

# Natural frames and interacting particles in three dimensions

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**Abstract**—Motivated by the problem of formation control for vehicles moving at unit speed in three-dimensional space, we are led to models of gyroscopically interacting particles, which require the machinery of curves and frames to describe and analyze. A Lie group formulation arises naturally, and we discuss the general problem of determining (relative) equilibria for arbitrary  $G$ -invariant controls (where  $G = SE(3)$  is a symmetry group for the control law). We then present global convergence (and non-collision) results for specific two-vehicle interaction laws in three dimensions, which lead to specific formations (i.e., relative equilibria). Generalizations of the interaction laws to  $n$  vehicles is also discussed, and simulation results presented.

## I. INTRODUCTION

This work is motivated by the problem of multi-vehicle formation (or swarm) control, e.g., for meter-scale UAVs (unmanned aerial vehicles), and builds on our earlier work on planar formation control laws [5], [6], [7] by extending the key results to the three-dimensional setting. Some objectives of our formation control laws are to avoid collisions between vehicles, maintain cohesiveness of the formation, be robust to loss of individuals, and scale favorably to large swarms.

In considering the problem of multi-vehicle formation control, there is special significance, both practically and theoretically, to modeling the vehicles as point particles moving at a common (constant) speed. In the language of mechanics, the individual particles are subject to *gyroscopic* forces; i.e., forces which alter the direction of motion of the particles, but which leave their speed (and hence their kinetic energy) unchanged. A formation control law is then a feedback law which specifies these gyroscopic forces based on the positions and directions of motion of the particles. In the planar setting, gyroscopic forces serve as steering controls [6]. For particles moving in three dimensional space, we need to introduce the notion of *framing of curves* to describe the effects of gyroscopic forces on particle motion [1], [2].

Recently, a growing literature has emerged on planar formation control for unit-speed vehicles, using tools from dynamical systems theory (including pursuit models [10] and phase-coupled oscillator models [11]), as well as graph-theoretic methods [3]. An early (discrete-time) unit-speed model for biological flocking behavior is the Vicsek model [12]. Interacting particle models similar to those described in

this paper have also found application in obstacle avoidance and boundary following [13].

## II. CURVES AND MOVING FRAMES

A single particle moving in three dimensional space traces out a trajectory  $\gamma : [0, \infty) \rightarrow \mathbb{R}^3$ , which we assume to be at least twice continuously differentiable, satisfying  $|\gamma'(s)| = 1, \forall s$ ; i.e.,  $s$  is the arc-length parameter of the curve (and the prime denotes differentiation with respect to  $s$ ). The direction of motion of the particle at  $s$  is the unit tangent vector to the trajectory,  $\mathbf{T}(s) = \gamma'(s)$ . If we further restrict the speed of particle motion to be unit speed, then the arclength parameter  $s$  is equivalent to time  $t$ , and  $\mathbf{T}(t) = \dot{\gamma}(t)$ . The gyroscopic force vector always lies in the plane perpendicular to  $\mathbf{T}$ , so to describe the effects of this force, we are compelled to introduce orthonormal unit vectors which span this *normal plane*. Taken together with  $\mathbf{T}$ , these unit vectors constitute a *framing* of the curve  $\gamma$  representing the particle trajectory.

There are different framings one can choose, as is best illustrated by examples (see figure 1). For a curve  $\gamma(s)$  which is three times continuously differentiable, and for which  $\gamma''(s) \neq 0$  for all  $s$ , the Frenet-Serret frame  $(\mathbf{T}, \mathbf{N}, \mathbf{B})$  is uniquely defined, and satisfies

$$\begin{aligned}\gamma'(s) &= \mathbf{T}(s), \\ \mathbf{T}'(s) &= \kappa(s)\mathbf{N}(s), \\ \mathbf{N}'(s) &= -\kappa(s)\mathbf{T}(s) + \tau(s)\mathbf{B}(s), \\ \mathbf{B}'(s) &= -\tau(s)\mathbf{N}(s).\end{aligned}\tag{1}$$

In (1),  $\mathbf{N}(s)$  is the unit normal vector to the curve  $\gamma$  at  $s$ , and  $\mathbf{B}(s)$  is the unit binormal vector (which completes the right-handed orthonormal frame). The curvature function  $\kappa$  and the torsion function  $\tau$  are given by expressions involving the derivatives of  $\gamma$ , and  $\gamma''(s) \neq 0$  is required for  $\tau(s)$  to be well-defined.

Although the Frenet-Serret frame for a curve (when it exists) has a special status (because it is uniquely defined by the derivatives of the curve), it is not the only choice of frame, nor is it necessarily the best choice. In particular, the requirement that  $\gamma''(s) \neq 0$  presents serious difficulties for the interaction laws we consider in this paper.

We therefore use an alternative framing of the curve  $\gamma$ , the natural Frenet frame, which is also referred to as the

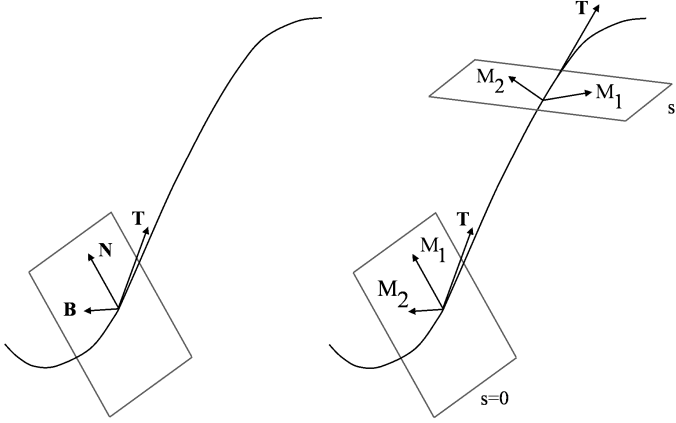


Fig. 1. The Frenet-Serret frame (left), and natural Frenet frame (right), illustrated for a three-dimensional curve.

Fermi-Walker frame or Relatively Parallel Adapted Frame (RPAF):

$$\begin{aligned}
 \gamma'(s) &= \mathbf{T}(s), \\
 \mathbf{T}'(s) &= k_1(s)\mathbf{M}_1 + k_2(s)\mathbf{M}_2, \\
 \mathbf{M}_1'(s) &= -k_1(s)\mathbf{T}(s), \\
 \mathbf{M}_2'(s) &= -k_2(s)\mathbf{T}(s).
 \end{aligned} \tag{2}$$

In (2),  $\mathbf{M}_1(s)$  and  $\mathbf{M}_2(s)$  are unit normal vectors which (along with  $\mathbf{T}(s)$ ) complete a right-handed orthonormal frame. However, there is freedom in the choice of initial conditions  $\mathbf{M}_1(0)$  and  $\mathbf{M}_2(0)$ ; once these are specified, the corresponding natural Frenet frame for a twice-continuously-differentiable curve  $\gamma$  is unique.

Both (1) and (2) can be packaged as control systems on the Lie group  $SE(3)$ , the group of rigid motions in three-dimensional space. (A modern reference for control systems on Lie groups is Jurdjevic [4].) Here we think of  $(\kappa, \tau)$  or the *natural curvatures*  $(k_1, k_2)$  as controls, which drive the evolution of the frame and the particle position  $\gamma$ .

### III. FORMATION MODEL

Figure 2 illustrates the trajectories of two vehicles moving at unit speed, and their respective natural Frenet frames. The particle (i.e., vehicle) positions are denoted by  $\mathbf{r}_1$  and  $\mathbf{r}_2$ , and the frames by  $(\mathbf{x}_1, \mathbf{y}_1, \mathbf{z}_1)$  and  $(\mathbf{x}_2, \mathbf{y}_2, \mathbf{z}_2)$ , so that

$$\begin{aligned}
 \dot{\mathbf{r}}_1 &= \mathbf{x}_1, & \dot{\mathbf{r}}_2 &= \mathbf{x}_2, \\
 \dot{\mathbf{x}}_1 &= \mathbf{y}_1 u_1 + \mathbf{z}_1 v_1, & \dot{\mathbf{x}}_2 &= \mathbf{y}_2 u_2 + \mathbf{z}_2 v_2, \\
 \dot{\mathbf{y}}_1 &= -\mathbf{x}_1 u_1, & \dot{\mathbf{y}}_2 &= -\mathbf{x}_2 u_2, \\
 \dot{\mathbf{z}}_1 &= -\mathbf{x}_1 v_1, & \dot{\mathbf{z}}_2 &= -\mathbf{x}_2 v_2.
 \end{aligned} \tag{3}$$

where the controls  $(u_1, v_1)$  and  $(u_2, v_2)$  may be feedback functions of the position and frame variables.

We consider control laws which depend only on relative vehicle positions and orientations; i.e., which depend only on the *shape* of the formation. The controls for the first vehicle are assumed to only be functions of the relative vehicle position,  $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$ , the heading direction of the second

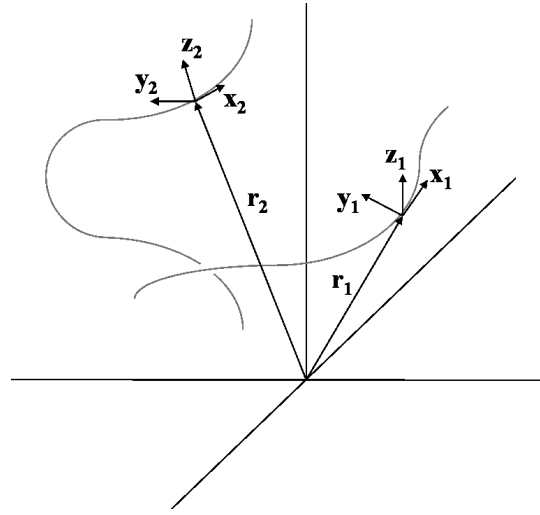


Fig. 2. Three-dimensional trajectories for two vehicles, and their respective frames.

vehicle,  $\mathbf{x}_2$ , and the frame variables for the first vehicle,  $(\mathbf{x}_1, \mathbf{y}_1, \mathbf{z}_1)$ . Thus,

$$\begin{aligned}
 u_1 &= u_1(\mathbf{r}, \mathbf{x}_1, \mathbf{y}_1, \mathbf{z}_1, \mathbf{x}_2), \\
 v_1 &= v_1(\mathbf{r}, \mathbf{x}_1, \mathbf{y}_1, \mathbf{z}_1, \mathbf{x}_2),
 \end{aligned} \tag{4}$$

and similarly,

$$\begin{aligned}
 u_2 &= u_2(\mathbf{r}, \mathbf{x}_2, \mathbf{y}_2, \mathbf{z}_2, \mathbf{x}_1), \\
 v_2 &= v_2(\mathbf{r}, \mathbf{x}_2, \mathbf{y}_2, \mathbf{z}_2, \mathbf{x}_1).
 \end{aligned} \tag{5}$$

Furthermore, because the overall motion of the first vehicle should be independent of  $\mathbf{y}_1$  and  $\mathbf{z}_1$ , we require

$$v_1(\mathbf{r}, \mathbf{x}_1, \mathbf{y}_1, \mathbf{z}_1, \mathbf{x}_2) = u_1(\mathbf{r}, \mathbf{x}_1, \mathbf{z}_1, -\mathbf{y}_1, \mathbf{x}_2), \tag{6}$$

and similarly,

$$v_2(\mathbf{r}, \mathbf{x}_2, \mathbf{y}_2, \mathbf{z}_2, \mathbf{x}_1) = u_2(\mathbf{r}, \mathbf{x}_2, \mathbf{z}_2, -\mathbf{y}_2, \mathbf{x}_1). \tag{7}$$

Finally, we require that our control laws have a discrete (relabelling) symmetry, which corresponds to the intuitive notion that both vehicles “run the same algorithm.” This implies

$$\begin{aligned}
 u_1(-\mathbf{r}, \mathbf{x}_1, \mathbf{y}_1, \mathbf{z}_1, \mathbf{x}_2) &= u_2(\mathbf{r}, \mathbf{x}_2, \mathbf{y}_2, \mathbf{z}_2, \mathbf{x}_1), \\
 v_1(-\mathbf{r}, \mathbf{x}_1, \mathbf{y}_1, \mathbf{z}_1, \mathbf{x}_2) &= v_2(\mathbf{r}, \mathbf{x}_2, \mathbf{y}_2, \mathbf{z}_2, \mathbf{x}_1).
 \end{aligned} \tag{8}$$

In this paper, the specific control laws we consider have the form

$$\begin{aligned}
 u_1 &= F(-\mathbf{r}, \mathbf{x}_1, \mathbf{y}_1, \mathbf{x}_2) - f(|\mathbf{r}|) \left( -\frac{\mathbf{r}}{|\mathbf{r}|} \cdot \mathbf{y}_1 \right), \\
 u_2 &= F(\mathbf{r}, \mathbf{x}_2, \mathbf{y}_2, \mathbf{x}_1) - f(|\mathbf{r}|) \left( \frac{\mathbf{r}}{|\mathbf{r}|} \cdot \mathbf{y}_2 \right), \\
 v_1 &= F(-\mathbf{r}, \mathbf{x}_1, \mathbf{z}_1, \mathbf{x}_2) - f(|\mathbf{r}|) \left( -\frac{\mathbf{r}}{|\mathbf{r}|} \cdot \mathbf{z}_1 \right), \\
 v_2 &= F(\mathbf{r}, \mathbf{x}_2, \mathbf{z}_2, \mathbf{x}_1) - f(|\mathbf{r}|) \left( \frac{\mathbf{r}}{|\mathbf{r}|} \cdot \mathbf{z}_2 \right),
 \end{aligned} \tag{9}$$

which is a further restricted class of laws consistent with (4) - (8). (We discuss later how  $F$  and  $f$  are chosen.)

#### IV. SHAPE VARIABLES AND EQUILIBRIA

The geometry of the problem of interacting particles moving at unit speed in the plane has been considered in earlier work [5], [6], [7]. The unit speed constraint leads to the study of gyroscopic interaction forces, and the identification of the constant kinetic energy hyper-surface with the group  $SE(2)$  of rigid motions in the plane. Formations or steady patterns of motion in the plane thus become relative equilibria for particle dynamics on  $SE(2)$  [5], [6], [7].

A key difficulty in extending the above geometric perspective to three dimensions arises from the fact that the corresponding constant kinetic energy hyper-surface cannot be identified with  $SE(3)$ , the rigid motion group in three dimensions. It is a homogeneous space  $SE(3)/SO(2)$ . However, there is considerable advantage, particularly in the multi-particle context, to formulating the dynamics in terms of interacting particles in  $SE(3)$ .

The dynamics (3) can be expressed in terms of the group variables  $g_1, g_2 \in G = SE(3)$  as a pair of left-invariant systems

$$\dot{g}_1 = g_1 \xi_1, \quad \dot{g}_2 = g_2 \xi_2, \quad (10)$$

where  $\xi_1, \xi_2 \in \mathfrak{g}$  = the Lie algebra of  $G$ . The dynamics for  $g = g_1^{-1} g_2$  are given by

$$\begin{aligned} \dot{g} &= -g_1^{-1} \dot{g}_1 g_1^{-1} g_2 + g_1^{-1} \dot{g}_2 \\ &= -g_1^{-1} g_1 \xi_1 g + g_1^{-1} g_2 \xi_2 \\ &= -\xi_1 g + g \xi_2 \\ &= g \xi, \end{aligned} \quad (11)$$

where  $\xi = \xi_2 - \text{Ad}_{g^{-1}} \xi_1 \in \mathfrak{g}$ .

Equation (11), where  $\xi$  incorporates the control inputs  $(u_1, v_1)$  and  $(u_2, v_2)$ , describes the evolution of the *relative* position and *relative* natural Frenet frame orientation of the pair of vehicles. It is thus natural to consider what equilibria of (11) exist, and then to design control laws which stabilize those equilibria. Equilibria of the shape dynamics (11) correspond to *relative equilibria* of the system (10) on  $G \times G$ .

##### A. Shape equilibria for a two-particle system on $SE(3)$

At an equilibrium shape  $g_e$  of the shape dynamics (11), we have

$$g_e \xi_2(g_e) = \xi_1(g_e) g_e. \quad (12)$$

To facilitate calculation, we define

$$\begin{aligned} g_e &= \begin{bmatrix} Q & \mathbf{b} \\ \mathbf{0} & 1 \end{bmatrix}, \quad \text{where } Q \in SO(3) \text{ and } \mathbf{b} \in \mathbb{R}^3, \\ \xi_1(g_e) &= \begin{bmatrix} \hat{\Omega}_1 & \mathbf{e}_1 \\ \mathbf{0} & 0 \end{bmatrix}, \quad \xi_2(g_e) = \begin{bmatrix} \hat{\Omega}_2 & \mathbf{e}_1 \\ \mathbf{0} & 0 \end{bmatrix}, \end{aligned} \quad (13)$$

where  $\mathbf{e}_1 = [1 \ 0 \ 0]^T$ ,

$$\Omega_1 = \begin{bmatrix} w_1 \\ -v_1 \\ u_1 \end{bmatrix}, \quad \Omega_2 = \begin{bmatrix} w_2 \\ -v_2 \\ u_2 \end{bmatrix}, \quad (14)$$

and for any 3-vector  $\Gamma = (\Gamma_1, \Gamma_2, \Gamma_3)$ ,  $\hat{\Gamma}$  is the skew-symmetric matrix defined by

$$\hat{\Gamma} = \begin{bmatrix} 0 & -\Gamma_3 & \Gamma_2 \\ \Gamma_3 & 0 & -\Gamma_1 \\ -\Gamma_2 & \Gamma_1 & 0 \end{bmatrix}. \quad (15)$$

Note that here we allow  $\Omega_1$  and  $\Omega_2$  to each have the full three degrees of freedom - not just the two corresponding to the natural curvatures. The reason for proceeding in this manner is that ultimately we recover not only the relative equilibria of (10) and (3), but also an interesting class of relative *periodic* solutions for (3).

From (12) we see that  $Q \hat{\Omega}_2 = \hat{\Omega}_1 Q$ , from which it follows that

$$\Omega_1 = Q \Omega_2. \quad (16)$$

From (12) we also obtain  $Q \mathbf{e}_1 = \hat{\Omega}_1 \mathbf{b} + \mathbf{e}_1$ . It can then be shown that  $w_1 = w_2$ , and  $u_1^2 + v_1^2 = u_2^2 + v_2^2$ .

Introducing new variables  $w$ ,  $a$ ,  $\psi_1$ , and  $\psi_2$ , we can express  $\Omega_1$  and  $\Omega_2$  as

$$\Omega_1 = \begin{bmatrix} w \\ a \sin \psi_1 \\ a \cos \psi_1 \end{bmatrix}, \quad \Omega_2 = \begin{bmatrix} w \\ a \sin \psi_2 \\ a \cos \psi_2 \end{bmatrix}. \quad (17)$$

If (for  $a^2 + w^2 \neq 0$ ) we further define

$$\cos \varphi = \frac{a}{\sqrt{a^2 + w^2}}, \quad \sin \varphi = \frac{w}{\sqrt{a^2 + w^2}}, \quad (18)$$

along with

$$\begin{aligned} R_{\psi_j} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \psi_j & -\sin \psi_j \\ 0 & \sin \psi_j & \cos \psi_j \end{bmatrix}, \quad R_\varphi = \begin{bmatrix} \cos \varphi & 0 & -\sin \varphi \\ 0 & 1 & 0 \\ \sin \varphi & 0 & \cos \varphi \end{bmatrix}, \\ R_\vartheta &= \begin{bmatrix} \cos \vartheta & -\sin \vartheta & 0 \\ \sin \vartheta & \cos \vartheta & 0 \\ 0 & 0 & 1 \end{bmatrix}, \end{aligned} \quad (19)$$

where  $\vartheta \in [0, 2\pi)$  is arbitrary, we see that (17) becomes

$$\Omega_j = \sqrt{a^2 + w^2} R_{\psi_j}^T R_\varphi^T \mathbf{e}_3, \quad j = 1, 2, \quad (20)$$

and from (16) we obtain  $Q = R_{\psi_1}^T R_\varphi^T R_\vartheta R_\varphi R_{\psi_2}$ . Note that  $R_\vartheta$ , for arbitrary  $\vartheta$ , is a rotation matrix that fixes the basis vector  $\mathbf{e}_3$ .

Defining  $\tilde{\mathbf{b}}$  by  $\mathbf{b} = R_{\psi_1}^T R_\varphi^T \tilde{\mathbf{b}}$ , after some calculation, one can show that

$$\begin{bmatrix} Q & \mathbf{b} \\ \mathbf{0} & 1 \end{bmatrix} = \begin{bmatrix} R_{\psi_1}^T & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} R_\varphi^T & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} R_\vartheta & \tilde{\mathbf{b}} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} R_\varphi & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} R_{\psi_2} & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix}, \quad (21)$$

$$\tilde{\mathbf{b}} = \begin{bmatrix} \frac{a}{a^2 + w^2} \sin \vartheta \\ \frac{a}{a^2 + w^2} (1 - \cos \vartheta) \\ \tilde{b}_3 \end{bmatrix}. \quad (22)$$

Thus,  $g_e$  can be decomposed as a product of five rigid motions (four of which represent pure rotations), and contains two free parameters -  $\vartheta$  and  $\tilde{b}_3$  - once the control vectors  $\Omega_1$  and  $\Omega_2$  are specified.

**Remark:** For purposes of interpretation of (21) in the context of particle trajectories, we may take  $R_{\psi_1} = R_{\psi_2} = I$ , so that (21) reduces to

$$\begin{bmatrix} Q & \mathbf{b} \\ \mathbf{0} & 1 \end{bmatrix} = \begin{bmatrix} R_{\varphi}^T & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} R_{\vartheta} & \tilde{\mathbf{b}} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} R_{\varphi} & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix}. \quad (23)$$

To see this, let  $\tilde{g}_e$  be defined by

$$g_e = \begin{bmatrix} R_{\psi_1}^T & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix} \tilde{g}_e \begin{bmatrix} R_{\psi_2} & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix}, \quad (24)$$

so that

$$\tilde{g}_e = \left( g_1 \begin{bmatrix} R_{\psi_1}^T & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix} \right)^{-1} \left( g_2 \begin{bmatrix} R_{\psi_2}^T & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix} \right). \quad (25)$$

Thus, if we exhibit a shape equilibrium  $\tilde{g}_e$  of the form (23), we can always write down a family of shape equilibria (24) parameterized by  $\psi_1$  and  $\psi_2$ , which differ only in the orientation of the unit normal vectors of the two frames (and are therefore indistinguishable if only the particle trajectories in  $\mathbb{R}^3$  are observed).  $\square$

**Proposition 1:** Consider the two-particle system on  $G \times G$  given by

$$\dot{g}_1 = g_1 \begin{bmatrix} \hat{\Omega}_1 & \mathbf{e}_1 \\ \mathbf{0} & 0 \end{bmatrix}, \quad \dot{g}_2 = g_2 \begin{bmatrix} \hat{\Omega}_2 & \mathbf{e}_1 \\ \mathbf{0} & 0 \end{bmatrix}, \quad (26)$$

where  $\Omega_1 = \Omega_1(g)$ ,  $\Omega_2 = \Omega_2(g)$ , and  $g = g_1^{-1}g_2$  (i.e., the controls  $\Omega_1$  and  $\Omega_2$  are arbitrary, but are  $G$ -invariant). Then there is a corresponding reduced system on  $G$  (the ‘‘shape space’’) given by

$$\dot{g} = - \begin{bmatrix} \hat{\Omega}_1 & \mathbf{e}_1 \\ \mathbf{0} & 0 \end{bmatrix} g + g \begin{bmatrix} \hat{\Omega}_2 & \mathbf{e}_1 \\ \mathbf{0} & 0 \end{bmatrix}, \quad (27)$$

(c.f. (11)) whose equilibria are given by (12). Solutions of (12), with (14), require that (17) hold.

- (1) If  $w = a = 0$ , then  $Q$  satisfies  $Q\mathbf{e}_1 = \mathbf{e}_1$ , and  $\mathbf{b}$  is arbitrary. Then  $Q$  yields one free parameter, and  $\mathbf{b}$  yields three free parameters.
- (2) If  $w^2 + a^2 \neq 0$ , then  $(Q, \mathbf{b})$  satisfies (21), with  $R_{\psi_1}$ ,  $R_{\psi_2}$ ,  $R_{\varphi}$ , and  $R_{\vartheta}$  given by (19) and with  $\tilde{\mathbf{b}}$  given by (22). The angle  $\varphi$  is related to  $w$  and  $a$  through (18), and  $\vartheta$  and  $\tilde{b}_3$  are free parameters.

The resulting  $(Q, \mathbf{b})$  then describe the shape equilibria (i.e., the relative equilibria) for (26).

**Proof:** Follows from the calculations outlined above.  $\square$

**Proposition 2:** Consider (26) as the underlying dynamics for the evolution of two particle trajectories in  $\mathbb{R}^3$  and their corresponding natural Frenet frames. Then relative equilibria  $(Q, \mathbf{b})$  for (26) correspond to the following steady-state formations of the two particles in  $\mathbb{R}^3$ :

- (1) If  $w = a = 0$ , then the two particles move in the same direction with arbitrary relative positions.
- (2) If  $w = 0$  but  $a \neq 0$ , then the particles move on circular orbits with a common radius, in planes perpendicular to a common axis.

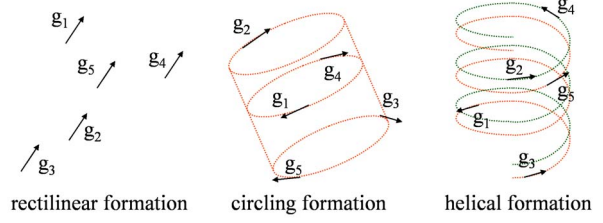


Fig. 3. Rectilinear, circling, and helical formations, illustrated for five particles. The arrows represent the unit tangent vectors to the particle trajectories.

- (3) If  $w \neq 0$  but  $a = 0$ , then the particles move in the same direction on collinear trajectories.
- (4) If  $w \neq 0$  and  $a \neq 0$ , then the particles follow circular helices with the same radius, pitch, axis, and axial direction of motion.

**Proof:** Omitted due to space constraints, but follows from **Proposition 1**, along with the **Remark** and calculations outlined above.  $\square$

### B. Shape equilibria for an $n$ -particle system on $SE(3)$

Our definition of the shape variable  $g$  for the two-particle problem extends naturally to the  $n$ -particle problem (under the assumption that the  $n$ -particle interaction law has  $G$  as a symmetry group). We define  $\tilde{g}_j = g_1^{-1}g_j$ ,  $j = 2, \dots, n$ , where  $g_1, g_2, \dots, g_n$  are the group variables (each representing one of the particles), and  $\tilde{g}_2, \tilde{g}_3, \dots, \tilde{g}_n$  are shape variables. (This is analogous to the approach taken in the planar problem, where the corresponding group is  $SE(2)$  [5], [6], [7].)

**Proposition 3:** Consider

$$\dot{g}_1 = g_1 \begin{bmatrix} \hat{\Omega}_1 & \mathbf{e}_1 \\ \mathbf{0} & 0 \end{bmatrix}, \quad \dots, \quad \dot{g}_n = g_n \begin{bmatrix} \hat{\Omega}_n & \mathbf{e}_1 \\ \mathbf{0} & 0 \end{bmatrix}, \quad (28)$$

where  $\Omega_1, \dots, \Omega_n$  are  $G$ -invariant controls, as the underlying dynamics for the evolution of  $n$  particle trajectories in  $\mathbb{R}^3$ . Then relative equilibria  $(Q_2, \mathbf{b}_2), \dots, (Q_n, \mathbf{b}_n)$  for (28) correspond to the following steady-state formations of the  $n$  particles in  $\mathbb{R}^3$  (see figure 3):

- (1) If  $w = a = 0$ , then the  $n$  particles all move in the same direction with arbitrary relative positions.
- (2) If  $w = 0$  but  $a \neq 0$ , then the particles move on circular orbits with a common radius, in planes perpendicular to a common axis.
- (3) If  $w \neq 0$  but  $a = 0$ , then the particles move in the same direction on collinear trajectories.
- (4) If  $w \neq 0$  and  $a \neq 0$ , then the particles follow circular helices with the same radius, pitch, axis, and axial direction of motion.

**Proof:** Omitted due to space constraints, but analogous to the proof of **Proposition 2**.  $\square$

**Remark:** When  $w \neq 0$  at a relative equilibrium for our model (26) of particles evolving in  $G \times G$ , the corresponding natural curvatures in (3) are then in fact periodic functions of time (or arc-length parameter).  $\square$

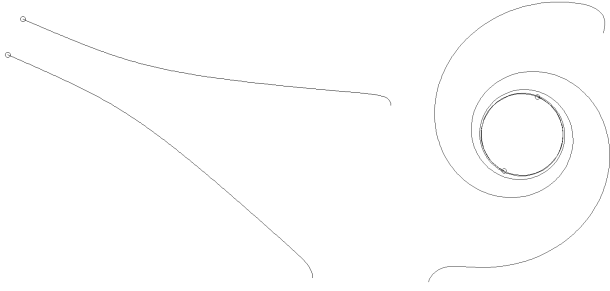


Fig. 4. Convergence to a rectilinear formation (left), and to a circling formation (right). The trajectories, which are three-dimensional, are viewed perpendicular to the plane of the equilibrium formation.

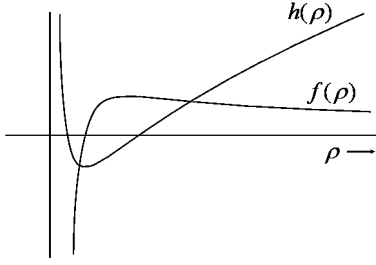


Fig. 5. An example of suitable functions  $f(\cdot)$  and  $h(\cdot)$  satisfying conditions (A1) and (A2) [6].

## V. RECTILINEAR FORMATION LAW

The two types of formations for which we consider specific stabilizing control laws (for a pair of vehicles) are rectilinear formations (in which both vehicles head in the same direction) and circling formations (in which both vehicles follow the same circular orbit). Figure 4 shows simulations which converge to these two types of formations. For concreteness, we use the variables  $(\mathbf{r}_1, \mathbf{x}_1, \mathbf{y}_1)$  and  $(\mathbf{r}_2, \mathbf{x}_2, \mathbf{y}_2)$ , rather than the group variables  $g_1$  and  $g_2$ .

Consider the Lyapunov function candidate

$$V_{rect} = -\ln(1 + \mathbf{x}_2 \cdot \mathbf{x}_1) + h(|\mathbf{r}|), \quad (29)$$

where we assume that

- (A1)  $dh/d\rho = f(\rho)$ , where  $f(\rho)$  is a Lipschitz continuous function on  $(0, \infty)$ , so that  $h(\rho)$  is continuously differentiable on  $(0, \infty)$ ;
- (A2)  $\lim_{\rho \rightarrow 0} h(\rho) = \infty$ ,  $\lim_{\rho \rightarrow \infty} h(\rho) = \infty$ , and  $\exists \tilde{\rho}$  such that  $h(\tilde{\rho}) = 0$ .

Figure 5 shows an example of functions  $f(\cdot)$  and  $h(\cdot)$  satisfying conditions (A1) and (A2). An example of a suitable function  $f(\cdot)$  is

$$f(|\mathbf{r}|) = \alpha \left[ 1 - (r_o/|\mathbf{r}|)^2 \right], \quad (30)$$

where  $\alpha$  and  $r_o$  are positive constants. Observe that the term  $-\ln(1 + \mathbf{x}_2 \cdot \mathbf{x}_1)$  in (29) penalizes heading-direction misalignment between the two vehicles, and the term  $h(|\mathbf{r}|)$  penalizes vehicle separations which are too large or too small.

We consider  $F$  to be of the form

$$F(\mathbf{r}, \mathbf{x}_2, \mathbf{y}_2, \mathbf{x}_1) = \mp \eta \left( \frac{\mathbf{r}}{|\mathbf{r}|} \cdot \mathbf{x}_2 \right) \left( \frac{\mathbf{r}}{|\mathbf{r}|} \cdot \mathbf{y}_2 \right) + \mu \mathbf{x}_1 \cdot \mathbf{y}_2, \quad (31)$$

where  $\mu$  and  $\eta$  satisfy

- (A3)  $\mu(\rho)$  and  $\eta(\rho)$  are Lipschitz continuous on  $(0, \infty)$ ;
- (A4)  $\mu(|\mathbf{r}|) > \frac{1}{2}\eta(|\mathbf{r}|) > 0, \forall |\mathbf{r}| \geq 0$ .

(For simplicity,  $\mu$  and  $\eta$  can be taken to be constants, rather than functions of  $|\mathbf{r}|$ .)

The control law given by (9) with (31) is the natural generalization to three dimensions of the planar two-vehicle rectilinear law analyzed in [5], [6], [7]. As in the planar setting, we can interpret the terms in (9) that involve  $f$  as steering the vehicles apart to avoid collisions (or steering them together into formation if they are too far apart). The terms in (9) that involve  $F$  serve to align the vehicle headings (with respect to each other and with respect to the baseline between them).

**Proposition 4:** Consider the system  $(\mathbf{r}, \mathbf{x}_1, \mathbf{x}_2)$  evolving on  $\mathbb{R}^3 \times S^2 \times S^2$ , where  $S^2$  is the two-sphere, according to (3), (9), and (31). In addition, assume (A1), (A2), (A3), and (A4). Define the set

$$\Lambda = \left\{ (\mathbf{r}, \mathbf{x}_1, \mathbf{x}_2) \mid \mathbf{x}_2 \cdot \mathbf{x}_1 \neq -1 \text{ and } |\mathbf{r}| > 0 \right\}. \quad (32)$$

Then any trajectory starting in  $\Lambda$  converges to the set

$$M = \left( \left\{ (\mathbf{r}, \mathbf{x}_1, \mathbf{x}_2) \mid \mathbf{x}_1 = \mathbf{x}_2, \mathbf{r} \cdot \mathbf{x}_1 = 0, f(|\mathbf{r}|) = 0 \right\} \cup \left\{ (\mathbf{r}, \mathbf{x}_1, \mathbf{x}_2) \mid \mathbf{x}_1 = \mathbf{x}_2 = \pm \frac{\mathbf{r}}{|\mathbf{r}|} \right\} \right) \cap \Lambda, \quad (33)$$

which is the the set of equilibria for the  $(\mathbf{r}, \mathbf{x}_1, \mathbf{x}_2)$ -dynamics contained in  $\Lambda$ .

**Proof:** Uses LaSalle's Invariance Principle [9]. See [8] for a detailed proof.  $\square$

**Remark:** If  $f$  is given by (30), then  $f(|\mathbf{r}|) = 0$  is equivalent to  $|\mathbf{r}| = r_o$ . Thus, the set of equilibria consists of formations with both vehicles heading in the same direction: either the motion is perpendicular to the baseline between the vehicles with an intervehicle separation equal to  $r_o$ , or else both vehicles follow the same straight-line trajectory with one leading the other by an arbitrary distance. The stability of these equilibria depend on the choice of parameters, and can be further analyzed using linearization.

## VI. CIRCLING FORMATION LAW

Consider the Lyapunov function candidate

$$V_{circ} = -\ln \left[ 1 - \mathbf{x}_2 \cdot \mathbf{x}_1 + 2 \left( \frac{\mathbf{r}}{|\mathbf{r}|} \cdot \mathbf{x}_2 \right) \left( \frac{\mathbf{r}}{|\mathbf{r}|} \cdot \mathbf{x}_1 \right) \right] + h(|\mathbf{r}|), \quad (34)$$

where we assume

- (A1')  $dh/d\rho = f(\rho) - 2/\rho$ , where  $f(\rho)$  is a Lipschitz continuous function on  $(0, \infty)$ , so that  $h(\rho)$  is continuously differentiable on  $(0, \infty)$ ;

and (A2). It can be shown that the argument of the natural log function in (34) is nonnegative, and the function  $f$  given by (30) can be used here, as well. The term  $h(|\mathbf{r}|)$  in (34) penalizes vehicle separations which are too large or too

small. The natural-log term in (34) involves the relative headings of the vehicles, as well as the relative orientations of the headings with respect to the baseline between the vehicles.

In place of (31), we use

$$F(\mathbf{r}, \mathbf{x}_2, \mathbf{y}_2, \mathbf{x}_1) = \pm \eta \left( \frac{\mathbf{r}}{|\mathbf{r}|} \cdot \mathbf{x}_2 \right) \left( \frac{\mathbf{r}}{|\mathbf{r}|} \cdot \mathbf{y}_2 \right) + \mu \left[ -\mathbf{x}_1 \cdot \mathbf{y}_2 + 2 \left( \frac{\mathbf{r}}{|\mathbf{r}|} \cdot \mathbf{x}_1 \right) \left( \frac{\mathbf{r}}{|\mathbf{r}|} \cdot \mathbf{y}_2 \right) \right], \quad (35)$$

where we assume (A3) and (A4).

**Proposition 5:** Consider the system  $(\mathbf{r}, \mathbf{x}_1, \mathbf{x}_2)$  evolving on  $\mathbb{R}^3 \times S^2 \times S^2$ , according to (3), (9), and (35). In addition, assume (A1'), (A2), (A3), and (A4). Define the set

$$\Lambda' = \left\{ (\mathbf{r}, \mathbf{x}_1, \mathbf{x}_2) \mid 1 - \mathbf{x}_2 \cdot \mathbf{x}_1 + 2 \left( \frac{\mathbf{r}}{|\mathbf{r}|} \cdot \mathbf{x}_2 \right) \left( \frac{\mathbf{r}}{|\mathbf{r}|} \cdot \mathbf{x}_1 \right) \neq 0 \text{ and } |\mathbf{r}| > 0 \right\}. \quad (36)$$

Then any trajectory starting in  $\Lambda'$  converges to the set

$$\tilde{M}' = \left( \left\{ (\mathbf{r}, \mathbf{x}_1, \mathbf{x}_2) \mid \mathbf{x}_1 = -\mathbf{x}_2, \mathbf{r} \cdot \mathbf{x}_1 = 0, f(|\mathbf{r}|) = \frac{2}{|\mathbf{r}|} \right\} \cup \left\{ (\mathbf{r}, \mathbf{x}_1, \mathbf{x}_2) \mid \mathbf{x}_1 = \mathbf{x}_2 = \pm \frac{\mathbf{r}}{|\mathbf{r}|} \right\} \right) \cap \Lambda'. \quad (37)$$

Note that elements of  $\tilde{M}'$  with  $\mathbf{x}_1 = -\mathbf{x}_2$  correspond to the two vehicles following the same circular orbit, separated by the diameter of the orbit, which is prescribed by the function  $f$ . Elements of  $\tilde{M}'$  with  $\mathbf{x}_1 = \mathbf{x}_2$  correspond to rectilinear formations in which one vehicle leads the other by an arbitrary distance.

**Proof:** Similar in approach to the proof of **Proposition 4** (see [8]).  $\square$

## VII. MULTI-VEHICLE FORMATIONS

One way to generalize the two-vehicle laws discussed above to  $n$  vehicles is to use an average of the pairwise interaction terms used for the two-vehicle problem [5], [6], [7], [8]. Figures 6 and 7 show simulation results for multi-vehicle interactions of this type. Their analysis is a topic of ongoing research.

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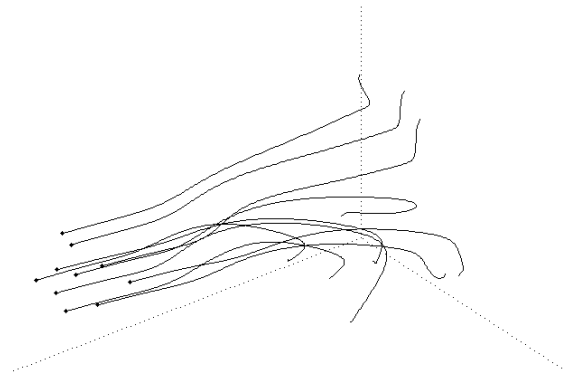


Fig. 6. Simulation results for ten vehicles using a generalization of the two-vehicle rectilinear formation control law (9) with (31) and (30).

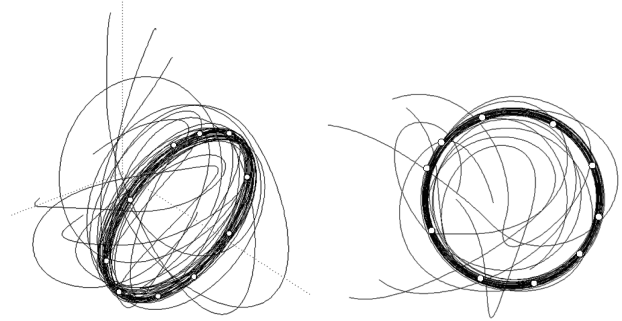


Fig. 7. Simulation results for ten vehicles using a generalization of the two-vehicle circling formation control law (9) with (35) and (30). (The same simulation results are viewed from two different angles.)

## REFERENCES

- [1] R.L. Bishop, "There is more than one way to frame a curve," *The American Mathematical Monthly*, **82**(3), 246-251, 1975.
- [2] A. Calini, "Recent developments in integrable curve dynamics," In *Geometric Approaches to Differential Equations*, Lecture Notes of the Australian Math. Soc., **15**, 56-99, Cambridge Univ. Press, 2000.
- [3] A. Jadbabaie, J. Lin, and A.S. Morse, "Coordination of groups of mobile autonomous agents using nearest neighbor rules," *IEEE Trans. Automatic Control*, **48**(6), 988-1001, 2003 (also in *Proc. IEEE Conf. Decision and Control*, **3**, 2953-2958, 2002).
- [4] V. Jurdjevic, *Geometric Control Theory*, Cambridge: Cambridge Univ. Press, 1997.
- [5] E.W. Justh and P.S. Krishnaprasad, "A Simple Control Law for UAV Formation Flying," Institute for Systems Research Technical Report TR 2002-38 (see <http://www.isr.umd.edu>), 2002.
- [6] E.W. Justh and P.S. Krishnaprasad, "Equilibria and steering laws for planar formations," *Systems and Control Lett.*, **52**, 25-38, 2004.
- [7] E.W. Justh and P.S. Krishnaprasad, "Steering laws and continuum models for planar formations," *Proc. IEEE Conf. Decision and Control*, 3609-3614, 2003.
- [8] E.W. Justh and P.S. Krishnaprasad, "Natural frames and interacting particles in three dimensions," arXiv:math.OA/0503390, <http://www.arxiv.org/abs/math.OA/0503390>, 2005.
- [9] H. Khalil. *Nonlinear Systems*. New York: Macmillan Publishing Co., 1992.
- [10] J.A. Marshall, M.E. Broucke, and B.A. Francis, "Formations of Vehicles in Cyclic Pursuit," *IEEE Trans. Automatic Control*, **49**(11), 1963-1974, 2004.
- [11] R. Sepulchre, D. Paley, and N. Leonard, "Collective motion and oscillator synchronization," *Lecture Notes in Control and Information Sciences*, **309**, "Cooperative Control," eds. V.J. Kumar, N.E. Leonard, and A.S. Morse, pp. 189-205, Springer-Verlag, 2004.
- [12] T. Vicsek, A. Czirók, E.B.-Jacob, I. Cohen, and O. Shochet, "Novel type of phase transitions in a system of self-driven particles," *Phys. Rev. Lett.*, **75**, 1226-1229, 1995.
- [13] F. Zhang, E.W. Justh, and P.S. Krishnaprasad, "Boundary-following using gyroscopic control," *Proc. IEEE Conf. Decision and Control*, 5204-5209, 2004.