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Dynamic Observer Error Linearization

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Abstract— **In this paper, a new framework of observer error linearization problem is proposed. The main idea of our approach is twofold. The one is to introduce an auxiliary dynamics whose input is the system output, and the other is to transform the augmented system into an observable linear system with an injection term which contains the system output as well as the state of the auxiliary dynamics. It is a natural extension of the recently developed dynamic observer error linearization where the injection term contains only newly defined output. It is also shown that whenever an n dimensional system is immersible into an $n+d$ dimensional linear system up to an output injection, then it can be also dynamically observer error linearizable in our sense with a d dimensional auxiliary dynamics. Moreover, we show that the converse is not true by providing a counterexample, which implies that our approach is applicable to a strictly wider class of systems than that of the system immersion method.**

I. INTRODUCTION

Originated from the pioneering works [1], [2], the observer error linearization problem has been studied extensively. It is on the equivalence of a system to an observable linear system with output injection, which is often called the nonlinear observer canonical form (NOCF for short). Various approaches and extensions to this problem are available in the literature. In particular, extensions to multi-output case were studied in [3]–[7], and the generalized characteristic equation was used in [8]. The output dependent time scaling was employed in [9], [10]. However, all of the results cited above have a fundamental restriction; simply for single output systems with dimension n, the nth derivative of the output should be a degree n polynomial (in the sense of [3]) of time derivatives of the output [3].

Recently, motivated by [11]–[13], some researchers relaxed this restriction by transforming the system into a higher dimensional NOCF. For example, a chain of integrators (whose input is the output of the given system) is included to linearize the system in the extended state space [14] and system immersion was employed in [15]–[17]. As can be seen in the examples of the papers, these approaches can overcome the 'polynomial' type restriction.

We note that the basic ideas of [14] and [15] are different. The former integrates the output several times and define the last signal as a new output of the augmented system so that the augmented system with the new output is observer error linearizable, while the latter differentiates the output until it satisfies a differential equation in a specific form.

In this paper, we generalize [14] to define a problem called dynamic observer error linearization (DOEL). Although our approach shares the same idea (integrating the output) with [14], the observer canonical forms involved are fundamentally different; not only the newly defined output but also the integrals of the original output can be included in the injection term, which is a sharp contrast to [14] where only the newly defined output is used in the injection term. One of the results of our generalization is that even in the single output case, there exists a system which is not diffeomorphic to NOCF but dynamically linearizable in our sense, which is not the case of [14]. Moreover, our approach is more 'natural' than that of [14] because the integrals of the output are known signals, and because if a system is dynamically linearizable in our sense with d dimensional chain of integrators then it holds with any chain of integrators whose length is larger than d, while [14] can't guarantee this. The most surprising result of this generalization is that under the observability assumption our framework covers the work of [15], [17]. That is to say, whenever an n dimensional observable system is immersible into $n+d$ dimensional NOCF, then it can be also dynamically observer error linearizable with a d dimensional auxiliary dynamics. Moreover, we show that the converse is not true by providing a counterexample, which means that our approach is applicable to a strictly larger class of systems than that of the system immersion method.

The paper is organized as follows. We formulate the problem and provide basic results in Section II. In Section III, it is shown that linearization by system immersion is a special case of our approach. The relation of the proposed approach with the output dependent time scaling and possible extensions are discussed in Section IV and Section V concludes the paper.

II. DYNAMIC OBSERVER ERROR LINEARIZATION

A. Motivating Example

We begin with a simple example which illustrates the main idea of this paper.

Example 1: Consider the system

$$\begin{aligned}&\dot{\xi}_1 = \xi_2,\ \dot{\xi}_2 = \xi_3,\ \dot{\xi}_3 = -4\xi_1\xi_3 - 3\xi_2^2 - 6\xi_1^2\xi_2 \\ &y = \xi_1.\end{aligned} \tag{1}$$

First, we note that the system is not diffeomorphic to 3 dimensional NOCF even though an output diffeomorphism

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is employed, because it is impossible to find functions a_1, $\cdots$, a_3 and a diffeomorphism ψ satisfying

$$L_f^3\psi(\xi_1) = L_f^2 a_1(\xi_1) + L_f a_2(\xi_1) + a_3(\xi_1),$$

which is a necessary and sufficient condition [3], [15], [17]. To see this, we expand the equation to obtain

$$\begin{aligned}\psi'''(\xi_1)\xi_2^3 + 3\psi''(\xi_1)\xi_2\xi_3 + \psi'(\xi_1)[-4\xi_1\xi_3 - 3\xi_2^2 - 6\xi_1^2\xi_2] \\ = a_1''(\xi_1)\xi_2^2 + a_1'(\xi_1)\xi_3 + a_2'(\xi_1)\xi_2 + a_3(\xi_1).\end{aligned}$$

Equating the coefficients of the polynomials of ξ_2 and ξ_3 yields that $\psi'(\xi_1) = 0$ which in turn implies that the characteristic equation has no solution except the trivial one.

Now, we introduce an auxiliary dynamics

$$\begin{aligned}\dot{w} &= y,\\ y_a &= w\end{aligned}$$

which integrates the output and whose output (denoted by y_a) is w. Since w is obtained by integrating the output, w as well as ξ_1 is a known signal. This system combined with the original system yields

$$\begin{aligned}&\dot{w} = \xi_1,\ \dot{\xi}_1 = \xi_2,\ \dot{\xi}_2 = \xi_3,\ \dot{\xi}_3 = -4\xi_1\xi_3 - 3\xi_2^2 - 6\xi_1^2\xi_2\\ &y_a = w.\end{aligned}$$

If we apply the state transformation

$$\begin{aligned}z_1 &= w, & z_2 &= \xi_1 e^w\\ z_3 &= [\xi_2 + \xi_1^2]e^w, & z_4 &= [\xi_3 + 3\xi_1\xi_2 + \xi_1^3]e^w\end{aligned}$$

which is clearly a diffeomorphism, then the augmented system is transformed into

$$\begin{aligned}&\dot{z}_1 = z_2 + y[1 - e^w],\ \dot{z}_2 = z_3,\ \dot{z}_3 = z_4,\ \dot{z}_4 = y^4 e^w\\ &y_a = z_1\end{aligned}$$

which is a linear system modulo a vector field of known signals. We will precisely define this canonical form shortly, and say that the system is dynamically observer error linearizable. $\diamondsuit$

B. Problem Formulation and Characterization

Consider a nonlinear system:

$$\begin{aligned}\dot{x} &= f(x), \quad x \in \mathbb{R}^n,\ x(0) = x_0\\ y &= h(x), \quad y \in \mathbb{R}.\end{aligned} \tag{2}$$

We assume that f and h are a smooth vector field and a smooth function, respectively, and that both are defined on a neighborhood $\mathcal{D} \subset \mathbb{R}^n$ of x_0. Throughout the paper, it is also assumed that the system is observable on $\mathcal{D}$, i.e.

$$\dim \operatorname{span}\{dh, dL_f h, \cdots, dL_f^{n-1}h\} = n,\ \forall x \in \mathcal{D}. \tag{3}$$

Definition 1: The system (2) is said to be dynamically observer error linearizable (DOEL) if there exist a dynamic system (called auxiliary dynamics)

$$\begin{aligned}\dot{w} &= p(w, y), \quad w \in \mathbb{R}^d\\ y_a &= q(w, y), \quad y_a \in \mathbb{R}\end{aligned} \tag{4}$$

and a diffeomorphism $\Phi(w, x)$ defined on V, a neighborhood of $(0, x_0)$ in the extended state space $\mathbb{R}^n \times \mathcal{D}$, such that $\Phi(w, x)$ transforms the augmented system

$$\begin{aligned}\begin{bmatrix}\dot{w}\\ \dot{x}\end{bmatrix} &= F(w, x) := \begin{bmatrix}p(w, h(x))\\ f(x)\end{bmatrix}\\ y_a &= q(w, y)\end{aligned} \tag{5}$$

into an $n + d$ dimensional generalized nonlinear observer canonical form (GNOCF) defined by

$$\begin{aligned}\dot{z} &= A_o z + a(w, y), \quad z \in \mathbb{R}^{n+d}, \quad d \geq 0\\ y_a &= C_o z, \quad y_a \in \mathbb{R}\end{aligned} \tag{6}$$

$$A_o := \begin{bmatrix}0 & I_{n+d-1}\\ 0 & 0\end{bmatrix}, \quad C_o := \begin{bmatrix}1 & 0 & \cdots & 0\end{bmatrix}$$

$$a(w, y) := \begin{bmatrix}a_1(w, y) & \cdots & a_{n+d}(w, y)\end{bmatrix}^T.$$

$\diamondsuit$

If a system is DOEL with an auxiliary dynamics like (4), then the Luenberger-type observer

$$\dot{\hat{z}} = A_o\hat{z} + a(w, y) + L(y_a - C_o\hat{z}) \tag{7}$$

with $(A_o - LC_o)$ being Hurwitz is a state observer for the system because w, y_a, and y are all known signals and because the dynamics of extended estimation error $e_E = \hat{z} - z$ becomes $\dot{e}_E = (A_o - LC_o)e_E$.

Now, we characterize the systems that are DOEL.

Theorem 1: Suppose that the system (2) is observable on $\mathcal{D}$. Then the system is DOEL if and only if

C1. There exist an auxiliary dynamics represented by (4) and functions $a_1, \cdots, a_{n+d}$ which constitute a solution to the differential equation

$$\begin{aligned}L_F^{n+d}q(w, h(x)) &= L_F^{n+d-1}a_1(w, h(x))\\ &+ L_F^{n+d-2}a_2(w, h(x)) + \cdots + a_{n+d}(w, h(x)).\end{aligned} \tag{8}$$

C2. The mapping $\Phi(w, x) : V \to \mathbb{R}^{n+d}$ defined by

$$\begin{aligned}\Phi(w, x) &:= \begin{pmatrix}\Phi_1(w, x) & \cdots & \Phi_{n+d}(w, x)\end{pmatrix}^T\\ \Phi_i(w, x) &:= L_F^{i-1}q(w, h(x)) - \sum_{k=1}^{i-1} L_F^{i-1-k}a_k(w, h(x))\end{aligned} \tag{9}$$

is a diffeomorphism.

$\diamondsuit$

Proof: Necessity: If the system (2) is DOEL with (4), then there exists a diffeomorphism $z = \overline{\Phi}(w, x)$ defined on V such that

$$\begin{aligned}\frac{\partial\overline{\Phi}}{\partial(w, x)}F(w, x) &= A_o\overline{\Phi}(w, x) + a(w, h(x)),\ \forall(w, x) \in V\\ q(w, h(x)) &= \overline{\Phi}_1(w, x),\end{aligned}$$

from which we obtain

$$\begin{aligned}\overline{\Phi}_1 &= q\\ \overline{\Phi}_2 &= L_F\overline{\Phi}_1 - a_1\\ &\vdots\\ \overline{\Phi}_{n+d} &= L_F\overline{\Phi}_{n+d-1} - a_{n+d-1}\\ a_{n+d} &= L_F\overline{\Phi}_{n+d}.\end{aligned} \tag{10}$$

Using these relations one realizes that $a_1, \cdots, a_{n+d}$ constitute a solution to (8). Indeed, by successive substituting of the relations in (10) into the last relation of (10), we have

$$
\begin{aligned}
a_{n+d} &= L_F^2 \overline{\Phi}_{n+d-1} - L_F a_{n+d-1} \\
&\vdots \\
&= L_F^{n+d} \overline{\Phi}_1 - L_F^{n+d-1} a_1 - \cdots - L_F a_{n+d-1} \\
&= L_F^{n+d} q - L^{n+d-1} a_1 - \cdots - L_F a_{n+d-1}
\end{aligned}
$$

which is nothing but (8). Realizing that $\overline{\Phi}_i(w,x) = \Phi_i(w,x)$, $i = 1, \cdots, n+d$, concludes the proof of necessity.

Sufficiency: Suppose that $a_1(w, h(x)), \cdots, a_{n+d}(w, h(x))$ satisfy the conditions C1 and C2. Then, $z = \Phi(w,x)$ transforms (5) into

$$
\begin{aligned}
y_a &= z_1 \\
\dot{z}_1 &= z_2 + a_1(w, y) \\
&\vdots \\
\dot{z}_{n+d-1} &= z_{n+d} + a_{n+d-1}(w, y) \\
\dot{z}_{n+d} &= a_{n+d}(w, y).
\end{aligned}
$$

The upper $n+d$ relations come from the definition of Φ and the last one is obtained by using the relation (8), which proves the sufficiency. ■

Remark 1: The paper [14] also considered a canonical form in the extended space, but it is a special case of our GNOCF. Indeed in that paper, the output injection term contains only y_a (i.e., $a(w,y) := a(y_a)$), while in our case, the term may have the output of the system y as well as the states of the auxiliary system w as its arguments. It is worth noting that in [14], the auxiliary dynamics should be observable with respect to y_a, while our approach does not require this. Moreover for single output systems with the auxiliary dynamics being linear, it has been shown that a system is observer error linearizable if and only if it is dynamically linearizable in the sense of [14]. ♢

Remark 2: When the auxiliary dynamics is just a chain of integrators whose input is y and if $y_a = w_1$, our approach can be regarded as observer error linearization up to an injection which is composed of the new output (y_a) and its derivatives (w and y) although they are actually y and its integrals up to d times. From this, one may recollect the approach of Generalized Input-Output Injection Algorithm with output time derivatives (GIOIAd) in [6]. However, two approaches are fundamentally different. Firstly, in [6], y and its derivatives are utilized which are sensitive to noise, while our approach uses y and its integrals. Secondly, the canonical form of GIOIAd itself contains a chain of d integrators if the output and its derivatives up to d times are included in the injection term, while our canonical form need not contain a chain of integrators. ♢

From now, we call (8) the characteristic equation (CE), which is reminiscent of the work [8].

Remark 3: Since CE (8) is a partial differential equation with unknowns a_i's, p, and q, it is difficult to find the solutions even if we fix the auxiliary dynamics p and the auxiliary output q. Thus, it may be helpful to restrict the class of auxiliary dynamics with which some possible algorithm can be obtained. In [18], which is a companion paper of this, provides an algorithm to find the unknowns to (8). ♢

Remark 4: Condition C2 of Theorem 1 is essential in our framework, because in some cases, a solution $\{a_1, \cdots, a_{n+d}\}$ trivially satisfies (8) but yields rank deficiency of Φ. This situation is due to the inclusion of w into the functions a_i's. For example, consider Example 1 again. The CE is (a_i's are functions of w and ξ_1)

$$L_F^4 y_a = L_F^3 a_1 + L_F^2 a_2 + L_F a_3 + a_4$$

which becomes

$$-4\xi_1\xi_3 - 3\xi_2^2 - 6\xi_1^2\xi_2 = L_F^3 a_1 + L_F^2 a_2 + L_F a_3 + a_4.$$

At least two solutions are available to this equation:

$$
\begin{aligned}
&\mathrm{S1}: \; a_1 = \xi_1, && a_2 = 0, a_3 = 0, a_4 = 0 \\
&\mathrm{S2}: \; a_1 = \xi_1 - \xi_1 e^w, && a_2 = 0, a_3 = 0, a_4 = \xi_1^4 e^w.
\end{aligned}
$$

S1 results in $z = [w, 0, 0, 0]^T$ (not diffeomorphism), while Φ with S2 is a diffeomorphism. ♢

Remark 5: DOEL covers a strictly wider class of systems than the conventional observer error linearization method even though output diffeomorphism is used. In addition, DOEL in the sense of [14] is a special case of DOEL simply because of Remark 1. One of the most important result of this paper is that if an observable system is immersible into NOCF with dimension $n+d$, then it is also DOEL with a d dimensional auxiliary dynamics. See Section III. Example 1 and 2 illustrate this point in detail. ♢

From now on, without loss of generality (because of the observability assumption), we suppose the system is written in observable form [3]:

$$
\begin{aligned}
&\dot{\xi}_1 = \xi_2, \cdots, \; \dot{\xi}_{n-1} = \xi_n, \; \dot{\xi}_n = f_n(\xi_1, \xi_2, \cdots, \xi_n) \\
&y = \xi_1.
\end{aligned}
\tag{11}
$$

C. DOEL with special linear systems

To apply DOEL to a given nonlinear system, one should design the auxiliary dynamics and the auxiliary output. Although this allows the designer to have lots of flexibility, it severely increases the computational complexity. Thus, as in the study of dynamic feedback linearization [19], [20] and in [14], we restrict (4) to the following forms:

$$
\begin{aligned}
&\dot{w}_i = \begin{cases} w_{i+1}, & 1 \le i \le d-1 \\ -\alpha_1 w_1 - \cdots - \alpha_d w_d + \phi(\xi_1), & i = d \end{cases} \\
&y_a = \psi(w_i), 1 \le i \le d, \text{ or } y_a = \psi(\xi_1).
\end{aligned}
\tag{12}
$$

This class of systems covers a chain of integrators (α_i's are zero) and a stable linear system (α_i's are the coefficients of a Hurwitz polynomial). For simplicity, the auxiliary output is confined to be a function of a single variable.

Although this dynamics makes CE simpler, it is still hard to solve. Fortunately, some algorithms are available. One is [18] which shows that when the nth time derivative of the output is a polynomial (f_n in our notation) with respect to $\dot{y}$ (ξ_2 in ours), then the unknowns can be solved one by one although one should solve a partial differential equation at each step. Another algorithm is from [6]. As mentioned in Remark 2, GIOIAd utilizes the output and its time derivatives. This helps us if we restrict the auxiliary dynamics to be a chain of integrators and y_a to be w_1 because in this case one may regard our problem as GIOIAd as if our system were the augmented system itself with y_a being its output. However in both algorithms, one should solve a partial differential equation at each step. Thus, it is still open how to solve the general case.

Moving to Condition C2, we present two lemmas.

Lemma 1: Consider the auxiliary dynamics (12) with $y_a = \psi(w_1)$. Suppose a_i's in GNOCF are such that $a_i := a_i(w_1, \cdots, w_{i+1})$, $i = 1, \cdots, d-1$ when $d \geq 2$. Then, $\Phi(w, x)$ generated by a_i's is a diffeomorphism if and only if

$$\Pi_1(w_1)\Pi_2(w_1, w_2)\cdots\Pi_d(w)\Pi_{d+1}(w, \xi_1) \neq 0, \quad \forall (w, \xi_1) \in \mathbb{R}^d \times \mathcal{D} \tag{13}$$

where

$$\Pi_i(w_1, \cdots, w_i) := \psi'(w_1) - \sum_{j=1}^{i-1} \frac{\partial a_j}{\partial w_{j+1}}, \ i = 1, \cdots, d$$

$$\Pi_{d+1}(w, \xi_1) := \Pi_d(w)\phi'(\xi_1) - \frac{\partial a_d}{\partial \xi_1}. \qquad \diamondsuit$$

Note that there is no restriction on $a_d, \cdots, a_{n+d}$.

Lemma 2: Consider the auxiliary dynamics (12) with $y_a = \psi(\xi_1)$. Suppose a_i's in GNOCF are such that $a_i := a_i(\xi_1)$, $i = 1, \cdots, n-1$ and $a_i := a_i(w_1, \cdots, w_{i-n+1}, \xi_1)$, $i = n, \cdots, n+d-1$. Then, $\Phi(w, x)$ generated by a_i's is a diffeomorphism if and only if

$$\Pi_1(w_1, \xi_1)\Pi_2(w_1, w_2, \xi_1)\cdots\Pi_d(w, \xi_1)\Pi_{d+1}(\xi_1) \neq 0, \quad \forall (w, \xi_1) \in \mathbb{R}^d \times \mathcal{D} \tag{14}$$

where

$$\Pi_i(w_1, \cdots, w_i, \xi_1) := \sum_{j=1}^{i} \frac{\partial a_{n+j-1}}{\partial w_j}, \ i = 1, \cdots, d$$

$$\Pi_{d+1}(\xi_1) := \psi'(\xi_1). \qquad \diamondsuit$$

Next result says that under some restriction, DOEL with a chain of integrators is equivalent to DOEL with a class of linear systems.

Lemma 3: The system (2) is DOEL with (12) where $\alpha_i = 0$, $y_a = \psi(\xi_1)$ and with $a_i := \overline{a}_i(\xi_1)$, $i = 1, \cdots, n-1$ and $a_i := \overline{a}_i(w_1, \cdots, w_{i-n+1}, \xi_1)$, $i = n, \cdots, n+d-1$ if and only if (2) is DOEL with (12) where $\alpha := [\alpha_1, \cdots, \alpha_d]^T$ is an arbitrary vector and $y_a = \psi(\xi_1)$ and with $a_i := \tilde{a}_i(\xi_1)$, $i = 1, \cdots, n-1$ and $a_i := \tilde{a}_i(w_1, \cdots, w_{i-n+1}, \xi_1)$, $i = n, \cdots, n+d-1$. $\diamondsuit$

We close this section with a special but interesting result related to the characteristic equation (8).

Proposition 1: Consider a system (2) with $n = 2$ and an auxiliary dynamic system $\dot{w} = \phi(\xi_1)$, $y_a = q(w, \xi_1), w \in \mathbb{R}$ where ϕ and q are chosen to make $L_F^3 q(w, \xi_1)$ a function of ξ. Then, there exists a solution $\{a_1(w, \xi_1),\ a_2(w, \xi_1),\ a_3(w, \xi_1)\}$ to the CE

$$L_F^3 q(w, \xi_1) = L_F^2 a_1(w, \xi_1) + L_F a_2(w, \xi_1) + a_3(w, \xi_1)$$

if and only if there the CE

$$L_F^3 q(w, \xi_1) = L_f^2 \overline{a}_1(\xi_1) + L_f \overline{a}_2(\xi_1) + \overline{a}_3(\xi_1)$$

admits a solution $\{\overline{a}_1(\xi_1), \overline{a}_2(\xi_1), \overline{a}_3(\xi_1)\}$. $\diamondsuit$

III. DOEL versus System Immersion

In this section, we will show that the class of n dimensional observable systems that are immersible into $n+d$ dimensional NOCF is DOEL with a d dimensional auxiliary dynamics. Moreover, it is shown that the converse is not true.

At first, we recall the system immersion into NOCF (Definition 2) and its characterization (Theorem 2). For proof and discussions, the readers are referred to [15], [17]. Let $X_t(x_0)$ (resp., $Z_t(z_0)$) be the solution of (2) (resp., the system in NOCF) starting from $x_0 \in \mathcal{D}$ (resp., $z_0 \in \mathbb{R}^N$) and let $\tau_{x_0} := \sup\{\ t \geq 0 \mid X_t(x_0) \in \mathcal{D}\}$.

Definition 2: The system (2) is said to be immersible into $n+d$ dimensional NOCF if there exists a smooth function $\mathcal{T} : D \to \mathcal{T}(D)$ with $\mathcal{T}(U) \subset \mathbb{R}^{n+d}$ such that if for every x_0 and every z_0 with $\mathcal{T}(x_0) = z_0$, then $\psi(h(X_t(x_0))) = C_o Z_t(z_0)$ for every t such that $0 \leq t < \tau_{x_0}$. $\diamondsuit$

Theorem 2: ([15], [17]) The system (2) is immersible into $n+d$ dimensional NOCF if and only if there exist $n+d$ smooth functions $a_1(h), \cdots, a_{n+d}(h)$ and a diffeomorphism $\psi(h)$ such that

$$L_f^{n+d}\psi(h) = L_f^{n+d-1}a_1(h) + L_f^{n+d-2}a_2(h) + \cdots + a_{n+d}(h). \tag{15}$$

$\diamondsuit$

The CE (15) is a special case of (8) if we consider the d dimensional system (12) with $\alpha_i = 0$, $\phi(s) := s$ and $y_a = \psi(h(x))$ because $L_F^k a(x) = L_f^k a(x)$ for any k and any smooth function a. However, this does not mean that the system is DOEL since if we take a_i's of (8) as those of a_i's of (15) then $\Phi(w, x)$ is not a diffeomorphism. However, fortunately and rather surprisingly, it can be proved that system immersion is a special case of DOEL, which is stated below.

Theorem 3: If the system (2) is immersible into $n+d$ dimensional NOCF, then it is DOEL with a d dimensional auxiliary dynamics. $\diamondsuit$

Proof: Suppose the system (2) is written in ξ coordinate. From Theorem 2, there exist functions $\overline{a}_1(\xi_1), \cdots, \overline{a}_{n+d}(\xi_1)$, and a diffeomorphism $\psi(\xi_1)$ such that

$$L_f^{n+d}\psi(\xi_1) = L_f^{n+d-1}\overline{a}_1(\xi_1) + L_f^{n+d-2}\overline{a}_2(\xi_1) \cdots + \overline{a}_{n+d}(\xi_1).$$

We choose the auxiliary dynamics as

$$\dot{w}_1 = w_2, \cdots, \dot{w}_{d-1} = w_d, \dot{w}_d = \xi_1, y_a = \psi(\xi_1).$$

The functions a_i's to the characteristic equation (8) are chosen as

$$a_i(w, \xi_1) = \begin{cases} \overline{a}_i(\xi_1),\ i = 1, \cdots, n+d,\ i \notin \{n, n+d\} \\ \overline{a}_i(\xi_1) + w_1, \quad i = n \\ \overline{a}_i(\xi_1) - \xi_1, \quad i = n+d. \end{cases}$$

These functions constitute a solution to (8) because we have (assume $d \geq 2$, the proof for $d = 1$ is clear from context)

$$\begin{aligned}
&L_F^{n+d-1} a_1(w, \xi_1) + \cdots + a_{n+d}(w, \xi_1) \\
=&L_f^{n+d-1} \overline{a}_1(\xi_1) + \cdots + L_f^{d+1} \overline{a}_{n-1}(\xi_1) + L_F^d (\overline{a}_n(\xi_1) + w_1) \\
&+ L_f^{d-1} \overline{a}_{n+1}(\xi_1) + \cdots + (\overline{a}_{n+d}(\xi_1) - \xi_1) \\
=&L_f^{n+d-1} \overline{a}_1(\xi_1) + \cdots + \overline{a}_{n+d}(\xi_1) \\
=&L_f^{n+d} \psi(\xi_1) \\
=&L_F^{n+d} \psi(\xi_1).
\end{aligned}$$

It remains to show that $z = \Phi(w, \xi)$ is a diffeomorphism. Simple computation yields

$$\begin{aligned}
z_1 &= \psi(\xi_1) \\
z_2 &= L_f \psi(\xi_1) - \overline{a}_1(\xi_1) \\
&\vdots \\
z_n &= L_f^{n-1} \psi(\xi_1) - \cdots - \overline{a}_{n-1}(\xi_1) \\
z_{n+1} &= L_f^n \psi(\xi_1) - \cdots - \overline{a}_n(\xi_1) - w_1 \\
&\vdots \\
z_{n+d} &= L_f^{n+d-1} \psi(\xi_1) - \cdots - \overline{a}_{n+d-1}(\xi_1) - w_d.
\end{aligned}$$

The map $\Phi(w, \xi)$ is indeed a diffeomorphism because its Jacobian matrix is nonsingular on $\mathcal{D}$. In fact, it is of the form:

$$\begin{aligned}
\frac{\partial \Phi}{\partial (w, \xi)} &= \begin{bmatrix} 0_{n \times d} & \Psi_\xi(\xi) \\ I_{d \times d} & \times \end{bmatrix} \\
\Psi_\xi(\xi) &:= \begin{bmatrix} \psi'(\xi_1) & 0 & 0 & \cdots \\ \times & \psi'(\xi_1) & 0 & \cdots \\ \times & \times & \psi'(\xi_1) & \ddots \\ \vdots & \vdots & \ddots & \ddots \end{bmatrix}.
\end{aligned}$$

Thus, the proof is complete. ■

Remark 6: Since the conventional observer error linearization problem with output diffeomorphism (OELOD) is a special case of the system immersion problem ($d = 0$) [15], [17], it is worth noting that if a system is OELOD, then it is DOEL for any d. Note that DOEL in the sense of [14] does not guarantee this. ♢

From the proof of the theorem, it follows that an n dimensional system that is immersible into $n+d$ dimensional NOCF is DOEL with a chain of integrators. Furthermore, since the a_i's chosen during the proof satisfy the condition of Lemma 3, we have the following result.

Corollary 1: If the system (2) is immersible into $n+d$ dimensional NOCF, then it is DOEL also with a d dimensional stable linear system. ♢

A natural question arising from Theorem 3 is that whether the converse is true or not. As a matter of fact, it is true under a very restrictive condition and false in general.

Proposition 2: Suppose that $n = 2$, $d = 1$ and that the auxiliary dynamics is of the form

$$\dot{w} = y, \ y_a = q(y) \tag{16}$$

where q is a diffeomorphism. Then, (2) is immersible into 3 dimensional NOCF if and only if it is DOEL with (16). ♢

Proof: Necessity follows from the proof of Theorem 3 and sufficiency from Proposition 1. ■

Example 2: (Counterexample to converse of Theorem 3) We revisit Example 1. Recall that the system (1) is DOEL with a 1 dimensional linear system. This system, however, is not immersible into 3 dimensional NOCF even if an output diffeomorphism is used. To see this, consider the CE with $n = 3, d = 1$:

$$L_f^4 \psi(\xi_1) = L_f^3 a_1(\xi_1) + L_f^2 a_2(\xi_1) + L_f a_3(\xi_1) + a_4(\xi_1).$$

Expanding the terms yields

$$\begin{aligned}
&\psi^{(4)} \xi_2^4 + 3\psi''' \xi_2^2 \xi_3 + 3\psi'' \xi_3^2 + 4\psi'' \xi_2 f_3 \\
&+ \psi' [\frac{\partial f_3}{\partial \xi_1} \xi_2 + \frac{\partial f_3}{\partial \xi_2} \xi_3 + \frac{\partial f_3}{\partial \xi_3} f_3] \\
&= a_1''' \xi_2^3 + 3a_1'' \xi_2 \xi_3 + a_1' f_3 + a_2'' \xi_2^2 + a_2' \xi_3 + a_3' \xi_2 + a_4.
\end{aligned}$$

Since $f_3 = -4\xi_1\xi_3 - 3\xi_2^2 - 6\xi_1^2\xi_2$ is a polynomial of degree 2 in the sense of [3], it follows that $\frac{\partial f_3}{\partial \xi_1}\xi_2 + \frac{\partial f_3}{\partial \xi_2}\xi_3 + \frac{\partial f_3}{\partial \xi_3} f_3(\xi)$ is at most of degree 3. This implies that $\psi''(\xi) = 0$ (it can be also deduced by substituting f into CE). Realizing this fact and equating the coefficients of the monomials of ξ_2 and ξ_3 conclude that there is no nontrivial solution to CE, which in turn implies that the system (1) is not immersible into 3 dimensional NOCF. ♢

IV. Remarks on Time Scaling and Extensions

A few recent results on the observer error linearization employ the output dependent time scaling as well as the coordinate change [9], [10]. As usual, the time scaling transformation is represented by

$$\frac{dt}{d\tau} = \rho(h(x(t)))$$

where ρ is a nonvanishing real-valued function.

In the language of the characteristic equation, it is easily seen that the system is transformable into NOCF in τ scale if and only if there exist functions $\psi(y)$, $\rho(y)$, $a_1(y)$, $\cdots$, $a_n(y)$ that satisfy

$$L_{\rho f}^n \psi(y) = L_{\rho f}^{n-1} a_1(y) + \cdots + L_{\rho f} a_{n-1}(y) + a_n(y) \tag{17}$$

where ψ is an output diffeomorphism and ρ is a nonvanishing smooth function. For detailed discussion on this topic, see [9], [10] and references therein.

In this section, two examples are given to illustrate that these two approaches are different.

Example 3: Consider the system in Example 1 which is DOEL in the large. The system is not diffeomorphic to NOCF even if output dependent time scaling transformation together with output diffeomorphism is applied. Indeed, expanding (17) with respect to the vector field (1) and equating the coefficients of $\xi_2\xi_3$, ξ_2^2 and ξ_3 yield

$$\begin{aligned}
\xi_2\xi_3: &\quad 4\psi'\rho' + 3\psi''\rho = 0 \\
\xi_2^2: &\quad a_1''\rho + a_1'\rho' = -3\psi'\rho \\
\xi_3: &\quad a_1' = -4\xi_1\psi'.
\end{aligned}$$

After eliminating a_1' from the second equation, we obtain $-4\xi_1\psi'\rho' = 4\xi_1\psi''\rho + \psi'\rho$. Using the first equation to eliminate ϕ, one has $4\xi_1\rho' = 3\rho$ and $\rho = C\xi_1^{\frac{3}{4}}$ (C is a constant). Hence, the system is not linearizable into NOCF on any neighborhood $\mathcal{N}$ containing the origin. $\diamondsuit$

Example 4: Consider

$$\begin{aligned} &\dot{\xi}_1 = \xi_2,\ \dot{\xi}_2 = \xi_3,\ \dot{\xi}_3 = -\frac{1}{8}\xi_2^3 + \xi_2\xi_3 \\ &y = \xi_1. \end{aligned} \tag{18}$$

This system is observer error linearizable via output dependent time scaling (Proposition 1 of [10]). But it is not DOEL with $d = 1$, $\alpha = 0$, $\phi = \xi_1$, $q = \psi(y)$ nor $d = 1$, $\alpha = 0$, $\phi = \xi_1$, $q = \psi(w)$. It is easy to check the first case. For the second case, we expand the corresponding CE

$$L_F^4\psi(w) = L_F^3 a_1 + L_F^2 a_2 + L_F a_3 + a_4,$$

and compare the coefficients of some monomials to have

$$\begin{aligned}
&E1 \quad \xi_2\xi_3: \quad 3\frac{\partial^2 a_1}{\partial \xi_1^2} + \frac{\partial a_1}{\partial \xi_1} = \psi'(w) \\
&E2 \quad \xi_2^3: \quad \frac{\partial^3 a_1}{\partial \xi_1^3} - \frac{1}{8}\frac{\partial a_1}{\partial \xi_1} = -\frac{1}{8}\psi'(w) \\
&E3 \quad \xi_2^2: \quad \frac{\partial^2 a_2}{\partial \xi_1^2} + 3\frac{\partial^3 a_1}{\partial w \partial \xi_1^2} + 3\frac{\partial^2 a_1}{\partial w \partial \xi_1} = 3\psi''(w) \\
&E4 \quad \xi_3: \quad \frac{\partial a_2}{\partial \xi_1} + 3\frac{\partial^2 a_1}{\partial w \partial \xi_1}\xi_1 + \frac{\partial a_1}{\partial w} = 4\xi_1\psi''(w).
\end{aligned}$$

$E4 - \frac{\partial}{\partial \xi_1}E3 - \frac{\partial}{\partial w}E1$ yields $\frac{\partial^3 a_1}{\partial w \partial \xi_1^2} = 0$, from which we set $\frac{\partial a_1}{\partial \xi_1} = A(\xi_1) + B(w)$. Substituting this into $E1$ yields

$$3A'(\xi_1) + A(\xi_1) = \psi'(w) - B(w) = C$$

where C is a constant. With this relation and $E2$, one deduces $A(\xi) = 0$. However, this means that $\psi'(w) = B(w)$, which implies that Condition C2 of Theorem 1 is violated. $\diamondsuit$

We note that DOEL can be adapted to include the output dependent time scaling. Moreover, the time scaling can be 'dynamic'; ρ can include w (states of auxiliary dynamics) as well as y (system output). Then, the characteristic equation is modified as

$$\begin{aligned} L_{\overline{F}}^{n+d} q(w,y) &= L_{\overline{F}}^{n+d-1} a_1(w,y) + \cdots + a_{n+d}(w,y), \\ \overline{F} &:= \rho(w,y)F(w,x), \end{aligned}$$

and Condition C2 of Theorem 1 as the one F replaced by $\overline{F}$.

Although this extension increases lots of computational complexity, it is worth investigating how much the class of systems can be broadened. Research on this topic is underway.

V. CONCLUSION

Recent results of observer error linearization problem which does not necessarily require the exact integrability condition includes the system immersion method and the dynamic observer error linearization. Although the basic ideas of these two approaches are different, we can mention a relation between them by generalizing the dynamic observer error linearization proposed recently. The dynamic observer error linearization presented in this paper covers not only the conventional observer error linearization but also the system immersion method, and we showed that the converse is not true with a counterexample. Besides these advantages, this approach can be easily adapted to include the generalized output dependent time scaling transformation. The price we have to pay is an increase of computational effort. It is thus an open question how to check the conditions for a given system. Researches on the extensions to SISO systems and MIMO systems are also open.

REFERENCES

[1] A. J. Krener and A. Isidori, "Linearization by output injection and nonlinear observers," *Systems & Control Letters*, vol. 3, pp. 47–52, 1983.

[2] D. Bestle and M. Zeitz, "Canonical form observer for non-linear time-variable systems," *Int. J. Control*, vol. 38, pp. 419–431, 1983.

[3] A. J. Krener and W. Respondek, "Nonlinear observers with linearizable error dynamics," *SIAM J. Control Optim.*, vol. 23, no. 2, pp. 197–216, 1985.

[4] X.-H. Xia and W.-B. Gao, "Nonlinear observer design by observer error linearization," *SIAM J. Control Optim.*, vol. 27, no. 1, pp. 199–216, 1989.

[5] A. R. Phelps, "On constructing nonlinear observers," *SIAM J. Control Optim.*, vol. 29, pp. 516–534, 1991.

[6] F. Plestan and A. Glumineau, "Linearization by generalized input-output injection," *Systems & Control Letters*, vol. 31, pp. 115–128, 1997.

[7] M. Hou and A. C. Pugh, "Observer with linear error dynamics for nonlinear multi-output systems," *Systems & Control Letters*, vol. 37, pp. 1–9, 1999.

[8] H. Keller, "Non-linear observer design by transformation into a generalized observer canonical form," *Int. J. Control*, vol. 46, no. 6, pp. 1915–1930, 1987.

[9] M. Guay, "Observer linearization by output-dependent time-scale transformations," *IEEE Trans. Automat. Contr.*, vol. 47, no. 10, pp. 1730–1735, 2002.

[10] W. Respondek, A. Pogromsky, and H. Nijmeijer, "Time scaling for observer design with linearizable error dynamics," *Automatica*, vol. 40, no. 2, pp. 277–285, 2004.

[11] M. Fliess and I. Kupka, "A finiteness criterion for nonlinear input-output differential systems," *SIAM J. Control Optim.*, vol. 21, no. 5, pp. 721–728, 1983.

[12] J. Levine and R. Marino, "Nonlinear system immersion, observers and finite-dimensional filters," *Systems & Control Letters*, vol. 7, pp. 133–142, 1986.

[13] D. Bossane, D. Rakotopara, and J. P. Gauthier, "Local and global immersion into linear systems up to output injection," in *28th IEEE Conference on Decision and Control*, 1989, pp. 2000–2004.

[14] D. Noh, N. H. Jo, and J. H. Seo, "Nonlinear observer design by dynamic observer error linearization," *IEEE Trans. Automat. Contr.*, no. 10, pp. 1746–1750, 2004.

[15] P. Jouan, "Immersion of nonlinear systems into linear systems modulo output injection," *SIAM J. Control Optim.*, vol. 41, no. 6, pp. 1756–1778, 2003.

[16] J. Back and J. H. Seo, "Immersion of nonlinear systems into linear systems up to output injection: characteristic equation approach," *Int. J. Control*, vol. 77, no. 8, pp. 723–734, 2004.

[17] ——, "A constructive algorithm for system immersion into nonlinear observer form," in *Proceedings of the IFAC Symposium on Nonlinear Control Systems, Stuttgart, Germany*, 2004, pp. 621–626.

[18] K. T. Yu, J. Back, and J. H. Seo, "An algorithm for dynamic linearization by generalized output injection," in *submitted to Joint IEEE CDC and ECC*, 2005.

[19] B. Carlet, J. Lévine, and R. Marino, "On dynamic feedback linearization," *Systems & Control Letters*, vol. 13, pp. 143–151, 1989.

[20] ——, "Sufficient conditions for dynamic state feedback linearization," *SIAM J. Control Optim.*, vol. 29, pp. 38–57, 1991.