Proceedings of the
44th IEEE Conference on Decision and Control, and
the European Control Conference 2005
Seville, Spain, December 12-15, 2005

A Class of Higher Order Algorithms for Computing Polynomial Zeros

Mohammed A. Hasan
Department of Electrical & Computer Engineering
University of Minnesota Duluth
E.mail:mhasan@d.umn.edu

Abstract

A general framework for deriving multi-parameter classes of methods of any presribed order is presented. In particular, two classes of cubically and fourth order convergent iterative methods for simple zeros of polynomial equations are derived. As special cases, the cubically convergent class includes the one-parameter Hansen-Patricks family and other well-known Laguerre's, Halley's and Ostrowski's methods. Additionally, methods of any order which are based on Taylor expansion of certain rational functions are developed. Modified methods for extracting multiple zeros are briefly discussed.

1. Introduction

The problem of extracting zeros of polynomials and eigenvalues of matrices play a very important role in many algorithmic developments. This problem is extensively investigated in the literature. Many efficient second order iterative methods such as the Newton method, and third-order methods such as Laguerre's [1], Halley's [2,3] and Ostrowski's methods [4] have been used for computing zeros of polynomials. Many aspects of higher order methods are explored in [5,6]. In particular, Hansen and Patrick presented a unified class of methods with cubic convergence. An excellent outline of the numerical implementation of Laguerre iteration for non symmetric eigenproblems can be found in Parlett [7]. It should also stated that analysis and derivation of Laguerres method can be found in [8] and [9]. Iterations of order of convergence at least four for simultaneous determination of all simple zeros of a polynomial were presented in [6] based on modifications of Laguerres method. In [10] the Laguerre iteration was derived from Halleys method. It is also shown that Hansen-Patricks methods are special cases of Laguerres method.

The purpose of this paper is to extend the work in [5]-[6] and develop new higher order zero-finding iterative methods for polynomials and examine their relation to some existing methods.

2. Definitions and Basic Concepts

Let p be a polynomial of degree m over the field of complex numbers $\mathcal{C}$ with zeros $\Lambda = \{z_j\}_{j=1}^m$. In what follows a sought zero $\xi \in \Lambda$ will be assumed simple unless otherwise indicated. Let z be an approximation of ξ, then the improved approximation $\hat{z}$ will be written as $\hat{z} = z - \Phi(z)$ for some function Φ which is a function of p and derivatives of p. The notation $tr(A)$ denotes the trace of the matrix A, $det(A)$ denotes the determinant of the matrix A, and $A^{-k} = (A^{-1})^k$ provided that A is invertible.

For each positive integer k and each complex number $z \neq z_1, \cdots, z_m$, define

$$S_k = \sum_{j=1}^{m} \frac{1}{(z - z_j)^k}, \; k = 1, 2 \cdots. \tag{1a}$$

The expression S_k can also be computed for a square matrix A of size m with characteristic polynomial $det(zI - A)$ so that for each complex number z that is not an eigenvalue of A

$$S_k = tr((zI - A)^{-k}), \tag{1b}$$

where I is an identity matrix of size m. The expression S_k can be seen as a scalar multiple of the kth logarithmic derivative of $p(z)$, specifically

$$S_k = \frac{(-1)^{k-1}}{(k-1)!} \frac{d^k(\log p(z))}{dz^k}. \tag{2}$$

Direct calculation of S_1, S_2 and S_3 yields

$$\begin{aligned} S_1 &= \frac{p'(z)}{p(z)}, \\ S_2 &= -(\frac{p'(z)}{p(z)})' = -\frac{p(z)p''(z) - p'(z)^2}{p(z)^2} \\ &= -\frac{p''(z)}{p(z)} + (\frac{p'(z)}{p(z)})^2 = -\frac{p''(z)}{p(z)} + S_1^2, \\ S_3 &= \frac{1}{2}(\frac{p'(z)}{p(z)})'' = \frac{2p'(z)^3 - 3p(z)p'(z)p''(z) - p(z)^2 p'''(z)}{2p(z)^3} \end{aligned} \tag{3}$$

Here, $p'(z) = \frac{d(p(z))}{dz}$, $p''(z) = \frac{d^2(p(z))}{dz^2}$, and $p'''(z) = \frac{d^3(p(z))}{dz^3}$.

Many well-known methods can be expressed in terms of $S_1, S_2, \cdots$. For example, the Newton method can be written as $\hat{z} = z - \frac{1}{S_1}$, and Halley's method as $\hat{z} = z - \frac{1}{S_1 + \frac{S_2}{S_1}}$. Let r be any positive integer, a root iteration method of order r is of the form $\hat{z} = z - \frac{1}{\sqrt[r]{S_r}}$.

3. A Family of Third Order Methods

In this section, a two-parameter family of Laguerre-type methods is derived. Let $\alpha, \beta \in \mathcal{C}$ such that $|\alpha| + |\beta| \neq 0$. Let $\xi \in \Lambda$ and for convenience assume that $\xi = z_1$. The expression $\alpha S_2 + \beta S_1^2$ can be expanded as follows:

$$\alpha S_2 + \beta S_1^2 = \sum_{j=1}^{m} \frac{\alpha+\beta}{(z-z_j)^2} + 2\beta \sum_{i<j} \frac{1}{(z-z_i)(z-z_j)}$$
$$= \frac{\alpha+\beta}{(z-\xi)^2}\{1 + \frac{2\beta}{\alpha+\beta}\{\frac{z-\xi}{z-z_2} + \frac{z-\xi}{z-z_3} + ... + \frac{z-\xi}{z-z_m}\}$$
$$+ (z-\xi)^2(S_2 - \frac{1}{(z-\xi)^2}) + \sum_{i<j} \frac{1}{(z-z_i)(z-z_j)}\}. \tag{4}$$

Assuming that z is a good approximation of ξ, the last expression can be written as

$$\alpha S_2 + \beta S_1^2 = \frac{\alpha+\beta}{(z-\xi)^2}\{1 + \frac{2\beta(z-\xi)}{\alpha+\beta}(S_1 - \frac{1}{z-\xi}) + O((z-\xi)^2)\}. \tag{5}$$

The notation $O((z-\xi)^2)$ denotes all terms of second degree or higher for the term $(z-\xi)$. The square root of the expression of (5) can be obtained using the Binomial theorem:

$$\{\alpha S_2 + \beta S_1^2\}^{\frac{1}{2}} = \frac{\sqrt{\alpha+\beta}}{z-\xi}\{1 + \frac{\beta(z-\xi)}{\alpha+\beta}(S_1 - \frac{1}{z-\xi}) + O((z-\xi)^2\} \tag{6}$$
$$= \frac{\sqrt{\alpha+\beta}}{z-\xi} + \frac{\beta}{\sqrt{\alpha+\beta}}(S_1 - \frac{1}{z-\xi}) + O(z-\xi).$$

By rearranging terms,

$$(\alpha S_2 + \beta S_1^2)^{\frac{1}{2}} - \frac{\beta}{\sqrt{\alpha+\beta}} S_1 = \frac{\sqrt{\alpha+\beta} - \frac{\beta}{\sqrt{\alpha+\beta}}}{z-\xi} + O(z-\xi).$$

or

$$\sqrt{\alpha+\beta}(\alpha S_2 + \beta S_1^2)^{\frac{1}{2}} - \beta S_1 = \frac{\alpha}{z-\xi} + O(z-\xi) \tag{7}$$

Therefore, the improved approximation $\hat{z}$ of ξ can be obtained by ignoring higher order terms and solving (7) for ξ:

$$\hat{z} = z - \frac{\alpha}{\sqrt{\alpha+\beta}(\alpha S_2 + \beta S_1^2)^{\frac{1}{2}} - \beta S_1}$$
$$= z - \frac{\alpha(\sqrt{\alpha+\beta}\sqrt{\alpha S_2 + \beta S_1^2} + \beta S_1)}{(\alpha+\beta)(\alpha S_2 + \beta S_1^2) - \beta^2 S_1^2} \tag{8}$$
$$= z - \frac{\sqrt{\alpha+\beta}(\alpha S_2 + \beta S_1^2)^{\frac{1}{2}} + \beta S_1}{(\alpha+\beta)S_2 + \beta S_1^2}.$$

Using the relations in (3) we obtain the following family of third-order iterations:

$$\hat{z} = z - \frac{\sqrt{\alpha+\beta}\{\alpha\frac{p'(z)^2 - p(z)p''(z)}{p(z)^2} + \beta(\frac{p'(z)}{p(z)})^2\}^{\frac{1}{2}} + \beta\frac{p'(z)}{p(z)}}{(\alpha+\beta)(\frac{p'(z)^2 - p(z)p''(z)}{p(z)^2}) + \beta(\frac{p'(z)}{p(z)})^2}$$
$$= z - \frac{\sqrt{\alpha+\beta}p(z)\{(\alpha+\beta)p'(z)^2 - \alpha p(z)p''(z)\}^{\frac{1}{2}} + \beta p(z)p'(z)}{(\alpha+2\beta)p'(z)^2 - (\alpha+\beta)p(z)p''(z)}. \tag{9}$$

3.1 Special Cases

1. It can be shown that if $\alpha + \beta \neq 0$, then Iteration (9) is at least third order. However, when $\alpha + \beta = 0$ and $\alpha \neq 0$, the method reduces to the standard Newton method.
2. If $\alpha = 0$, then (8) reduces to the following iteration:
$$\hat{z} = z - \frac{2S_1}{S_2 + S_1^2} = \frac{p(z)}{p'(z) - \frac{p(z)p''(z)}{2p'(z)}}. \tag{10}$$
This corresponds to the rational form of Halley's method.
3. If $\alpha = (m-1)m,\ \beta = -(m-1)$, then $\alpha + \beta = m^2 - 2m + 1$ and hence (8) simplifies to
$$\hat{z} = z - \frac{(m-1)m}{(m-1)S_1 \pm (m-1)\sqrt{(m-1)mS_2 - (m-1)S_1^2}}$$
$$= z - \frac{m}{\frac{p'(z)}{p(z)} \pm \sqrt{(m-1)^2(\frac{p'(z)}{p(z)})^2 - m(m-1)\frac{p''(z)}{p(z)}}}, \tag{11}$$
which is known as the Laguerre formula. The choice of the sign in front of the square root is performed according to Henrici's criterion [8, p. 532], so that the argument of the square root appearing in (13) is to be chosen to differ from the argument of $p'(z)$ by less than $\pi/2$.
4. Let $\alpha = 1 + \gamma$ and $\beta = -\gamma$, where $\gamma \neq -1$, then $\alpha + \beta = 1$ and thus Iteration (9) is equivalent to the Hansen-Patrick family [5] given by
$$\hat{z} = z - \frac{(1+\gamma)p(z)}{\gamma p'(z) \pm \sqrt{p'^2(z) - (1+\gamma)p(z)p''(z)}}. \tag{12}$$
This family of methods are derived in Hansen and Patrick [5, 6]. It is shown in [10] that this family can be obtained from the classical form of Laguerres method by a special choice of a parameter.
5. If $\beta = 0$, then (9) reduces to the square root iteration
$$\hat{z} = z - \frac{p(z)}{\sqrt{p'^2(z) - p(z)p''(z)}}. \tag{13}$$
Generalization of (13) which includes higher order root iterations are given in [10].

3.2 Multiple Zeros

If $\alpha+\beta \neq 0$, the methods described in (8) converge cubically only if the sought zero ξ is simple. If ξ is of multiplicity m_j, the methods converge only linearly. In general if the zeros are not simple, i.e., if $p(z)$ has M distinct zeros so that z_j is of multiplicity m_j for $j = 1, \cdots, M$, then $\sum_{j=1}^{M} m_j = m$ and

$$S_k = \sum_{j=1}^{M} \frac{m_j}{(z - z_j)^k}.$$

For convenience, assume that $\xi = z_1$, then

$$\alpha S_2 + \beta S_1^2 = \sum_{j=1}^{M} \frac{\alpha m_j + \beta m_j^2}{(z - z_j)^2} + \sum_{i<j} \frac{2\beta m_i m_j}{(z - z_i)(z - z_j)}$$
$$= \frac{\alpha m_1 + \beta m_1^2}{(z - \xi)^2}\{1 + \frac{2\beta m_1 (z - \xi)}{\alpha m_1 + \beta m_1^2}(S_1 - \frac{m_1}{z - \xi})$$
$$+ O(z - \xi)^2\}.$$

Taking the square root of both sides, it follows that

$$\sqrt{\alpha S_2 + \beta S_1^2} = \frac{\sqrt{\alpha m_1 + \beta m_1^2}}{z - \xi}\{1 + \frac{\beta m_1 (z - \xi)}{\alpha m_1 + \beta m_1^2}(S_1 - \frac{m_1}{z - \xi})$$
$$+ O(z - \xi)^2\}.$$

By ignoring higher order terms and solving for ξ we obtain

$$\hat{z} = z - \frac{\alpha m_1}{\sqrt{\alpha m_1 + \beta m_1^2}} \frac{1}{\frac{\beta m_1}{\sqrt{\alpha m_1 + \beta m_1^2}} S_1 + \sqrt{\alpha S_2 + \beta S_1^2}}. \quad (14)$$

Note that if $\beta = 0$ and $\alpha = 1$, we obtain a modified third order square root iteration:

$$\hat{z} = z - \frac{\sqrt{m_1} p(z)}{\sqrt{p'^2(z) - p(z)p''(z)}}.$$

If ξ is of multiplicity m_i, then replacing $\frac{p'(z)}{p(z)}$ with $\frac{p'(z)}{m_i p(z)}$ in Halley's formula (12) yields

$$\hat{z} = z - \frac{p(z)}{\frac{1+m_i}{2m_i} p'(z) - \frac{p(z)p''(z)}{2p'(z)}}. \quad (15)$$

If the multiplicity of ξ is not known, one can replace $p(z)$ with $\frac{p(z)}{p'(z)}$ (whose zeros are all simple) to obtain the following:

$$\hat{z} = z - \frac{p(z)}{\sqrt{p'(z)^2 - p(z)p''(z) - p(z)^2(\frac{p''^2(z) - p'(z)p'''(z)}{p'(z)^2})}}. \quad (16)$$

Note that the modified square root iteration (16) converges cubically to a multiple root, however this is achieved at the cost of computing the third order derivative $p'''(z)$. Generally, if the square root iteration is applied to the function $\frac{p(z)}{g(z)}$, where $g(z)$ is any polynomial which is not a scalar multiple of p, then

$$\hat{z} = z - \frac{p(z)}{\sqrt{p'(z)^2 - p(z)p''(z) - p(z)^2(\frac{g'^2(z) - g(z)g''(z)}{g^2(z)})}}. \quad (17)$$

Now assume that the zero $\xi = z_1$ has been computed, the factor $(z - \xi)^{m_1}$ can be removed from $p(z)$. In computing the next zero, one can apply (14) to $\frac{p(z)}{(z-\xi)^{m_1}}$ in which case (17) reduces to:

$$\hat{z} = z - \frac{\sqrt{m_2} p(z)}{\sqrt{p'(z)^2 - p(z)p''(z) - \frac{p(z)^2}{(z-\xi)^{2m_1}}}}. \quad (18)$$

This procedure can be generalized further. Assume the zeros $z_1, \cdots, z_{j-1}$ have been computed along with their multiplicities, then to compute z_j one can apply the following

$$\hat{z} = z - \frac{\sqrt{m_j} p(z)}{\sqrt{p'(z)^2 - p(z)p''(z) - \sum_{i=1}^{j-1} \frac{p(z)^2}{(z-z_i)^{2m_i}}}}. \quad (19)$$

Generally, the multiplicity m_k of a zero z_k is not known. Practically, m_k can be obtained approximately as $m_k \approx (z - z_k)\frac{p'(z)}{p(z)}$, where z is an approximation of z_k.

4. A Family of Fourth Order Methods

Let $\alpha, \beta, \gamma \in \mathcal{C}$ and consider the expression

$$\alpha S_3 + \beta S_1 S_2 + \gamma S_1^3 = (\alpha + \beta + \gamma) \sum_{j=1}^{m} \frac{1}{(z - z_i)^3}$$
$$+ \sum_{i<j} \frac{\beta + 3\gamma}{(z - z_i)^2 (z - z_j)} + \sum_{i<j} \frac{\beta + 3\gamma}{(z - z_i)(z - z_j)^2}$$
$$+ \sum_{i<j<k} \frac{6\gamma}{(z - z_i)(z - z_j)(z - z_k)}.$$

Assuming that z is a good approximation of ξ, the quantity $\alpha S_3 + \beta S_1 S_2 + \gamma S_1^3$ can be rewritten as

$$\alpha S_3 + \beta S_1 S_2 + \gamma S_1^3 = \frac{\alpha + \beta + \gamma}{(z - \xi)^3}$$
$$\times \{1 + \frac{(\beta + 3\gamma)(z - \xi)}{\alpha + \beta + \gamma}(S_1 - \frac{1}{z - \xi})$$
$$+ O(z - \xi)^2\}.$$

Using the Binomial Theorem, we obtain:

$$\{\alpha S_3 + \beta S_1 S_2 + \gamma S_1^3\}^{\frac{1}{3}} = \frac{(\alpha + \beta + \gamma)^{1/3}}{z - \xi}$$
$$+ \frac{\beta + 3\gamma}{3(\alpha + \beta + \gamma)^{2/3}}(S_1 - \frac{1}{z - \xi}) + O(z - \xi).$$

The last equation can be rewritten as

$$\{\alpha S_3 + \beta S_1 S_2 + \gamma S_1^3\}^{\frac{1}{3}} - \frac{\beta + 3\gamma}{3(\alpha + \beta + \gamma)^{2/3}} S_1$$
$$= \frac{(\alpha + \beta + \gamma)^{1/3} - \frac{\beta + 3\gamma}{3(\alpha + \beta + \gamma)^{2/3}}}{z - \xi} + O(z - \xi).$$

or equivalently

$$(\alpha + \beta + \gamma)^{2/3}(\alpha S_3 + \beta S_1 S_2 + \gamma S_1^3)^{1/3} - \frac{\beta + 3\gamma}{3} S_1$$
$$= \frac{\alpha + \beta + \gamma - \frac{\beta + 3\gamma}{3}}{z - \xi} + O(z - \xi) = \frac{\alpha + \frac{2}{3}\beta}{z - \xi} + O(z - \xi). \quad (20)$$

An approximation of ξ is obtained by solving (20) for ξ

$$\hat{z} = z - \frac{\alpha + \frac{2\beta}{3}}{(\alpha + \beta + \gamma)^{2/3}(\alpha S_3 + \beta S_1 S_2 + \gamma S_1^3)^{1/3} - \frac{\beta + 3\gamma}{3} S_1} \quad (21)$$

Note that if $\alpha + \beta + \gamma = 0$ and $\alpha + \frac{2\beta}{3} \neq 0$, then $\frac{\beta + 3\gamma}{3} = \frac{\beta + 3(-\alpha - \beta)}{3} = \frac{-2\beta - 3\alpha}{3} = -(\frac{\alpha + 2\beta}{3})$. Thus (21) reduces to the Newton Method. In similar fashion, one can show by using the relations (3) along with the identity $\frac{p'''}{p} = S_1^3 -$

$3S_1S_2 + 2S_3$, that when $\beta = \gamma = 0$, $\alpha \neq 0$, (21) reduces to the cube root iteration

$$\hat{z} = z - \frac{p(z)}{\sqrt[3]{p'(z)^3 - \frac{3}{2}p(z)p'(z)p''(z) + \frac{1}{2}p(z)^2p'''(z)}}, \qquad (22)$$

which is a fourth order method.

One can also verify that the rational iteration

$$\hat{z} = z - \frac{p(z)}{p'(z) - \frac{1}{2}\frac{p(z)p''(z)}{p'(z)} + p(z)^2\{\frac{1}{6}\frac{p'''(z)}{p'(z)^2} - \frac{1}{4}\frac{p''(z)^2}{p'(z)^3}\}} \qquad (23)$$

is fourth order. The formula (23) can be derived by rationalizing (22) using the Binomial Theorem. It should be stated that additional analysis is needed to examine the rate of convergence of (21) as a function of α, β, and γ.

It should be stated that this class of fourth order methods can be modified to deal with multiple zeros by replacing p with $\frac{p}{p'}$ as described in Section 3.2.

5. Miscellaneous Methods

The previous analysis can be applied to different linear combinations of powers of S_k's to obtain different families of higher order methods. For example, for any nonzero $\alpha, \beta, \gamma, \kappa \in \mathcal{C}$,

$$\alpha S_4 + \beta S_1 S_3 + \gamma S_2^2 + \kappa S_1^4 = (\alpha + \beta + \gamma + \kappa) \sum_{j=1}^{m} \frac{1}{(z - z_i)^4}$$
$$+ (\beta + 4\kappa) \sum_{i<j} \frac{1}{(z - z_i)^3 (z - z_j)} + \cdots = \frac{\alpha + \beta + \gamma + \kappa}{(z - \xi)^4}$$
$$\times \{1 + \frac{(\beta + 4\kappa)(z - \xi)}{\alpha + \beta + \gamma + \kappa}(S_1 - \frac{1}{(z - \xi)}) + O((z - \xi)^2)\}^{1/4}.$$

It can be verified that

$$\{\alpha S_4 + \beta S_1 S_3 + \gamma S_2^2 + \kappa S_1^4\}^{\frac{1}{4}}$$
$$= \frac{(\alpha + \beta + \gamma + \kappa)^{1/4}}{z - \xi}$$
$$+ \frac{(\beta + 4\kappa)}{4(\alpha + \beta + \gamma + \kappa)^{3/4}}(S_1 - \frac{1}{z - \xi}) + O(z - \xi)$$

Therefore,

$$\hat{z} = z - \frac{\alpha + \frac{3}{4}\beta + \gamma}{-\frac{\beta + 4\kappa}{4} S_1 + (\alpha + \beta + \gamma + \kappa)^{3/4} (\alpha S_4 + \beta S_1 S_3 + \gamma S_2^2 + \kappa S_1^4)^{1/4}}. \qquad (24)$$

Note that the last formula reduces to the Newton method if $\alpha + \beta + \gamma + \kappa = 0$ and $\beta + 4\kappa \neq 0$.

5.1 Families Based on First and Second Order Derivatives

Similar analysis on the expressions $\alpha S_2^2 + \beta S_1^4$, $\alpha S_1 S_2 + \beta S_1^3$, $\alpha S_1^5 + \beta S_1^3 S_2$, $\alpha S_1^6 + \beta S_1^4 S_2 + \gamma S_1^2 S_2^2 + \kappa S_2^3$ yields respectively the following formulas which only require first and second order derivatives:

$$\hat{z} = z - \frac{\frac{2\alpha}{3}}{-\frac{\alpha + 3\beta}{3} S_1 + (\alpha + \beta)^{2/3} (\alpha S_1 S_2 + \beta S_1^3)^{1/3}} \qquad (25)$$

$$\hat{z} = z - \frac{\alpha}{-\beta S_1 + (\alpha + \beta)^{3/4} (\alpha S_2^2 + \beta S_1^4)^{1/4}} \qquad (26)$$

$$\hat{z} = z - \frac{\frac{2\beta}{5}}{-(\alpha + \frac{3}{5}\beta) S_1 + (\alpha + \beta)^{4/5} (\alpha S_1^5 + \beta S_1^3 S_2)^{1/5}} \qquad (27)$$

$$\hat{z} = z - \frac{\frac{2\beta + 4\gamma + 6\kappa}{6}}{-\frac{6\alpha + 4\beta + 2\gamma}{6} S_1 + \bar{\alpha} (\alpha S_1^6 + \beta S_1^4 S_2 + \gamma S_1^2 S_2^2 + \kappa S_2^3)^{1/6}} \qquad (28)$$

where $\bar{\alpha} = (\alpha + \beta + \gamma + \kappa)^{5/6}$. Further analysis is needed to examine the actual rate of convergence of each of the families (25)-(28). The conjecture is that these iterations are at most third order.

6. Power Series Expansion Method

Let w_n be an approximation of a zero of p and consider the Taylor expansion of $\frac{1}{p(w_n+h)}$ around w_n. Let D be any disk centered at w_n and does not contain any zero of p, then $g(h) = \frac{1}{p(w_n+h)}$ is analytic in D. Assume that

$$g(h) = \frac{1}{p(w_n + h)} = \sum_{k=0}^{\infty} a_k h^k, \qquad (28)$$

is the Taylor expansion of g around $h = 0$. It can be shown that $a_0 = \frac{1}{p(w_n)}$, $a_1 = \frac{-p'(w_n)}{p(w_n)^2}$, $a_2 = \frac{2p'(w_n)^2 - p(w_n)p''(w_n)}{2p'(w_n)^3}$. Generally, the a_i's can be recursively determined from the relation

$$\sum_{j=0}^{r} a_{r-j} p^{(j)}(w_n) j!(r - j)! = 0, \text{ for } r = 1, 2, \cdots. \qquad (29)$$

and $a_0 p(w_n) = 1$ for $r = 0$. If the poles of g are simple, then the radius of convergence of the power series (28) is $|h_0|$ where $|h_0|$ is the magnitude of the smallest pole of g. Moreover, $g(h)$ has a pole on the boundary of D. If there is only one pole (counting multiplicities) of magnitude $|h_0|$, then the sequence $\frac{a_k}{a_{k+1}}$ converges to h_0 as $k \to \infty$.

For each positive integer r, a fixed point function $\phi_r(z)$ can be defined as $\phi_r(z) = z - \frac{a_{r-1}}{a_r}$. It can be shown that ϕ is an $(r+1)$th order fixed point function for any simple zero of p. For example, $r = 1$ yields $\phi_1(z) = z - \frac{a_0}{a_1} = z - \frac{-p(z)}{p'(z)}$ which is the Newton iteration, and $\phi_2(z) = z - \frac{a_1}{a_2} = z - \frac{\frac{-p'(z)}{p(z)^2}}{\frac{2p'(z)^2 - p(z)p''(z)}{2p(z)^3}} = \frac{2p(z)p'(z)}{2p'(z)^2 - p(z)p''(z)}$ which is the Halley's iteration.

6.1 Zeros of Equal Modulus

Let D be the region of convergence of the power series (28) and let $\bar{D} = D \cup \partial D$. If the closed disk $\bar{D}$ contains more than one pole of g, these poles must be of equal modulus and lie on the boundary of D. Assume that there are only two poles on ∂D, then these poles can be obtained simultaneously as indicated in the following result.

Theorem 1. *Let z be an approximation of a zero of p and let a_i be defined by the expansion of $g(h) = \frac{1}{p(z+h)}$ around $h = 0$ as in (28). Assume that the radius of convergence* *of the Taylor series (28) is $|h_0|$ and that g has exactly two poles of magnitude $|h_0|$. Then for sufficiently large r, the sequence $[c_{1r}\, c_{2r}]$ defined by the equation*

$$\begin{bmatrix} a_r & a_{r+1} \\ a_{r+1} & a_{r+2} \end{bmatrix}^{-1} \begin{bmatrix} a_{r+2} \\ a_{r+3} \end{bmatrix} = - \begin{bmatrix} c_{2r} \\ c_{1r} \end{bmatrix}, \qquad (30)$$

exists and the zeros of $z^2 - c_{1r}z + c_{2r} = 0$ converge to a quadratic factor of the polynomial $p(z+h)$.

The proof of similar results can be found in [11].

The last result can be generalized for the case in which there are exactly s poles of equal modulus of radius $|h_0|$ as follows. Assume that these poles are $h_1, \cdots, h_s$, then an sth order polynomial for these poles can be obtained from the relation:

$$\begin{bmatrix} a_r & a_{r+1} & \cdots & a_{r+s-1} \\ a_{r+1} & a_{r+2} & \cdots & a_{r+s} \\ \cdots & \cdots & \cdots & \cdots \\ a_{r+s-1} & a_{r+s} & \cdots & a_{r+2s-1} \end{bmatrix}^{-1} \begin{bmatrix} a_{r+s} \\ \vdots \\ a_{r+s+1} \\ a_{r+2s} \end{bmatrix} = - \begin{bmatrix} c_{sr} \\ c_{s-1,r} \\ \vdots \\ c_{1r} \end{bmatrix}, \qquad (31)$$

Then $z + h_j$, $j = 1, \cdots, s$ are improved approximations of the zeros of p, where $h_1, \cdots, h_s$ are zeros of the polynomial $z^s - c_{1r}z^{s-1} + \cdots \pm c_{s,r}$. This polynomial converges to an sth order factor of $p(z+h)$ as $r \to \infty$.

6.2 Multiple Zeros

It can be shown that the iterations obtained in Section 3 converge only linearly if they converge to a multiple zero. The framework outlined in Section 6.1 of using power series expansion can be applied to polynomials with multiple zeros. However, to apply (31) the multiplicity of the particular zero should be is known which is not always possible. Thus, to obtain fixed point iterations which are of specific order regardless of multiplicities of zeros of p, consider the Taylor expansion of $\frac{p'(z+h)}{p(z+h)}$ so that

$$\frac{p'(z+h)}{p(z+h)} = \sum_{i=0}^{\infty} b_i h^i. \qquad (32)$$

Direct calculations of the b_i shows that $b_0 = \frac{p'(z)}{p(z)}$, $b_1 = \frac{p(z)p''(z) - p'(z)^2}{p(z)^2}$, $b_2 = \frac{2p'(z)^3 - 3p(z)p'(z)p''(z) + p^2 p'''(z)}{2p'(z)^3}$. The b_i's can be recursively computed using the equation

$$\sum_{j=0}^{r} b_{r-j} p^{(j)(z)} j!(r-j)! = p^{(r+1)}(z), \text{ for } r = 1, 2, \cdots. \qquad (33)$$

and $b_0 p(z) = p'(z)$ for $r = 0$.

It can be shown that $\psi_r(z) = z - \frac{a_{r-1}}{a_r}$ defines an r order iteration. For $r = 1$, we have

$$\frac{b_1}{b_2} = \frac{p(z)p'(z)}{p(z)^2 - p(z)p''(z)}$$

and thus $\psi_1 = z - \frac{p(z)p'(z)}{p(z)^2 - p(z)p''(z)}$ which is the Newton method applied to $\frac{p}{p'}$. For $r = 2$, we have

$$\begin{aligned} \frac{b_2}{b_3} &= \frac{p(z)}{p'(z) - \frac{p(z)p'(z)p''(z) - p(z)^2 p'''(z)}{2p'(z)^2 - 2p(z)p''(z)}} \\ &= \frac{p(z)}{p'(z) - \frac{p'(z)p''(z) - p(z)p'''(z)}{2p'(z)^2 - 2p(z)p''(z)} p(z)}. \end{aligned} \qquad (34)$$

Therefore, $\psi_2 = z - \frac{p(z)}{p'(z) - \frac{p'(z)p''(z) - p(z)p'''(z)}{2p'(z)^2 - 2p(z)p''(z)} p(z)}$, which is a generalized third order Halley's method.

Remark: More higher order fixed point iterations for the zeros of p can be obtained from the Taylor expansion of $\frac{\sqrt{p'(z+h)}}{p(z+h)}$ so that

$$\frac{\sqrt{p'(z+h)}}{p(z+h)} = \sum_{i=0}^{\infty} c_i h^i \qquad (35)$$

then $c_0 = \frac{\sqrt{p'(z)}}{p(z)}$, $c_1 = \frac{p(z)p''(z) - 2p'(z)^2}{2p(z)^2 \sqrt{p'(z)}}$,

$$c_2 = -\frac{1}{4}(p'(z))^{\frac{-3}{2}} \frac{p^2 p'' + 8pp'^2 p'' - 2p^2 p' p''' - 4p'^4}{p^3}.$$

The c_i's can be recursively computed by differentiating both sides of the equation

$$p'(z+h) = p(z+h)^2 \left(\sum_{i=0}^{\infty} c_i h^i\right)^2$$

Fixed point functions can be developed using ratios of c_i's, i.e.,

$$\xi_r(z) = z - \frac{a_{r-1}}{a_r}$$

defines an $(r+2)$th order iteration for simple zeros of p.

Hence,

$$\xi_1(z) = z - \frac{c_0}{c_1} = z - \frac{2pp'}{2p'^2 - pp''} \qquad (36)$$

and

$$\begin{aligned} \xi_2(z) &= z - \frac{c_1}{c_2} \\ &= z - 2pp' \frac{p(z)p''(z) - 2p'(z)^2}{p^2 p'' + 8pp'^2 p'' - 2p^2 p' p''' - 4p'^4} \end{aligned} \qquad (37)$$

Note that the function ξ_r is an improvement over ϕ_r and ψ_r in that it generates higher order iteration for a given r provided zeros are simple.

7. Relation Between Third Order Methods

In [8], the Laguerre iteration function for solving $p(z) = 0$ is given by

$$g(z) = z - \frac{\nu p(z)}{p'(z) + \{(\nu-1)^2 [p'(z)]^2 - \nu(\nu-1)p(z)p''(z)\}^{\frac{1}{2}}}, \qquad (39a)$$

where the argument of the root is to be chosen to differ by less than $\frac{\pi}{2}$ from the argument of $(\nu-1)p'(z)$ and $\nu \neq 0, 1$. By inspection one can form the following iteration function

$$\phi(z) = z - \frac{p(z)}{(1-a)p'(z) \pm \sqrt{a^2 p'(z)^2 - ap(z)p''(z)}}. \quad (39b)$$

This is a second order iteration for any $a \in \mathcal{C}$ and it is third order for any $a \neq 0$. Laguerre iteration can be modified so that fourth order convergence can be obtained by choosing $\nu = \frac{4}{3}$ and adding the term $\frac{p'p'''}{6p'^3}$, so that

$$g(z) = z - \frac{p(z)}{\frac{3}{4}p'(z) + \frac{1}{4}\{p'(z)^2 - 4p(z)p''(z)\}^{\frac{1}{2}} + \frac{p'p'''}{6p'^3}p^2}. \quad (40)$$

Remark: Using the binomial theorem it follows that this iteration can be expressed as

$$\Phi(z) = z - \frac{p(z)}{p'(z) + p_1(z)p(z) + O(p^2)}, \quad (41)$$

where $p_1 = \frac{-p''}{2p'}$ and thus the iteration is at least third order. Clearly (41) reduces to Halley's method if we ignore the term $O(p^2)$.

We also remark that every higher order method can be expressed in the form given in Equation (41). For example, the one-parameter Hansen-Patricks family (12), Laguerre's, Halley's and the square root or Ostrowski's method, can all be expressed as in (41) where $p_1 = \frac{-p''}{2p'}$. These methods differ only in the $O(p(z)^2)$ term. If $p_1 \neq \frac{-p''}{2p'}$, then Φ is only a second order iteration.

One can also show that fourth order methods can be expressed as

$$\Phi(z) = z - \frac{p(z)}{p'(z) + p_1(z)p(z) + p_2(z)p(z)^2 + O(p(z)^3)}, \quad (42)$$

where $p_1 = \frac{-p''}{2p'}$, and $p_2 = \frac{1}{6}\frac{p'''(z)}{p'(z)^2} - \frac{1}{4}\frac{p''(z)^2}{p'(z)^3}$.

8. Conclusion

We propose several approaches for generating families of third, fourth, and higher-order iterations. These include well known methods such as the Newton, Laguerre and Halley methods as special cases. Although these methods are derived for polynomials, they are also applicable to entire functions and arbitrary transcendental equations including analytic and rational functions. There remain many issues regarding the proposed methods. For example, third-order and fourth order families require the computation of a square root and a cubic root, respectively. The question here is which root should be chosen. Additionally, numerical stability and relative complexity of the methods need to be examined. These issues among others will be reported in a forthcoming paper.

References

[1] Foster LX., Generalizations of Laguerre's method: Higher order methods, SIAM J. Numer. Anal., 18 (1981), pp. 1004-1018.

[2] Davies, M., Dawson, B., On the global convergence of Halley's iteration formula. Numer. Math., 24 (1975), 133135.

[3] Bateman H., Halley's method for solving equations, Amer. Math. Monthly, 45 (1938), pp. 11-17.

[4] Ostrowski, A. M., Solution of equations in Euclidean and Banach space. New York: Academic Press, 1973.

[5] Hansen, E., Patrick, M., A family of root finding method. Numer. Math., 27 (1977), 257-269.

[6] Hansen, E., Patrick, M., Rusnak, J., Some modifications of Laguerre's method. BIT 17 (1977), 409-417.

[7] Parlett B., Laguerre's method applied to the matrix eigenvalue problem, Math. Comp., 18 (1964), pp. 464-485.

[8] Henrici, P., Applied and computational complex analysis, Vol. I, New York: John Wiley and Sons Inc., 1974.

[9] Ljiljana D. Petkovic, Miodrag S. Petkovic, Dragan .Zivkovic, "HANSENPATRICKS FAMILY IS OF LAGUERRES TYPE," Novi Sad J. Math., Vol. 33, No. 1, 2003, 109-115.

[10] Root Iterations and the Computation of Minimum and Maximum Zeros of Polynomials Hasan, M.A.; Circuits and Systems, 2005. ISCAS 2005. IEEE International Symposium on 23-26 May 2005 Page(s):2259 - 2262

[11] M. A. Hasan and A. A. Hasan, Hankel matrices of finite rank with applications to signal processing and polynomials, J. Math. Anal. Appls., vol. 208, pp. 218242, 1997.