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Log-optimal currency portfolios and control Lyapunov exponents

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***Abstract*— P. Algoet and T. Cover characterized log-optimal portfolios in a stationary market without friction. There is no analogous result for markets with friction, of which a currency market is a typical example. In this paper we restrict ourselves to simple static strategies. The problem is then reduced to the analysis of products of random matrices, the top-Lyapunov exponent giving the growth rate. New insights to products of random matrices will be given and an algorithm for optimizing top-Lyapunov exponents will be presented together with some key steps of its analysis. Simulation results will also be given.**

Let $X = (X_n)$ be a stationary process of $k \times k$ real-valued matrices, depending on some vector-valued parameter $\theta \in \mathbb{R}^p$, satisfying $\mathrm{E} \log^+ ||X_0(\theta)|| < \infty$ for all θ. The top-Lyapunov exponent of X is defined as

$$\lambda(\theta) = \lim_n \frac{1}{n} \mathrm{E} \log ||X_n \cdot X_{n-1} \dots \cdot X_0||.$$

We develop an iterative procedure for the optimization of $\lambda(\theta)$. In the case when X is a Markov-process, the proposed procedure is formally within the class defined in [3]. However the analysis of the general case requires different techniques: an ODE method defined in terms of asymptotically stationary random fields. The verification of some standard technical conditions, such as a uniform law of large numbers for the error process is hard. For this we need some auxiliary results which are interesting in their own right.

I. INTRODUCTION

We consider the problem of maximizing the long-term profit of an investor who is trading in a stock or currency market. Instead of maximizing the expected value of short-term returns the problem is to optimize the growth rate of the portfolio. This problem was studied in [4] for a simple stock market model where daily returns were supposed to be independent and identically distributed. An algorithm was presented to determine the optimal constant proportion of wealth held in the assets, called the relative portfolio. A shortcoming of this algorithm is that the distribution of stock returns should be known in advance.

This work has then been generalized in various ways: Algoet and Cover [2] proved the existence and asymptotic optimality of "logoptimal" portfolios for stationary ergodic stock returns, see [2], Algoet gave universal schemes producing an asymptotically optimal growth rate without a priori knowledge of the stock's distribution, see [1] and also [5] and [7]. Existence of optimal portfolios was shown in [10] for certain models with transaction costs, though without an explicit construction or algorithm.

The present paper provides a framework which is pertinent to a wide range of market models with or without transaction costs. *Parametrized families* of investment strategies are considered and an algorithm for maximizing the logarithmic growth rate of portfolios is presented. The results apply whenever the portfolio position at time $t+1$ is computable from that of time t by multiplication with a random matrix $X_t(\theta)$, where θ is a parameter for a given class of strategies. The sequence $X_t(\theta)$ is supposed to be strictly stationary for each θ. An algorithm will be proposed to maximize the top-Lyapunov exponent with respect to θ.

The structure of the paper is as follows: in section 2 we present a simple example to motivate our line of thought, and the basic operations of a currency portfolio are given. In Section 3 and 4 classical and new results on random matrix products are given. In Section 5 an algorithm for the maximization of the top-Lyapunov exponent is presented. In Section 6 the so-called state-dependent random products are introduced and the initial steps for their analysis are given. Finally, in Section 7 simulation results are presented.

II. THE BASICS OF CURRENCY PORTFOLIOS

We start with a simple case of the model given in [4] and [2]. Suppose that an agent may invest in a bond and in a stock. For simplicity we assume that a unit of bond is worth \$1 all the time (i.e. interest rate is 0), the price of one unit of stock at time t is denoted by S_t, we have $S_0 = 1$ and $S_{t+1} = Y_{t+1} S_t$ where Y_t is a strongly stationary ergodic sequence of positive random variables with values close to 1.

His overall wealth at time t will be denoted by V_t. An investor seeking long-term profit wants to find a strategy that maximizes the logarithmic growth rate, which is defined as the limit of $\log V_t / t$, if it exists. If the Y_t-s are i.i.d. then it is known that an optimal strategy is to keep a fixed proportion $0 \leq \alpha \leq 1$ of the total wealth in the stock and the rest in the bond. In other words the *relative portfolio* is kept constant. In this case the dynamics of the wealth process is

$$V_{t+1} = (1-\alpha)V_t + \alpha V_t Y_t = [1 - \alpha + \alpha Y_t] V_t,$$

and the logarithmic growth rate is

$$\lim_{t \to \infty} \frac{\log V_t}{t} = E \log(\alpha Y_1 + 1 - \alpha), \qquad (1)$$

by the strong law of large numbers. To find the value of α an algorithm has been proposed by Cover in [4] for the case when the distribution of the Y_t is known. This algorithm requires the the computation of an expectation in each step.

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There seems to be very little known about optimal strategies for more complex examples with stationary (Y_t). However, the strategy of keeping a fixed proportion $0 \le \alpha \le 1$ of the total wealth in the stock and the rest in the bond can be also applied in the general case of (Y_t).

A simple extension of Cover's problem is obtained by allowing proportional transaction costs. This is a much harder problem, for which the optimal strategy has been found only very recently. It is defined in terms of two fixed target relative portfolios, say $\alpha_s < \alpha_b$. If the relative value of the stock falls below α_s then we re-balance our portfolio to bring it up exactly to α_s. On the other hand if the relative value of the bond exceeds α_b then we re-balance our portfolio to bring it down exactly to α_b. We do nothing in the intermediate zone.

The objective of this paper is to develop a general method for finding the log-optimal investment within a class of parametric models. Our focus will be on currency markets, which are markets with transaction costs.

Let us consider a currency portfolio $\phi = (\phi_n)$ consisting of k currencies. Thus $\phi_n = (\phi_{i,n}),\ i = 1,..,k$, where $\phi_{i,n}$ denotes the physical size of the portfolio held in the i-th currency at time n. At any time n the *exchange rates* are collected in a $k \times k$ matrix P_n. For any fixed $P = P_n$ the entry p_{ij} gives the amount of currency i that can be purchased for 1 unit of currency j. It is reasonable to suppose $p_{ii} = 1$ for all i and

$$p_{ij}p_{jl} \le p_{il}$$

for all i, j, l.

A strategy at any time n for *purchasing currency* j is a vector $b_j = (b_{jr})$, $r = 1, \ldots k$ such that

$$\sum_{j=1}^{k} b_{jr} = 1, \qquad b_{jr} \ge 0.$$

It gives the proportion of volume of currency r that is converted into currency j. The overall strategy is then represented by a matrix $B = (b_{ij})$.

If the current portfolio is $\phi = (\phi_1, \ldots, \phi_k)$ then the amount of currency i at the next period will be

$$\phi_i^+ = \sum_{j=1}^{k} p_{ij} b_{ij} \phi_j. \tag{2}$$

Write for the matrix with elements $x_{ij} = p_{ij}b_{ij}$

$$X = P \odot B.$$

Then the dynamics for the portfolio is

$$\phi_{n+1} = (P_{n+1} \odot B_{n+1})\phi_n =: X_{n+1}\phi_n$$

where B_n is the strategy selected at time n.

We consider now parametric strategies $B = (B_n(\theta))$, which may depend on the past of (P_n). Assuming that $\infty < n < \infty$, and that $B = (B_n(\theta))$ depends on the past of (P_n) in a stationary manner, the process (X_n) will become a strictly stationary process.

The simplest case is a constant strategy $B_n(\theta) = B(\theta)$ for all n. The wealth or the value of the portfolio at time t *expressed in currency* i will be obtained from a scalar product of the form

$$V_{n,i} = \sum_{j=1}^{k} p_{ij,n}\phi_{n,j}.$$

Then the growth rate of the wealth will be

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \log V_{n,i},$$

which, under appropriate initialization, is equal to the top-Lyapunov exponent of (X_n).

Example 1. We include an adaptation of the example at the beginning of this section to the case of currency markets. (The case of a stock market with transaction costs can be treated in much the same way.) Let P_t be as in Example 1. Suppose that the investor wishes to keep a fixed proportion α of his or her wealth (computed in dollars) in dollars and the rest in euros. Suppose that his current portfolio (in currency units) is (ϕ_1, ϕ_2). We have to distinguish two cases, depending on the direction of the transaction :

Case 1. If

$$\phi_1/\alpha > p_{12}\phi_2/(1-\alpha), \tag{3}$$

then some wealth (say, β dollars) must be transferred from dollars to euros. This β should satisfy

$$\frac{\phi_1 - \beta}{\alpha} = \frac{\phi_2 p_{12} + p_{21}\beta p_{12}}{1-\alpha}.$$

From here one can easily compute β as well as the new positions ϕ_1^+, ϕ_2^+.

Case 2. If

$$\phi_1/\alpha < p_{12}\phi_2/(1-\alpha), \tag{4}$$

then some units (say, γ) of euros must be converted into dollars such that

$$\frac{\phi_1 + \gamma p_{12}}{\alpha} = \frac{\phi_2 p_{12} - \gamma p_{12}}{1-\alpha}.$$

Putting together these two cases and taking $p_{12} := p_{12,t}$ and $p_{21} := p_{21,t}$ we get a strictly stationary random transformations $X_t(\alpha)$ which is piecewise linear. One can check that these transformations are linear if and only if $p_{12} = 1/p_{21}$, i.e. if the market is frictionless.

With the notation at the beginning of this section we may write

$$X_t(\alpha) = P_t \odot B_t(\alpha),$$

where

$$B_t(\alpha) = \begin{pmatrix} I_{A_t}(1-\beta/\phi_1) + I_{A_t^C} & I_{A_t^C}\gamma/\phi_2 \\ I_{A_t}\beta/\phi_1 & I_{A_t} + I_{A_t^C}(1-\gamma/\phi_2) \end{pmatrix},$$

and

$$A_t = \{(\phi_1, \phi_2) \in \mathbb{R}^2 :\ \phi_1/\alpha > \phi_2 p_{12}/(1-\alpha)\}.$$

III. Random matrix-products. A survey

In this section we describe some basic results on random matrix products. Let $X = (X_n), n = 0, 1, ...$ be a stationary process of $k \times k$ real-valued matrices over some probability space $(\Omega, \mathcal{F}, \mathcal{P})$, satisfying

$$\mathrm{E} \log^+ ||X_0|| < \infty \tag{5}$$

where $\log^+ x$ denotes the positive part of $\log x$. It is well-known (see [6]) that under the above condition

$$\lambda = \lim_n \frac{1}{n} \mathrm{E} \log ||X_n \cdot X_{n-1} \cdots X_0|| \tag{6}$$

exists. Here $\lambda = -\infty$ is allowed. The following result is fundamental in multiplicative ergodic theory (see [6]):

Proposition 1: Assume that the process $X = (X_n)$ described above satisfies (5) and in addition it is ergodic. Then P-almost surely

$$\lambda = \lim_n \frac{1}{n} \log ||X_n \cdot X_{n-1} \cdots X_0||.$$

The number λ, the exponential growth rate of the product $||X_n \cdot X_{n-1} ... \cdot X_0||$, is called the *top Lyapunov-exponent* of the process $X = (X_n)$. Now we can ask what happens if we apply the above random matrix products to a fixed vector. An answer is given by Oseledec's theorem (see [11] and [9]):

Proposition 2: Under the conditions of Theorem 1 there exists a subset $\Omega' \subset \Omega$ of probability 1 and a set of deterministic numbers $\lambda = \lambda_1 > \lambda_2 > ... > \lambda_p \geq -\infty$, and a set of random subspaces of fixed dimension

$$\mathbb{R}^k = V_0 \supset V_1(\omega) \supset ... \supset V_{p-1}(\omega) \supset V_p = 0,$$

with strict inclusion such that for all $\omega \in \Omega'$ and $v \in V_i(\omega) \setminus V_{i-1}(\omega)$ we have

$$\lim_n \frac{1}{n} \log |X_n(\omega) X_{n-1}(\omega) \cdots X_1(\omega) v| = \lambda_i.$$

The numbers λ_i are called Lyapunov-exponents. The theorem above implies that for $v \notin V_1(\omega)$ we have

$$\lim_n \frac{1}{n} \log \|X_n(\omega) X_{n-1}(\omega) \cdots X_0(\omega) v\| = \lambda.$$

Following [9], let us consider the singular-value decomposition of the random product $X_n \cdot X_{n-1} ... \cdot X_0$, and write

$$X_n \cdot X_{n-1} ... \cdot X_0 = U_n D_n V_n^T \tag{7}$$

where U_n, V_n are orthogonal $k \times k$ matrices, and $D_n = \mathrm{diag}(d_{ni})$ is a diagonal matrix, with d_{ni} being the singular values in decreasing order: $d_{n1} \geq ... \geq d_{nk}$. In particular $d_{n1} = ||X_n \cdot X_{n-1} ... \cdot X_0||$. The following extension of the Fürstenberg-Kesten theorem holds:

Theorem 1: Assume that the process $X = (X_n)$ described above satisfies (5) and in addition it is ergodic. Then P-almost surely the following limit exists:

$$\lambda_i = \lim_n \frac{1}{n} \log d_{ni}. \tag{8}$$

To characterize the asymptotic behavior of the orthogonal matrices U_n, V_n^T let us decompose the set $\{1, ..., k\}$ into disjoint "intervals" $I_1, .., I_r$ such that for $i, i' \in I_m$ $\lambda_i = \lambda_{i'}$ but for $i \in I_m, i' \in I_{m'}$ with $m \neq m'$ $\lambda_i > \lambda_{i'}$. Let us now consider the corresponding decomposition of the column-indices of V_n^T, $V_n^T = (V_{n1}^T, ..., V_{nr}^T)$.

Proposition 3: Assume that the process $X = (X_n)$ described above satisfies (5) and, in addition, it is ergodic. Then the linear subspaces spanned by the column-vectors of $V_{nm}^T, m = 1, ..., r$ converge P-almost surely in the gap-metric when n tends to ∞.

Assume now that $\lambda_1 > \lambda_2$. Then the above result implies that the first column of V_n^T, denoted by v_{n1}^T converges P-almost surely to some limit that will be denoted by v_1^{*T}

IV. Random matrix products. New results

A theoretical expression for λ is given in [6] as follows. Let us define the normalized products

$$Z_k = X_k ... X_1 / ||X_k ... X_1|| = \pi_m(X_k ... X_1)$$

where we set $\pi_m(A) = A/||A||$ for a non-singular matrix A.

A constructive result for computing λ is available if there is a gap between the first and second Lyapunov-exponent, i.e. if the co-dimension of V_1 is 1.

Proposition 4: Assume that the process $X = (X_n)$ described above satisfies (5), it is ergodic, and $\lambda_1 > \lambda_2$. Then for some $\varepsilon > 0$ we have for $\omega \in \Omega'$

$$\pi_m(X_k ... X_1) = u_k^* (v_1^*)^T + O(e^{-\varepsilon k}), \tag{9}$$

where (u_k^*) is a strictly stationary sequence of unit vectors, v_1^* is a fixed random unit vector, and the error term is a random variable bounded by $C(\omega) e^{-\varepsilon k}$ with some finite $C(\omega)$.

Let us now take random vector $\xi \in \mathbb{R}^k$ such that $\xi \notin V_1(\omega)$ almost surely, say, for $\omega \in \Omega$". E.g. take ξ independently of (X_n), having uniform distribution over $S_k = \{v \in \mathbb{R}^k, |v| = 1\}$. Redefine Ω' as $\Omega' \cap \Omega$". Assume that $\lambda > -\infty$. Then $X_n \cdot X_{n-1} \cdots X_1 \xi \neq 0$ for all n and $\omega \in \Omega'$. Let us define an $\mathbb{R}^k$-valued process $z = (z_n), n \geq 0$ as follows: $z_0 = \xi / |\xi|$, and for $n \geq 1$

$$z_n = \frac{X_n \cdot X_{n-1} \cdots X_1 \xi}{|X_n \cdot X_{n-1} \cdots X_1 \xi|}.$$

Obviously, $z = (z_n)$ can be defined recursively as follows:

$$z_{n+1} = \frac{X_{n+1} z_n}{|X_{n+1} z_n|} = \Pi_v(X_{n+1} z_n) \tag{10}$$

with initial condition $z_0 = \xi / |\xi|$, where now

$$\Pi_v(y) = y / |y|.$$

It is easily seen that

$$\log |X_n \cdot X_{n-1} \cdots X_1 \xi| = \sum_{k=0}^{n-1} \log |X_{k+1} z_k| + \log |\xi|.$$

Thus Theorem 1 implies

$$\lambda = \lim_n \frac{1}{n} \sum_{k=0}^{n-1} \log |X_{k+1} z_k|, \tag{11}$$

for $\omega \in \Omega'$.

The following result immediately follows from 4, but in fact, it is the statement below which should be proved first and then one can deduce Theorem 4.

Theorem 2: Assume that the process $X = (X_n)$ described above satisfies (5) and in addition it is ergodic, and $\lambda_1 > \lambda_2$. Let $\xi \notin V_1(\omega)$ almost surely, say for, $\omega \in \Omega'$, where Ω' was defined above. Then for some $\varepsilon > 0$ we have for $\omega \in \Omega'$

$$\pi_m(X_k...X_1\xi) = u_k^* + O(e^{-\varepsilon k}). \quad (12)$$

The above result indicates that for all initial conditions $\xi \notin V_1(\omega)$ the process (X_n, z_n) is asymptotically stationary. A stationary initialization can be constructed as follows:

Theorem 3: Assume that the process $X = (X_n)$ satisfies (5), it is ergodic, and $\lambda_1 > \lambda_2$. Let V_1 have co-dimension 1. Let ξ be uniformly distributed over the unit sphere, let it be independent from (X_n) and define

$$z_0^* = \lim_n \Pi_v(X_0 X_{-1} \dots X_{-n}\xi).$$

Then z_0^* is a stationary initialization for (10), i.e. defining

$$z_{n+1}^* = \Pi_v(X_{n+1} z_n^*), \qquad n \geq 0,$$

the process (X_n, z_n^*) is stationary. Moreover, we have

$$\mathrm{E} \log |X_1 z_0^*| = \lambda_1.$$

The fact that the process (z_n) forgets its initial condition exponentially fast can be expressed also in an infinitesimal form:

Theorem 4: Assume that the process $X = (X_n)$ satisfies (5), it is ergodic, and $\lambda_1 > \lambda_2$. Assume that $\xi = z_0(\omega) \notin V_1(\omega)$ for $\omega \in \Omega'$. Then

$$\|\frac{\partial z_n}{\partial \xi}\| \leq C(\omega, \xi) e^{-\gamma n}$$

where $C(\omega, \xi)$ is finite and $\gamma > 0$ is constant.

V. Maximization of the top-Lyapunov exponent

Assume now that the matrices $X_n, n = 0, 1...$ depend on a common parameter, say θ, where $\theta \in D \subset \mathbb{R}^p$, and D is an open domain. θ is considered as a control-parameter, and the top Lyapunov-exponent $\lambda = \lambda(\theta)$ will be a function of θ, and will be called a *controlled* Lyapunov-exponent. The problem that we consider is:

$$\max_\theta \lambda(\theta). \quad (13)$$

To compute the gradient of λ with respect to θ consider first a pair of smooth curves $(X(t), z(t)), t \geq 0$ in $\mathbb{R}^{k \times k}$ and $\mathbb{R}^k$, respectively, with $X(0) = X, z(0) = z$, such that $Xz \neq 0$. Then it is easy to see that

$$\frac{d}{dt} \log\ |X(t)z(t)| = \frac{1}{|Xz|^2}(z^T X^T X \dot{z} + z^T X^T \dot{X} z).$$

A shorthand notation will be

$$\frac{d}{dt} \log\ |X(t)z(t)| = \dot{H}(X, z, \dot{X}, \dot{z}).$$

Let us now consider the case when $X_n = X_n(\theta)$ is a smooth function of θ for $\theta \in D \subset \mathbb{R}^p$, where D is an open domain.

Differentiating the k-th term of (11) and setting

$$\begin{aligned} H_i(X, z, X_{\theta_i}, z_{\theta_i}) &= \dot{H}(X, z, X_{\theta_i}, z_{\theta_i}) \\ H(X, z, X_\theta, z_\theta) &= (H_1(\dots), \dots, H_p(\dots)) \quad (14) \end{aligned}$$

we get formally the following expression for the gradient of λ, denoted by λ_θ:

$$\lambda_\theta = \lim_n \frac{1}{n} \sum_{k=0}^{n-1} H(X_{k+1}, z_k, X_{\theta,k+1}, z_{\theta,k}). \quad (15)$$

It is assumed that the partial derivatives $X_{\theta_i,k+1}$ are available *explicitly*. On the other hand the partial derivatives $z_{\theta_i k}$ will be computed recursively, taking into account the recursive definition of z_n given in (10). For this purpose consider the mapping of $\mathbb{R}^{k \times k} \times \mathbb{R}^k$ into $\mathbb{R}^{k \times k}$ defined by

$$f(X, z) = Xz/|Xz|$$

assuming that $Xz \neq 0$. Consider first a pair of smooth curves $(X(t), z(t)), t \geq 0$ in $\mathbb{R}^{k \times k}$ and $\mathbb{R}^k$, respectively, with $X(0) = X, z(0) = z$. Then it is easy to see that

$$\frac{d}{dt} f(X(t), z(t)) = f_X \dot{X} + f_z \dot{z} = g(X, z, \dot{X}, \dot{z}),$$

where

$$f_X \dot{X} = \left(\dot{X} - X \frac{\mathrm{tr}\ (\dot{X} z z^T X^T))}{|Xz|^2}\right) \frac{z}{|Xz|},$$

and

$$f_z \dot{z} = \frac{X}{|Xz|} \left(I - z z^T X^T X \frac{1}{|Xz|^2}\right) \dot{z}.$$

Applying the above notations we can express the derivatives $z_{\theta_i,n}(\theta)$ in a recursive manner as follows:

$$z_{\theta_i,n+1} = g(X_{n+1}, z_n, X_{\theta_i,n+1}, z_{\theta_i,n}).$$

The iterative scheme. Assume, that at time n we have already computed θ_n and also $X_n, X_{\theta,n}, z_n$ and $z_{\theta_i,n}$. Observe $X_{n+1} = X_{n+1}(\theta_n)$ and $X_{\theta,n+1} = X_{\theta,n+1}(\theta_n)$. Then set

$$\begin{aligned} z_{n+1} &= X_{n+1} z_n / |X_{n+1} z_n| \\ z_{\theta_i n+1} &= g(X_{n+1}, z_n, X_{\theta_i,n+1}, z_{\theta_i,n}) \\ H_n &= H(X_{n+1}, X_{\theta,n+1}, z_n, z_{\theta,n}) \\ \theta_{n+1} &= \theta_n + \frac{1}{n} H_n. \quad (16) \end{aligned}$$

While the above method works well in simulation, its convergence analysis is yet incomplete. The algorithm formally falls within the class of recursive estimation methods described in [3] if X is a Markov-process, e.g. if (P_n) is i.i.d, but the application of the results of [3] does not seem to be straightforward. In particular, the verification of Conditions A4 and A5 of Section 1.2 Part II of [3] seems to be hard.

VI. State dependent random products

Consider now the problem when the random matrix X can be written in the form

$$X_n = X(P_n, \phi_{n-1}),$$

where $X(P, \phi)$ is a fixed function of P and ϕ, which is continuous in ϕ, (P_n) is an strictly stationary ergodic random matrix-process satisfying (5), and $\phi_n \in \mathbb{R}^p$ is a sequence of vectors computed recursively by

$$\phi_{n+1} = X(P_{n+1}, \phi_n)\phi_n.$$

A standard example we have in mind is

$$X = P \odot B(\phi)$$

where $B(\phi)$ is a redistribution matrix depending on the current portfolio ϕ. Assuming that all elements of P_n are positive for all n, it follows that for any non-negative, non-zero initial portfolio ϕ_0 the portfolios ϕ_n will be non-negative and non-zero for all n. Thus we can define the normalized portfolio sequence

$$z_n = \phi_n / |\phi_n|.$$

Note that $B(\phi)$ is scale-invariant, thus we can write $B(\phi_n) = B(z_n)$. This will be a general assumption for state-dependent products, i.e. we assume that

$$X(P, \phi) = X(P, z) \quad \text{with} \quad z = \phi/|\phi|. \tag{17}$$

The process z_n satisfies the usual recursion

$$z_{n+1} = \frac{X_{n+1} z_n}{|X_{n+1} z_n|} = \Pi_v(X(P_{n+1}, z_n) z_n) \tag{18}$$

with initial condition $z_0 = \xi/|\xi|$. The the growth-rate can be expressed using the usual identity

$$\log |X_n \cdot X_{n-1} \cdots X_1 \xi| = \sum_{k=0}^{n-1} \log |X_{k+1} z_k| + \log |\xi|.$$

Note, however, that we can not directly apply the results of the previous section, since the sequence of matrices $X(P_{n+1}, \phi_n)$ is not necessarily a stationary sequence. However, by a basic result of Has'minskii (see [8]) we get:

Proposition 5: Let $X_n = P_n \odot B(\phi_{n-1})$, where the process (P_n) is stationary, ergodic and satisfies

$$\mathrm{E} \log^+ ||P_0|| < \infty,$$

and $B(\phi)$ is bounded and continuous in ϕ. Then there exists an initialization z_0^* such that the resulting sequence (P_{n+1}, z_n^*) defined by (18) is stationary.

Since (X_n^*, z_{n-1}^*) is stationary, ergodic we have

$$\lim_n \frac{1}{n} \log |X_n^* \cdot X_{n-1}^* \cdots X_1^* z_0^*| = \lambda' = \mathrm{E} \log |X_1^* z_0^*|$$

almost surely. A key open problem is to find conditions under which the support of the marginal distribution of z_0^*, say $\mu(d\xi)$ is the full sphere.

VII. Simulation results

We take the model of Example 1 in Section 2 and suppose that $p_{12}(t), p_{21}(t)$ satisfy

$$p_{12}(t+1) := p_{12}(t)\xi_{t+1}(1 - d\varepsilon_{t+1}),$$
$$p_{21}(t+1) := \frac{1}{p_{12}(t)\xi_{t+1}}(1 - d\varepsilon_{t+1}),$$

where ξ_t are independent and identically distributed random variables with distribution

$$P(\xi_t = c) = P(\xi_t = 1/c) = 1/2,$$

and (ε_t) are independent and uniformly distributed random variables on $[0, 1]$. Here $c > 1$ and $0 \le d < 1$ are arbitrary constants. The price evolution is supposed to be driven by ξ_t while the ε_t are responsible for the marge of a dealer, i.e. they are thought to represent transaction costs.

We remark that if we choose $d = 0$ (no transaction costs) then the model reduces to Example 1 of Section 2. Choosing $c := 2$ we know from the Example on p. 370 of [4] that the optimal value of α is $\alpha^* = 0.5$ and this gives an asymptotic logarithmic growth rate $\lambda(\alpha^*) = 0.0626$.

In our simulations we took $c := 2,\ d := 0.05$ and found that in this case the optimal value of α is $\alpha^* = 0.54$. The corresponding growth rate $\lambda(\alpha^*)$ decreases to 0.04861 due to the presence of transaction costs. The growth rate λ as a function of α is shown on Figure 1, in both cases. The thin line corresponds to the no-transaction case, while thick line shows the result when 5% transaction cost is present. .

Figure 2 shows the convergence of our algorithm for the model with the above parameters, starting from $\alpha_0 := 0.3$. The horizontal axis shows the number of iteration on α. It is worth noting that about 30 *iterations* already gave a fairly good approximation of the optimal value.

We may conclude that the algorithm converges fairly fast in a model class which could not be treated by previous methods.

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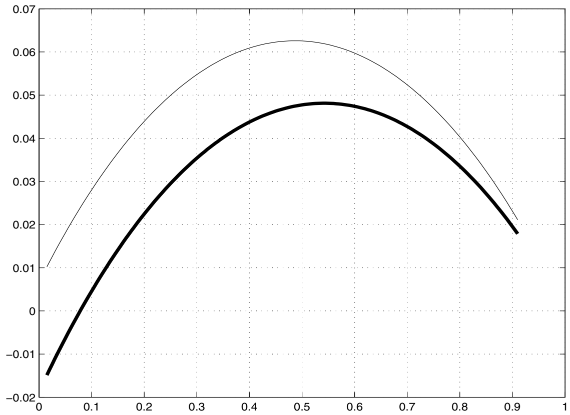

Fig. 1. Growth rate as a function of α

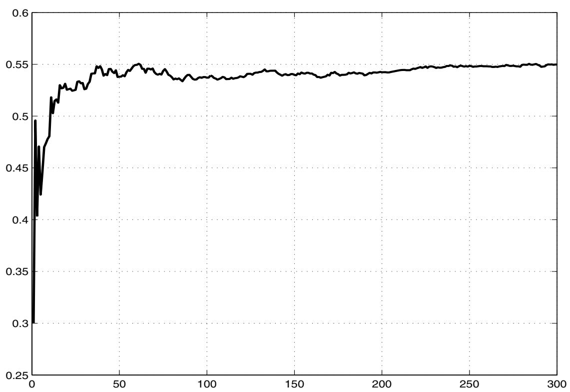

Fig. 2. Convergence for $\alpha_0 := 0.3$.

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