t) \end{aligned} \tag{30}$$

for some nonnegative function $C(t)$, functions $\varphi_{1j}^i(t), \varphi_{2j}^i \in L_1^+[0,\infty)$, then, the feedback interconnected system of H_1 and H_2 with

$$u_{\sigma_1} = -l_{\sigma_2}(z), \quad v_{\sigma_2} = h_{\sigma_1}(x) \tag{31}$$

is stable. If in addition, all S_{1i}, S_{2i} are globally defined radially unbounded, there exists at least one interconnected subsystem having infinite activated time intervals with positive dwell time, and all subsystems of H_1 and H_2 are asymptotically zero state detectable, then, the feedback interconnected system is globally asymptotically stable.

Proof. Applying the method used by Hill and Moylan for non-switched systems[10], [19], we can easily complete the proof.

Remark 5.4. It is not surprising that in addition to the usual small-gain condition $\gamma_1\gamma_2 < 1$, we still need the condition (30), which is a constraint on cross supply rates. This constraint can be regarded as a generalization of the small-gain condition between inactivated coupled subsystems. In particular, when cross supply rates are the same as supply rates, this condition is automatically satisfied. In general, the cross supply rates are allowed to take any forms.

VI. Concluding Remarks

We have established a framework of dissipativity theory for switched systems. Multiple storage functions and multiple supply rates are adopted to describe dissipativity. The introduction of cross supply rates relates the active subsystem and inactivated subsystems. As in classic notion of dissipativity, where the supply rate may represent abstract energy, cross supply rates in dissipativity of switched systems are abstract "exchanged energy" which simply represents the change of storage functions of inactivated subsystems cause by the activated subsystem. Therefore, we do not limit the form of cross supply rates in the description of dissipativity to cover more general situations. In particular, cross supply rates are allowed to be positive even though all supply rates are negative. This feature makes the dissipativity theory established here different to Branicky's theory of multiple Lyapunov functions, in which the non-decreasing condition of a Lyapunov function on the "switched on" time sequence is a key prerequisite.

It is worthwhile pointing out that sometimes this general cross supply rates may "weaken" the results. For example, the L_2-gain property only describes the input-output gain over the active time intervals. In order to have the input-output gain over the infinite time domain, which is related to H_∞ control, certain constraints must be imposed on the cross supply rates. This will be considered in our separate papers.

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