Proceedings of the
44th IEEE Conference on Decision and Control, and
the European Control Conference 2005
Seville, Spain, December 12-15, 2005

A Simple Condition for Checking Non-vanishing Basin of Attraction Stability for a Class of Positive Nonlinear Systems

H. Shim and Nam H. Jo

Abstract— **When dealing with positive nonlinear systems, conventional stability requires too much for the equilibrium points located on the boundary of the positive orthant, which encourages the consideration of 'stability with respect to the positive orthant.' In addition, since this case arises often when the bifurcation occurs, variation of stability property (regarding the size of basin of attraction) along the variation of the parameter becomes of interest. Motivated by this fact, NvBA(Non-vanishing Basin of Attraction)-stability was recently proposed and investigated. In particular, it was claimed that NvBA-stability holds if and only if the same property holds for the reduced order system on a parametrized center manifold. However, the verification of NvBA-stability is not easy in general, because a solution to the partial differential equation for the center manifold needs to be found. In this paper, we present a readily verifiable condition for the NvBA-stability by restricting the system structure and by utilizing the information about the location of another equilibria that split from or merge into the equilibrium of interest due to the parameter variation. The proposed condition requires neither the solution to the center manifold equation nor the construction of Lyapunov functions.**

Notation: A function is said to be of class C^k if it is continuously differentiable k times. For a (column) vector x and a matrix A, the *i*-th component of x and the *i*-th row of A are denoted by $x_{(i)}$ and $A_{(i)}$, respectively. ($x_{(i)}$ is replaced by x_i if there is no confusion.) We denote by e_k the column vector $[0 \cdots 0\ 1\ 0 \cdots 0]^T$ with the entry 1 in the *k*-th place. The elementary matrix obtained by interchanging the first and the *k*-th row of the identity matrix is denoted by E_k. The $r \times 1$ zero vector is denoted by 0_r and when there is no confusion, 0_r is abbreviated to 0. $||x||$ stands for the Euclidean norm of a vector x and, for some $r > 0$ and $x^* \in \mathbb{R}^n$, $B(x^*, r) := \{x \in \mathbb{R}^n \ : \ ||x - x^*|| < r\}$. Let $\bar{\mathbb{R}}^n_+ := \{x \in \mathbb{R}^n : x_1 \geq 0, \cdots, x_n \geq 0\}$ and $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n : x_1 > 0, \cdots, x_n > 0\}$. The order of magnitude notation o is used as follows: we say $f(x) = o(g(x))$ if, for each $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x)| \leq \epsilon |g(x)|$ for $|x| < \delta$. Finally, the intervals $\{x \in \mathbb{R} : a \leq x \leq b\}, \{x \in \mathbb{R} : a \leq x < b\}$, and $\{x \in \mathbb{R} : a < x \leq b\}$ will be denoted by $[a, b], [a, b)$, and $(a, b]$, respectively.

I. INTRODUCTION

A class of nonlinear systems that often appear in biology, chemical kinetic network, and finance is *positive systems* [3], whose property is that all the states of the system remain in the positive orthant of the state-space if initiated there. This is because, in those applications, the state is meaningful only when it is positive, and therefore, the corresponding dynamic system, if appropriately modeled, has such property. As a result, stability analysis of a certain equilibrium located on the boundary of the positive orthant needs to take into account the fact that the positive orthant is positively invariant. This implies that such an equilibrium may be asymptotically stable even when the corresponding Jacobian has an eigenvalue with nonnegative real part because the feasible perturbation of the initial condition is restricted to the positive orthant. More interesting case arises when the Jacobian is marginally stable, and thus one wants to utilize the center manifold theory. In this case, the positive orthant plays an important role in the stability analysis of the reduced system.

On the other hand, for a system under a parameter variation (or, having a constant input), the location of an equilibrium and its stability property often depend on the parameter. In fact, the afore-mentioned case where the Jacobian loses its rank, is often resulted from the bifurcation with the parameter (or, the constant input) as the bifurcation variable. Assuming that the bifurcation occurs at a certain equilibrium, the authors of [6] have studied the variation of the size of basin of attraction along the variation of the parameter. In addition, they characterized the case where the basin of attraction never vanishes under the variation of the bifurcation parameter, and called the property by 'Non-vanishing Basin of Attraction(NvBA)-stability.'[1] It was also shown that the NvBA-stability holds if and only if the same property holds for the corresponding reduced order system on a parametrized center manifold.

Generally speaking, if a bifurcation occurs at a certain equilibrium where two or more equilibria split or merge according to the variation of the bifurcation variable, then the NvBA-stability cannot hold (because every neighborhood of stable equilibrium contains another equilibrium). However, if it occurs on the boundary of the positive orthant, there are still chances for NvBA-stability, which yields the concept of 'NvBA-stability with respect to (*w.r.t.*) the positive orthant.' Although this concept is investigated in [6], determining NvBA-stability on the parametrized center manifold is not easy in general because the partial differential equation for the center manifold needs to be solved.

In this paper, we present a readily verifiable condition for

This work was supported by the Basic Research Program of the Korea Science and Engineering Foundation (Grant No. KOSEF R01-2002-000-00227-0).

H. Shim is with ASRI, School of Electrical Engineering and Computer Science, Seoul National Univ., Kwanak P.O.Box 34, Seoul, 151-600, Korea. Nam H. Jo is with the School of Electrical Engineering, Soongsil Univ., Sangdo5-dong, Dongjak-gu, Seoul, 156-743, Korea. Corresponding author: nhjo@ssu.ac.kr

[1]One of the motivations of studying the variation of the basin of attraction is for applying the slowly-varying control approach in order to solve the HIV infection control problem of [7], for example. See [2] for the application of the NvBA-stability.

the NvBA-stability *w.r.t.* the positive orthant by restricting our attention to the class of systems[2] such as

$$\begin{aligned}
\dot{x}_1 &= f_1(x,u) \\
&\vdots \\
\dot{x}_k &= f_k(x,u) = x_k\phi(x,u) \\
&\vdots \\
\dot{x}_n &= f_n(x,u)
\end{aligned} \tag{1}$$

where u is a parameter (or, a constant input), and by focusing on a particular equilibrium x^* (with a certain u^*) such that

$$x_k^* = 0 \qquad \text{and} \qquad \phi(x^*, u^*) = 0. \tag{2}$$

Clearly, x^* is located on the boundary of the positive orthant and, because of (2), the Jacobian at x^* has at least one zero eigenvalue. Thus, it is natural to apply the center manifold theory in order to verify the stability (with respect to the positive orthant). In contrast to the previous work [6], the proposed condition in this paper does not require the solution to the center manifold equation, although relying on the center manifold approach. Thus, it becomes very straightforward to check NvBA-stabilty *w.r.t.* the positive orthant.

The paper is organized as follows: In Section 2, the notion of stability *w.r.t.* the positive orthant and NvBA-stabilty is presented, and an easily checkable condition is proposed that guarantees the NvBA-stability. Section 3 contains the proof of the main result of Section 2 and some concluding remarks are given in Section 4.

II. Main Results

A. Verifying Stability with respect to $\mathbb{R}^n_+$

Consider the following nonlinear system

$$\dot{x} = f(x), \qquad x \in \mathbb{R}^n, \tag{3}$$

where f is of class C^2. It is assumed that $\mathbb{R}^n_+$ is a positively invariant set for system (3). We are interested in an isolated equilibrium point x^* of the system, (i.e., $f(x^*) = 0$ and there is no other equilibrium in a neighborhood of x^*) which is located on the boundary of $\mathbb{R}^n_+$ and satisfies the following assumption.

Assumption 1: Let A be the Jacobian of the system at x^*, that is,

$$A := \frac{\partial f}{\partial x}(x^*).$$

The matrix A has one eigenvalue at zero and $n-1$ eigenvalues with negative real parts. Moreover, there exists an integer k ($1 \le k \le n$) such that

$$x^*_{(k)} = 0 \quad \text{and} \quad \frac{\partial f_{(k)}}{\partial x}(x^*) = 0. \tag{4}$$

$\Diamond$

[2]There are several systems such as chemical reaction networks [1], biological systems [5], and HIV-infection model [7], that conform to the class considered in this paper.

It is easily seen that the whole class of systems that satisfy the condition (4) can be represented by (2) (without u-term).

We are now interested in the stability of x^* under the assumption that the state remains in $\mathbb{R}^n_+$.

Definition 1: The equilibrium point x^* is locally stable with respect to the set $\mathbb{R}^n_+$ if, for each $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that

$$x(0) \in B(x^*, \delta) \cap \mathbb{R}^n_+ \;\Rightarrow\; x(t) \in B(x^*, \epsilon) \cap \mathbb{R}^n_+.$$

Moreover, it is locally asymptotically stable *w.r.t.* $\mathbb{R}^n_+$ if it is stable *w.r.t.* $\mathbb{R}^n_+$ and δ can be chosen such that

$$x(0) \in B(x^*, \delta) \cap \mathbb{R}^n_+ \;\Rightarrow\; \lim_{t\to\infty} x(t) = 0.$$

$\Diamond$

The following proposition provides a Lyapunov-type characterization of the stability *w.r.t.* $\mathbb{R}^n_+$ [6].

Proposition 1: The equilibrium point x^* is locally asymptotically stable *w.r.t.* $\mathbb{R}^n_+$ if there exists a C^1 function $V : \mathbb{R}^n \to \mathbb{R}$ and a positive constant R such that, for all $x \in B(x^*, R) \cap \mathbb{R}^n_+$,

$$\begin{aligned}
\alpha_1(\|x - x^*\|) \le V(x) &\le \alpha_2(\|x - x^*\|) \\
\frac{\partial V}{\partial x}(x) f(x) &\le -\alpha_3(\|x - x^*\|),
\end{aligned}$$

where $\alpha_i(\cdot),\ i = 1,2,3$ are class-$\mathcal{K}$ functions. $\Diamond$

In contrast to Proposition 1, we now provide a simple sufficient condition for the stability *w.r.t.* $\mathbb{R}^n_+$ that does not require a Lyapunov function. Under Assumption 1, since $A_{(k)} = [0\ 0\ \cdots\ 0]$, exchanging the first and the k-th row and exchanging the first and the k-th column (i.e., $E_k A E_k^{-1}$) result in

$$E_k A E_k^{-1} = \begin{bmatrix} 0 & 0 \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix}, \tag{5}$$

where $\bar{A}_{21} \in \mathbb{R}^{(n-1)\times 1}$ and $\bar{A}_{22} \in \mathbb{R}^{(n-1)\times(n-1)}$ is a Hurwitz matrix because of Assumption 1.

Theorem 1: Under Assumption 1, suppose that

$$\frac{\partial^2 \psi}{\partial y^2}(0) < 0, \tag{6}$$

where the function $\psi : \mathbb{R} \to \mathbb{R}$ is defined as

$$\psi(y) := f_{(k)}\left(E_k \begin{bmatrix} 1 \\ -\bar{A}_{22}^{-1}\bar{A}_{21} \end{bmatrix} y + x^*\right), \tag{7}$$

where $\bar{A}_{21} \in \mathbb{R}^{(n-1)\times 1}$ and $\bar{A}_{22} \in \mathbb{R}^{(n-1)\times(n-1)}$ are defined by (5). Then, the isolated equilibrium point x^* is locally asymptotically stable *w.r.t.* $\mathbb{R}^n_+$. $\Diamond$

The proof of Theorem 1 follows as a corollary from Theorem 2. As addressed in the introduction, the verification of the stability with respect to $\mathbb{R}^n_+$ is much more simpler than the previous work [6] since there is no need to solve the center manifold equation.

B. Verifying NvBA-stability with respect to $\mathbb{R}^n_+$

In the previous section, we studied the stability *w.r.t.* $\mathbb{R}^n_+$ of a single equilibrium point. In contrast, we now deal with a family of equilibrium points and their stability property. Consider the following nonlinear system

$$\dot{x} = f(x,u), \quad x \in \mathbb{R}^n,\ u \in \mathbb{R}, \tag{8}$$

where f is of class C^3 and u is a constant input (or, a parameter) such that $u \in \Gamma = [0, \bar{u}]$, $\bar{u} > 0$. For each $u \in \Gamma$, we assume that the set $\mathbb{R}^n_+$ is positively invariant for the system (8) and that the system has at least one isolated equilibrium in $\bar{\mathbb{R}}^n_+$. More specifically, we assume that there exists a C^2 function $\mathcal{X} : \Gamma \to \bar{\mathbb{R}}^n_+$ such that $f(\mathcal{X}(u), u) = 0$.

Definition 2: The system (8) is *Non-vanishing Basin of Attraction(NvBA)-stable w.r.t.* $\mathbb{R}^n_+$ *on the locus of equilibrium points* $\mathcal{X}(u)$, $u \in \Gamma$, if there exists a constant $R > 0$ such that, for each $u \in \Gamma$, $\mathcal{X}(u)$ is asymptotically stable *w.r.t.* $\mathbb{R}^n_+$ and $B(\mathcal{X}(u), R) \cap \mathbb{R}^n_+$ is contained in the region of attraction for $\mathcal{X}(u)$. $\Diamond$

With regard to Definition 2, it should be noticed that R is independent of u. When there is no confusion, *NvBA-stability w.r.t.* $\mathbb{R}^n_+$ *on the locus of equilibrium points* $\mathcal{X}(u)$, $u \in \Gamma$, will be abbreviated to *NvBA-stability*. Moreover, for all $u \in \Gamma$, the largest $R \in (0, \infty]$ such that $B(\mathcal{X}(u), R) \cap \mathbb{R}^n_+$ is contained in the basin of attraction for $\mathcal{X}(u)$ will be referred to as *the size of the basin of attraction w.r.t.* $\mathbb{R}^n_+$.

The following lemma provides a simple condition to guarantee NvBA-stability on the locus of equilibrium, whose proof is omitted since it follows directly from [4, Lemma 9.8].

Lemma 1: If $\frac{\partial f}{\partial x}(\mathcal{X}(u), u)$ is Hurwitz for each $u \in \Gamma$, then the system (8) is NvBA-stable (*w.r.t.* $\mathbb{R}^n$) on the locus of $\mathcal{X}(u)$, $u \in \Gamma$. $\Diamond$

The assumption of Lemma 1 is quite restrictive in the sense that it guarantees the NvBA-stability with respect to not only $\mathbb{R}^n_+$ but also $\mathbb{R}^n$. We emphasize that there are many cases when the assumption of Lemma 1 does not hold, especially when the system undergoes the bifurcation at $\mathcal{X}(u)$ with u as the bifurcation parameter. For example, consider the following system

$$\dot{x}_1 = -x_1(x_1 + u)(x_1 - u), \qquad \dot{x}_2 = -x_2. \tag{9}$$

Let $\mathcal{X}(u) = [u, 0]^T$ and $\Gamma = [0, 1]$. Then, it is seen that the Jacobian at $\mathcal{X}(u)$ is Hurwitz for all $u \in (0, 1]$ but is not at $u = 0$. For each $u \in (0, 1]$, the size of basin of attraction *w.r.t.* $\mathbb{R}^2$ is equal to $|u|$, and hence it shrinks to zero as $u \to 0$. Thus, the system (9) is not NvBA-stable *w.r.t.* $\mathbb{R}^2$. However, it is clear from Fig. 1 that the system

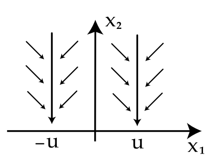

Fig. 1. Phase plane for $\dot{x}_1 = -x_1(x_1 + u)(x_1 - u)$, $\dot{x}_2 = -x_2$

(9) is NvBA-stable *w.r.t.* $\mathbb{R}^2_+$. As a matter of fact, the size of basin of attraction *w.r.t.* $\mathbb{R}^2_+$ can be as large as desired. This example indicates that there are some class of systems that are NvBA-stable *w.r.t.* $\mathbb{R}^n_+$ although it does not satisfy the assumption of Lemma 1.

The following definition will be helpful for the presentation of the NvBA-stability theorem that requires the weaker condition than Lemma 1.

Definition 3: If there exist a positive constant δ and a locus of equilibria $\bar{\mathcal{X}}(u)$ such that $\bar{\mathcal{X}}(\cdot)$ is continuous, $\bar{\mathcal{X}}(0) = \mathcal{X}(0)$, and $\bar{\mathcal{X}}(u) \neq \mathcal{X}(u)$ for $u \in (0, \delta]$, we call $\bar{\mathcal{X}}(u)$ by *the locus of split equilibria* of $\mathcal{X}(u)$. $\Diamond$

A locus of split equilibria of $\mathcal{X}(u)$ exists typically when the bifurcation occurs at $u = 0$. For example, in the system (9), $\bar{\mathcal{X}}(u) = [-u, 0]^T$ and $\bar{\mathcal{X}}(u) = [0, 0]^T$ are the loci of split equilibria of $\mathcal{X}(u) = [u, 0]^T$.

Now we state the main theorem.

Theorem 2: Assume that the following conditions are satisfied:
(H1) the Jacobian $\frac{\partial f}{\partial x}(\mathcal{X}(u), u)$ is Hurwitz for $u \in (0, \bar{u}]$,
(H2) the equilibrium $\mathcal{X}(0)$ and the corresponding Jacobian $\frac{\partial f}{\partial x}(\mathcal{X}(0), 0)$ satisfy Assumption 1,
(H3) all the loci of split equilibria of $\mathcal{X}(u)$, if any, are contained in the set

$$\{x \in \mathbb{R}^n : x_{(k)} \leq 0\}, \tag{10}$$

(that is, $\bar{\mathcal{X}}_{(k)}(u) \leq 0$ for all $u \in \Gamma$,)
(H4) $\frac{\partial^2 \psi}{\partial y^2}(0) < 0$, where the function $\psi : \mathbb{R} \to \mathbb{R}$ is defined by

$$\psi(y) := f_{(k)}\left(E_k \begin{bmatrix} 1 \\ -\bar{A}_{22}^{-1}\bar{A}_{21} \end{bmatrix} y + \mathcal{X}(0), 0\right), \tag{11}$$

in which $\bar{A}_{21} \in \mathbb{R}^{(n-1)\times 1}$ and $\bar{A}_{22} \in \mathbb{R}^{(n-1)\times(n-1)}$ are obtained from

$$E_k \frac{\partial f}{\partial x}(\mathcal{X}(0)) E_k^{-1} =: \begin{bmatrix} 0 & 0 \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix}. \tag{12}$$

Then, the system (8) is NvBA-stable *w.r.t.* $\mathbb{R}^n_+$ on the locus of $\mathcal{X}(u)$, $u \in \Gamma$. $\Diamond$

Compared to Theorem 1, a new condition (H3) is added in the assumption list. To appreciate (H3), consider the following system

$$\dot{x}_1 = -x_1(x_1 - u)(x_1 - 2u), \qquad \dot{x}_2 = -x_2. \tag{13}$$

Let $\mathcal{X}(u) = [2u, 0]^T$ and $\Gamma = [0, 1]$. Then, it can be seen that $\mathcal{X}(u)$ is locally asymptotically stable *w.r.t.* $\mathbb{R}^2_+$ for $u \in (0, 1]$ and is globally asymptotically stable even when $u = 0$. But, the size of basin of attraction *w.r.t.* $\mathbb{R}^2_+$ is $|u|$ for $u \in (0, 1]$ and as a result, (13) is not NvBA-stable *w.r.t.* $\mathbb{R}^2_+$. Note that $[u, 0]^T$ is a locus of split equilibria of $\mathcal{X}(u)$ and it violates (10) (with $k = 1$).

III. Proof of Theorem 2

Let $\bar{x} := x - \mathcal{X}(u)$. Then, we have

$$\dot{\bar{x}} = f(\bar{x} + \mathcal{X}(u), u) =: \bar{f}(\bar{x}, u). \tag{14}$$

Let

$$A(u) := \frac{\partial \bar{f}}{\partial \bar{x}}(0, u) = \frac{\partial f}{\partial x}(\mathcal{X}(u), u). \tag{15}$$

Claim 1: From (H2), there exists a nonsingular matrix T_0 such that

$$T_0 A(0) T_0^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & A_s \end{bmatrix}, \tag{16}$$

$$(T_0)_{(1)} = e_k^T \tag{17}$$

$$(T_0^{-1})^{(1)} = E_k \begin{bmatrix} 1 \\ -\bar{A}_{22}^{-1}\bar{A}_{22} \end{bmatrix} \tag{18}$$

where $A_s \in \mathbb{R}^{(n-1)\times(n-1)}$ is a Hurwitz matrix and $(T_0)^{(1)}$ is the first column of T_0. ◊

Proof: Let $N := -\bar{A}_{22}^{-1}\bar{A}_{21}$ and pick any nonsingular matrix $M \in \mathbb{R}^{(n-1)\times(n-1)}$. Define

$$\bar{T} := \begin{bmatrix} 1 & 0 \\ -MN & M \end{bmatrix},$$

then we obtain

$$\bar{T}^{-1} = \begin{bmatrix} 1 & 0 \\ N & M^{-1} \end{bmatrix}. \tag{19}$$

Then, since $\bar{A}_{21} + \bar{A}_{22}N = 0$, it can be verified by (12) and (19) that

$$(E_k A(0) E_k^{-1})\bar{T}^{-1} = \bar{T}^{-1} \begin{bmatrix} 0 & 0 \\ 0 & M\bar{A}_{22}M^{-1} \end{bmatrix},$$

which results in

$$(\bar{T}E_k)A(0)(\bar{T}E_k)^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & M\bar{A}_{22}M^{-1} \end{bmatrix}, \tag{20}$$

where $M\bar{A}_{22}M^{-1}$ is Hurwitz since $\bar{A}_{22}$ is.

Let $T_0 := \bar{T}E_k$. Then, since E_k exchanges the first and the k-th column of $\bar{T}$ and $E_k^{-1} = E_k$, we have

$$(T_0)_{(1)} = (\bar{T}E_k)_{(1)} = e_k^T$$
$$(T_0^{-1})^{(1)} = (E_k^{-1}\bar{T}^{-1})^{(1)} = E_k \begin{bmatrix} 1 \\ N \end{bmatrix}.$$

Also, from (20), we complete the proof. ■

Claim 2: From the matrix T_0 obtained above, there exist $\bar{u}_1$ such that $0 < \bar{u}_1 \le \bar{u}$ and a C^2 function $T : [0, \bar{u}_1] \to \mathbb{R}^{n\times n}$ such that $T(0) = T_0$ and

$$T(u)A(u)T^{-1}(u) = \begin{bmatrix} A_1(u) & 0 \\ 0 & A_2(u) \end{bmatrix}, \tag{21}$$

where $A_1(u) \in \mathbb{R}$ is such that $A_1(0) = 0$ and $A_1(u) < 0$ for $u \in (0, \bar{u}_1]$, and $A_2(u) \in \mathbb{R}^{(n-1)\times(n-1)}$ satisfies that $A_2(u)$ is Hurwitz for $u \in [0, \bar{u}_1]$ and $A_2(0) = A_s$. ◊

Proof: Proof of the claim is omitted due to the page restriction. ■

Now, by changing the coordinates as

$$z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} := T(u)\bar{x}, \qquad z_1 \in \mathbb{R}^1, \quad z_2 \in \mathbb{R}^{n-1},$$

the system (14) is transformed into

$$\begin{aligned} \dot{z} &= T(u)\bar{f}(T^{-1}(u)z, u) \\ &=: T(0)A(0)T^{-1}(0)z + g(z, u), \end{aligned}$$

where

$$g(z, u) = T(u)\bar{f}(T^{-1}(u)z, u) - T(0)A(0)T^{-1}(0)z. \tag{22}$$

That is, referring to (16), the system is written as

$$\dot{z}_1 = g_1(z_1, z_2, u) \tag{23a}$$
$$\dot{z}_2 = A_s z_2 + g_2(z_1, z_2, u), \tag{23b}$$

where the C^2 function g satisfies

$$g(0, 0, u) = 0, \quad \frac{\partial g}{\partial z}(0, 0, 0) = 0, \quad {}^{\forall}u \in [0, \bar{u}_1]. \tag{24}$$

Claim 3: There exist positive constants $\bar{r}_1$ and $\bar{u}_2$ ($\le \bar{u}_1$), and a C^1 function $(z_1, u) \mapsto \pi(z_1, u)$ defined for all $|z_1| \le \bar{r}_1$ and $0 \le u \le \bar{u}_2$ such that

$$\pi(0, u) = 0, \quad \frac{\partial \pi}{\partial z_1}(0, 0) = 0, \tag{25}$$

and

$$\begin{aligned} A_s\pi(z_1, u) &+ g_2(z_1, \pi(z_1, u), u) \\ &= \frac{\partial \pi}{\partial z_1} g_1(z_1, \pi(z_1, u), u). \end{aligned}$$

◊

This claim is basically the center manifold theorem. Indeed, it is observed that the set $\{(z_1, z_2) : z_2 = \pi(z_1, u)\}$ becomes the center manifold if $u = 0$. However, in contrary to the fact that the standard center manifold theorem (by augmenting the dynamics with $\dot{u} = 0$) results in the property $\pi(0, 0) = 0$, the above claim yields $\pi(0, u) = 0$ for all $u \in [0, \bar{u}_2]$ whose utility will be seen shortly. Proof of this claim is found in [6] and we do not repeat it here.

From (25), π can be written as

$$\pi(z_1, u) = \pi_1(u)z_1 + o(|z_1|), \tag{26}$$

where $\pi_1 : [0, \bar{u}_2] \to \mathbb{R}^{n-1}$ is continuous and $\pi_1(0) = 0$. Moreover, it follows from (21), (22), and (24) that $g_1(z_1, z_2, u)$ is of the form

$$\begin{aligned} g_1(z_1, z_2, u) = A_1(u)z_1 &+ c_1(u)z_1^2 + z_1 c_2(u) z_2 \\ &+ z_2^T c_3(u) z_2 + o(|(z_1, z_2)|^2) \end{aligned} \tag{27}$$

for $u \in [0, \bar{u}_2]$ and $|z_1| \le \bar{r}_2$, $\|z_2\| \le \bar{r}_2$ with some $0 < \bar{r}_2 \le \bar{r}_1$, where $c_i(\cdot)$ is continuous functions of appropriate dimensions[3]. This, together with (26), leads to

$$\begin{aligned} g_1(z_1, \pi(z_1, u), u) = A_1(u)z_1 &+ [c_1(u) + c_2(u)\pi_1(u) \\ &+ \pi_1^T(u)c_3(u)\pi_1(u)]z_1^2 + o(z_1^2). \end{aligned} \tag{28}$$

Claim 4: Under (H4), it holds that

$$c_1(0) =: -c < 0. \tag{29}$$

◊

[3]The vector norm $\|\cdot\|$ in the proof of Section III is defined by

$$\|w\| = \sqrt{w^T P w}$$

where $P = P^T > 0$ is the solution of $PA_s + A_s^T P = -I$. This is just for convenience and, due to the norm equivalence, the flow of the proof is not altered.

Proof: From (17), it happens that $z_1 = \bar{x}_{(k)}$ when $u = 0$. Therefore,

$$\begin{aligned} g_1(z_1, z_2, 0) = \dot{z}_1 = \dot{\bar{x}}_{(k)} &= f_{(k)}(\bar{x} + \mathcal{X}(0), 0) \\ &= f_{(k)}(T(0)^{-1}z + \mathcal{X}(0), 0) \end{aligned} \tag{30}$$

Putting $z_2 = 0$ and referring to (11), (18) and (30), it follows that

$$\psi(z_1) = g_1(z_1, 0, 0),$$

which, by virtue of (27), results in

$$\psi(z_1) = c_1(u)z_1^2 + o(z_1^2).$$

Hence, from (H4), the claim is proved. ■

Let $w := z_2 - \pi(z_1, u)$. Then, we obtain, from (23),

$$\dot{z}_1 = g_1(z_1, \pi(z_1, u), u) + N_1(z_1, w, u) \tag{31a}$$
$$\dot{w} = A_s w + N_2(z_1, w, u) \tag{31b}$$

where

$$\begin{aligned} N_1(z_1, w, u) &= g_1(z_1, w + \pi(z_1, u), u) - g_1(z_1, \pi(z_1, u), u) \\ N_2(z_1, w, u) &= g_2(z_1, w + \pi(z_1, u), u) - g_2(z_1, \pi(z_1, u), u) \\ &\quad - \frac{\partial \pi}{\partial z_1}(z_1, u) N_1(z_1, w, u), \end{aligned}$$

so that $N_i(z_1, 0, u) = 0$ and $\frac{\partial N_i}{\partial w}(0, 0, u) = 0$, for $i = 1, 2$.

Up to now, we have defined several transformations in order to convert the system (8) to (31). The coordinate change from the state (z_1, w) to the state x is written as

$$x = \phi(z_1, w, u) := T^{-1}(u) \begin{bmatrix} z_1 \\ w + \pi(z_1, u) \end{bmatrix} + \mathcal{X}(u), \tag{32}$$

which is valid for $|z_1| \le \bar{r}_3$, $\|w\| \le \bar{r}_3$ and $u \in [0, \bar{u}_2]$, where $\bar{r}_3 > 0$ is such that

$$|z_1| \le \bar{r}_3, \|w\| \le \bar{r}_3 \quad \Rightarrow \quad |z_1| \le \bar{r}_2, \|w + \pi(z_1, u)\| \le \bar{r}_2$$

for all $u \in [0, \bar{u}_2]$. Note that, for each u, the map ϕ is a C^1 diffeomorphism, so that it maps an open set in x-coordinates to an open set in (z_1, w)-coordinates, and vice versa.

Define

$$\Omega(u, r) := \{(z_1, w) : \phi_{(k)}(z_1, w, u) > 0, z_1 \le r, \|w\| \le r\}. \tag{33}$$

Claim 5: Let

$$\gamma(w, u) := \inf z_1 \quad \text{subject to} \quad \phi_{(k)}(z_1, w, u) > 0.$$

Then, there exist positive constants $\bar{r}_4 \le \bar{r}_3$ and $\bar{u}_3 \le \bar{u}_2$ such that, for $\|w\| \le \bar{r}_4$ and $u \in [0, \bar{u}_3]$, the function $\gamma(w, u)$ is C^1 and

$$\gamma(w, 0) = 0, \tag{34}$$
$$\gamma(0, u) \le 0, \tag{35}$$
$$\phi_{(k)}(\gamma(w, u), w, u) = 0. \tag{36}$$

◊

Proof: We note that $(T^{-1}(0))_{(k)} = [1, 0, \cdots, 0]$ because $x_{(k)} = (T^{-1}(0)z)_{(k)} = z_1$. From this fact, let the scalar functions $t_i(u)$, $i = 1, \cdots, n$, be defined by

$$(T^{-1}(u))_{(k)} =: [1 + t_1(u),\ t_2(u),\ \cdots,\ t_n(u)],$$

and let $\bar{t}_2(u) := [t_2(u), \cdots, t_n(u)]^T$ for simplicity. Obviously, $t_i(u)$ is C^1 with $t_i(0) = 0$. Then, we have

$$\begin{aligned} \phi_{(k)}(z_1, w, u) &= (1 + t_1(u))z_1 \\ &\quad + \bar{t}_2^T(u)(w + \pi(z_1, u)) + \mathcal{X}_{(k)}(u). \end{aligned} \tag{37}$$

The function $\phi_{(k)}$ is C^1, and it is seen referring to (26) that

$$\frac{\partial \phi_{(k)}}{\partial z_1}(0, w, u) = 1 + t_1(u) + \bar{t}_2^T(u)\pi_1(u). \tag{38}$$

Since $\phi_{(k)}(0, 0, 0) = 0$ and $\frac{\partial \phi_{(k)}}{\partial z_1}(0, 0, 0) = 1$, by the implicit function theorem, there exist a C^1 function $\bar{\gamma}(w, u)$ defined on $\|w\| \le \bar{r}_4$ and $u \in [0, \bar{u}_3]$ with positive constants $\bar{r}_4 (\le \bar{r}_3)$ and $\bar{u}_3 (\le \bar{u}_2)$, such that

$$\phi_{(k)}(\bar{\gamma}(w, u), w, u) = 0.$$

Without loss of generality, (38) is positive for $\|w\| \le \bar{r}_4$ and $u \in [0, \bar{u}_3]$, which implies that $\bar{\gamma}(w, u)$ is the infimum of z_1 satisfying $\phi_{(k)}(z_1, w, u) > 0$, *i.e.*, $\bar{\gamma}(w, u) = \gamma(w, u)$.

On the other hand, since $\phi_{(k)}(z_1, w, 0) = z_1$ from (37), it follows from the definition of $\gamma(w, u)$ that $\gamma(w, 0) = 0$.

Furthermore, from (38), it can also be seen that $\gamma(0, u) \le 0$ because $\phi_{(k)}(0, 0, u) = \mathcal{X}_{(k)}(u) \ge 0$. ■

From the definition of $\gamma(w, u)$, it follows that

$$\Omega(u, r) = \{(z_1, w) : \gamma(w, u) < z_1 \le r, \|w\| \le r\}$$

for all $r \in (0, \bar{r}_4]$ and $u \in [0, \bar{u}_3]$. Note that $\Omega(u, r)$ is a *bounded* set.

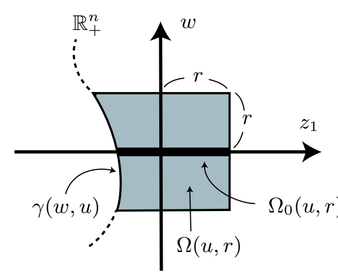

Fig. 2. Geometric representation of the set $\Omega(u, r)$ and $\Omega_0(u, r)$.

Claim 6: There exist positive $\bar{r}_5 \le \bar{r}_4$ and $\bar{u}_4 \le \bar{u}_3$ such that, for all $r \in (0, \bar{r}_5]$, $u \in [0, \bar{u}_4]$, the system (31a) has no equilibrium except the origin on the set

$$\begin{aligned} \Omega_0(u, r) &:= \Omega(u, r)\big|_{w=0} \\ &= \{(z_1, 0) : \gamma(0, u) < z_1 \le r\}. \end{aligned}$$

In addition, every solution trajectory of system (31a) starting in $\Omega_0(u, r)$ converges to the origin. ◊

Proof: By virtue of (28) and $N_1(z_1, 0, u) = 0$, the system (31a) on the set $\Omega_0(u, r)$ is given by

$$\dot{z}_1 = g_1(z_1, \pi(z_1, u), u) \tag{39a}$$
$$= A_1(u)z_1 + \bar{g}_1(u)z_1^2 + o(z_1^2), \tag{39b}$$

where $\bar{g}_1(u) = c_1(u) + c_2(u)\pi_1(u) + \pi_1^T(u)c_3(u)\pi_1(u)$. It follows from Claim 4 and $\pi_1(0) = 0$ that there exists a $\bar{u}_4$ such that $\bar{g}_1(u)$ is negative for $u \in [0, \bar{u}_4]$. Thus, there exists a positive $\bar{r}_5 \le \bar{r}_4$ such that $\bar{g}_1(u)z_1^2 + o(z_1^2)$ is negative for all $0 < z_1 \leq \bar{r}_5$. Since $A_1(u) \leq 0$, we conclude that there is no equilibrium of (39a) in the interval $0 < z_1 < \bar{r}_5$, and that every solution trajectory starting from $z_1(0)$ such that $0 < z_1(0) \leq \bar{r}_5$ converges to zero.

On the other hand, suppose that $\gamma(0,u) < 0$ for some $u \in (0, \bar{u}_4]$, and that there exists an equilibrium point, say z_1^*, of system (39a) in the interval $\gamma(0,u) < z_1 < 0$. (Note that $\gamma(0,u) < 0$ is impossible when $u = 0$ due to (34).) Then, the point $(z_1, w) = (z_1^*, 0)$ is an equilibrium (other than the origin) of system (31). The equilibrium is expressed in the x-coordinates as

$$x^* = \phi(z_1^*, 0, u) = T^{-1}(u) \begin{bmatrix} z_1^* \\ \pi(z_1^*, u) \end{bmatrix} + \mathcal{X}(u).$$

Therefore, we have

$$x_{(k)}^* = \phi_{(k)}(z_1^*, 0, u).$$

Since $z_1^* > \gamma(0,u)$ and $\phi_{(k)}(\gamma(0,u), 0, u) = 0$ by virtue of (36), it follows from the definition of $\gamma(w,u)$ that

$$x_{(k)}^* > 0.$$

This is a contradiction to (H3), which implies there is no equilibrium in $\gamma(0,u) < z_1 < 0$. In addition, because of $A_1(u) < 0$, we have $g_1(z_1, \pi(z_1,u), u) > 0$ for $\gamma(0,u) < z_1 < 0$. Thus, every solution with $\gamma(0,u) < z_1(0) < 0$ converges to zero. ■

Claim 7: There exist positive $r^* \leq \bar{r}_5$ and $\bar{u}_5 \leq \bar{u}_4$ such that, for each $u \in [0, \bar{u}_5]$, $\Omega(u, r^*)$ is positively invariant for (31), and every solution initiated in $\Omega(u, r^*)$ converges to $\Omega_0(u, r^*)$. ◇

Proof: Since $N_2(z_1, 0, u) = 0$ and $\frac{\partial N_2}{\partial w}(0,0,u) = 0$, it follows that there exists a nonnegative continuous function $k(z_1, w, u)$ such that $k(0,0,u) = 0$ and that

$$\|N_2(z_1, w, u)\| \leq k(z_1, w, u)\|w\|$$

for sufficiently small $|z_1|$ and $\|w\|$. Let $P(= P^T)$ be the solution of $PA_s + A_s^T P = -I$. Then, with $V(w) = w^T P w = \|w\|^2$, we have

$$\dot{V} = -p_1\|w\|^2 + 2p_2 k(z_1, w, u)\|w\|^2 \tag{40}$$

where p_i's are some positive constants. Therefore, there exists $r^* \leq \bar{r}_5$ such that $\dot{V}(z_1, w, u)$ is negative for all (z_1, w) such that $|z_1| \leq r^*$ and $0 < \|w\| \leq r^*$. In addition, by virtue of (34), there exists a $\bar{u}_5 \leq \bar{u}_4$ such that $|\gamma(w,u)| < r^*$ for each $u \in [0, \bar{u}_5]$. Thus, for each $u \in [0, \bar{u}_5]$, $\dot{V}(z_1, w, u)$ is negative for all $(z_1, w) \in \Omega(u, r^*)$ and $\|w\| \neq 0$. Consequently, the solution starting in $\Omega(u, r^*)$ does not leave it through the boundary $\|w\| = r^*$.

On the other hand, the boundary of $\Omega(u, r^*)$ such that $z_1 = \gamma(w,u)$ is in fact the boundary $\{x : x_{(k)} = 0\}$ of $\mathbb{R}_+^n$ in x-coordinates, which can be seen from (36) and (32). Since system (8) is a positive system, no solution $x(t)$ can touch the boundary of $\mathbb{R}_+^n$. Therefore, no solution $(z_1(t), w(t))$ starting in $\Omega(u, r^*)$ can cross the boundary $z_1 = \gamma(w,u)$.

Finally, we note that $\dot{z}_1 < 0$ at $(z_1, w) = (r^*, 0)$ by Claim 6. Hence, by the continuity of solutions with respect to the initial condition [4], the solution does not leave $\Omega(u, r^*)$ (reduce r^* if necessary) through the boundary $z_1 = r^*$ (and $\|w\| \leq r^*$). In summary, the set $\Omega(u, r^*)$ is positively invariant for (31).

Now, on the set $\Omega(u, r^*)$, $\dot{V}$ of (40) is negative except where $w = 0$, which implies that every solution initiated in $\Omega(u, r^*)$ converges to $\Omega_0(u, r^*)$. ■

From Claims 6 and 7, and by the LaSalle's Invariance Theorem, we conclude that, for each $u \in [0, \bar{u}_5]$, the origin of (31) is asymptotically stable and its basin of attraction contains $\Omega(u, r^*)$. The following claim is needed to show that the NvBA-stability also holds in x-coordinates for each $u \in [0, \bar{u}_5]$.

Claim 8: There exists an $R > 0$ such that, for each $u \in [0, \bar{u}_5]$,

$$\mathbb{R}_+^n \cap B(\mathcal{X}(u), R) \quad \subset \quad \phi(\Omega(u, r^*), u).$$

◇

Proof: Recalling that $\phi(0,0,u) = \mathcal{X}(u)$ and that the manifold $z_1 = \gamma(w,u)$ corresponds to the boundary $\{x : x_{(k)} = 0\}$ of $\mathbb{R}_+^n$, the proof is trivial. ■

It is left to show that the NvBA-stability holds for all $u \in [0, \bar{u}]$. Because of Claims 6, 7, and 8, the system (8) is NvBA-stable on $u \in [0, \bar{u}_5]$. Further, it follows from (H1) and Lemma 1 that it is also NvBA-stable on $u \in [\bar{u}_5, \bar{u}]$. This completes the proof.

IV. Conclusions

The main contribution of the paper is to prove that the system (1) is NvBA-stable *w.r.t.* the positive orthant under the assumption that the equilibrium at the bifurcation point and its corresponding Jacobian satisfy (2), the Jacobian is Hurwitz except at the bifurcation point, another equilibrium that splits from or merges into the equilibrium of interest does not move into the positive set in x_k-coordinate, and the negativity of a certain function is satisfied.

The benefit of the proposed condition is its simplicity, which is called for especially in systems biology where the model complexity usually causes much difficulty. For example, NvBA-stability of the HIV infection model [7], which is a fifth order nonlinear system, can be easily verified with the help of the proposed condition.

References

[1] D.S. Bernstein and S.P. Bhat, "Nonnegativity, reducibility, and semistability of mass action kinetics," in *Proc. of Conf. on Decision and Control,* pp. 2206–2211, 1999.

[2] H.J. Chang, H. Shim, and J.H. Seo, "Control of immune response of HIV infection model by gradual reduction of drug dose," in *Proc. of Conf. on Decision and Control*, pp. 1048–1052, 2004.

[3] W. Haddad, V. Chellaboina, and E. August, "Stability and dissipativity theory for nonnegative dynamic systems: a thermodynamic framework for biology and physiological systems," in *Proc. of Conf. on Decision and Control,* pp. 442–458, 2001.

[4] H.K. Khalil, *Nonlinear Systems. 3rd Ed.* Prentice-Hall, 2002.

[5] J.D. Murray, *Mathematical Biology I. An Introduction. 3rd Ed.* Berlin: Springer-Verlag, 2002.

[6] H. Shim, H.J. Chang, and Jin H. Seo, "Non-vanishing basin of attraction with respect to a parametric variation and center manifold," in *Proc. of Conf. on Decision and Control*, pp. 2984–2989, 2004.

[7] D. Wodarz, "Helper-dependent vs. helper-independent CTL responses in HIV infection: implications for drug therapy and resistance," *J. of Theoretical Biology,* vol. 213, pp. 447–459, 2001.