Proceedings of the
44th IEEE Conference on Decision and Control, and
the European Control Conference 2005
Seville, Spain, December 12-15, 2005

A very non-smooth maximum principle with state constraints

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Abstract— **We present a version of the Pontryagin Maximum Principle with state-space constraints and very weak technical hypotheses. The result does not require the time-varying vector fields corresponding to the various control values to be continuously differentiable, Lipschitz, or even continuous with respect to the state, since all that is needed is that they be "co-integrably bounded integrally continuous." This includes the case of vector fields that are continuous with respect to the state, as well as large classes of discontinuous vector fields, containing, for example, rich sets of single-valued selections for almost semicontinuous differential inclusions. Uniqueness of trajectories is not required, since our methods deal directly with multivalued maps. The reference vector field and reference Lagrangian are only required to be "differentiable" along the reference trajectory in a very weak sense, namely, that of possessing suitable "variational generators." The conclusion yields finitely additive measures, as in earlier work by other authors, and a Hamiltonian maximization inequality valid also at the jump times of the adjoint covector.**

I. Introduction

In a series of previous papers (cf. [1], [2], [3], [4]), we have developed a "primal" approach to the non-smooth Pontryagin Maximum Principle, based on generalized differentials, flows, and general variations. The method used is essentially the one of classical proofs of the Maximum Principle such as that of Pontryagin and his coauthors, based on the construction of packets of needle variations, but with a refinement of the "topological argument," and with concepts of differential more general than the classical one, and usually set-valued.

In this note we describe the result of applying this approach to optimal control problems with state-space constraints. The paper is organized as follows. In §II we introduce some of our notations, and in particular briefly recall the simple but not widely known basic concepts about finitely additive vector-valued measures on an interval. In §III we review the notion of Generalized Differential Quotient (GDQ), define the two types of variational generators that will occur in the maximum principle, and state theorems asserting that various classical generalized derivatives—such as classical differentials, Clarke generalized Jacobians, Michel-Penot subdifferentials, and, for functions defining state-space constraints, the object often referred to as $\partial_x^{>} g$ in the literature—are special cases of our variational generators. In §IV we discuss the discontinuous vector fields studied in detail in [5]. Finally, in §V we state the main theorem.

Research supported in part by NSF Grant DMS01-03901

Remark 1.1: For lack of space, we will omit the proofs, which will be given in a much longer self-contained paper. We point out, however, that the proof strategy is quite straightforward, except for one crucial technical detail. We make variations, which as usual are families of controls depending on a finite-dimensional parameter $\vec{\varepsilon}$; we associate to each variation an "augmented terminal point map" $\mathcal{E}$; we differentiate $\mathcal{E}$ at $\vec{\varepsilon} = 0$ in the sense of GDQ theory; we use a set separation theorem to infer the existence of an adjoint covector for the variation and, finally, we use a compactness argument to get an adjoint covector that works for all variations. The "technical detail" referred to above is that, instead of dealing with the usual packets of needle variations, we have had to introduce "chattering variations," which approximately convexify the set of velocities at each point. ∎

II. Notational preliminaries and background

The abbreviations "FDRLS" and "FDNRLS" will stand for "finite-dimensional real linear space," and "finite-dimensional normed real linear space," respectively. If X and Y are real linear spaces, then $Lin(X,Y)$ will denote the set of all linear maps from X to Y. We use $X^\dagger$ to denote $Lin(X,\mathbb{R})$, i.e., the dual space of X.

Single- and set-valued maps. A ***set-valued map*** is a triple $F = (A,B,G)$ such that A and B are sets and G is a subset of $A\times B$. If $F = (A,B,G)$ is a set-valued map, we say that *F is a set-valued map from A to B*. In that case, the sets A, B, G are the *source*, *target*, and *graph* of F, respectively, and we write $A = \mathrm{So}(F)$, $B = \mathrm{Ta}(F)$, $G = \mathrm{Gr}(F)$. If $x \in \mathrm{So}(F)$, we write $F(x) = \{y : (x,y) \in \mathrm{Gr}(F)\}$. The set $\mathrm{Do}(F) = \{x \in \mathrm{So}(F) : F(x) \neq \emptyset\}$ is the ***domain*** of F. If A, B are sets, we use $SVM(A,B)$ to denote the set of all set-valued maps from A to B, and write $F : A \mapsto\!\!\!\rightarrow B$ to indicate that $F \in SVM(A,B)$. A ***ppd map from** A **to** B* (where "ppd" stands for "possibly partially defined') is an $F \in SVM(A,B)$ such that $F(x)$ has cardinality zero or one for every $x \in A$. We write $F : A \hookrightarrow B$ to indicate that F is a ppd map from A to B. If $F : A \mapsto\!\!\!\rightarrow B$, and $C \subseteq A$, then the ***restriction*** of F to C is the set-valued map $F\lceil C$ defined by $F\lceil C \stackrel{\mathrm{def}}{=} (C,B,\mathrm{Gr}(F)\cap(C\times B))$.

Epimaps and constraint indicator maps. If $f : S \hookrightarrow \mathbb{R}$ is a ppd function, then the ***epimap*** of f is the set-valued map $\check{f} : S \mapsto\!\!\!\rightarrow \mathbb{R}$ whose graph is the epigraph of f, so that

$\check{f}(s) = \{f(s)+v : v \in \mathbb{R}, v \geq 0\}$ whenever $s \in \mathrm{Do}(f)$, and $\check{f}(s) = \emptyset$ if $s \in S \backslash \mathrm{Do}(f)$. The ***constraint indicator map*** of f is the set-valued map $\chi_f^{co} : S \mapsto\!\!\!\to \mathbb{R}$ such that $\chi_f^{co}(s) = \emptyset$ if $f(s) \leq 0$ or $s \in S \backslash \mathrm{Do}(f)$, and $\chi_f^{co}(x) = [0, +\infty[$ if $f(x) > 0$.

Cones. A ***cone*** in a FDRLS X is a nonempty subset C of X such that $r \cdot c \in C$ whenever $c \in C, r \in \mathbb{R}$ and $r \geq 0$. The ***polar*** of a cone $C \subseteq X$ is the closed convex cone $C^\dagger = \{\lambda \in X^\dagger : \lambda(c) \leq 0 \text{ for all } c \in C\}$.

If X is a FDRLS, $S \subseteq X$, and $x \in S$, a ***Boltyanskii approximating cone to*** S ***at*** x is a convex cone C in X such that there exist an $n \in \mathbb{Z}_+$, a closed convex cone D in $\mathbb{R}^n$, a neighborhood U of 0 in $\mathbb{R}^n$, a continuous map $F : U \cap D \mapsto S$, and a linear map $L : \mathbb{R}^n \mapsto X$, such that $F(h) = x + L \cdot h + o(\|h\|)$ as $h \to 0$ via values in D, and $C = L \cdot D$. A ***limiting Boltyanskii approximating cone to*** S ***at*** x is a closed convex cone C such that C is the closure of an increasing union $\bigcup_{j=1}^{\infty} C_j$ such that each C_j is a Boltyanskii approximating cone to S at x.

Tubes. If X is a FDNRLS, $a, b \in \mathbb{R}$, $a \leq b$, $\xi \in C^0([a,b], X)$ and $\delta > 0$, we use $\mathcal{T}^X(\xi, \delta)$ to denote the δ***-tube about*** ξ ***in*** X, defined by

$$\mathcal{T}^X(\xi, \delta) \stackrel{\mathrm{def}}{=} \{(x,t) : x \in X, a \leq t \leq b, \|x - \xi(t)\| \leq \delta\}. \qquad (1)$$

Finitely additive measures. If $a, b \in \mathbb{R}$, $a < b$, and X is a FDNRLS, we use $\mathcal{P}c([a,b];X)$ to denote the set of all piecewise constant X-valued functions on $[a,b]$, so that $f \in \mathcal{P}c([a,b];X)$ iff $f : [a,b] \mapsto X$ and there exists a finite partition $\mathcal{P}$ of $[a,b]$ into intervals such that f is constant on each $I \in \mathcal{P}$. We let $\overline{\mathcal{P}c}([a,b];X)$ denote the set of all uniform limits of members of $\mathcal{P}c([a,b];X)$, so $\overline{\mathcal{P}c}([a,b];X)$ is a Banach space, endowed with the sup norm. Furthemore, $\overline{\mathcal{P}c}([a,b];X)$ is exactly the space of all $f : [a,b] \mapsto X$ such that the left limit $f(t-) = \lim_{s \to t, s<t} f(s)$ exists for all $t \in]a,b]$, and the right limit $f(t+) = \lim_{s \to t, s>t} f(s)$ exists for all $t \in [a,b[$.

We define $\overline{\mathcal{P}c}_0([a,b];X)$ to be the set of all $f \in \overline{\mathcal{P}c}([a,b];X)$ that vanish on the complement of a countable (i.e., finite or countably infinite) set. (Then $\overline{\mathcal{P}c}_0([a,b];X)$ is the closure in $\overline{\mathcal{P}c}([a,b];X)$ of the space $\mathcal{P}c_0([a,b];X)$ of all $f \in \mathcal{P}c([a,b];X)$ such that f vanishes on the complement of a finite set.)

We let $pc([a,b];X)$ be the quotient space $\overline{\mathcal{P}c}([a,b];X)/\overline{\mathcal{P}c}_0([a,b];X)$. Then every equivalence class $F \in pc([a,b];X)$ has a unique left-continuous member F_-, and a unique right-continuous member F_+, and of course $F_- \equiv F_+$ on the complement of a countable set. So $pc([a,b];X)$ can be identified with the set of all pairs (f_-, f_+) of X-valued functions on $[a,b]$ such that f_- is left-continuous, f_+ is right-continuous, and $f_- \equiv f_+$ on the complement of a countable set.

If X is a FDNRLS, an ***additive*** X***-valued interval function of bounded variation*** on $[a,b]$ is a member of the dual space $pc([a,b];X^\dagger)^\dagger \stackrel{\mathrm{def}}{=} bvadd([a,b];X)$. A member μ of $bvadd([a,b];X)$ gives rise to a set function $\hat{\mu} : \mathcal{I}([a,b]) \mapsto X$ (where $\mathcal{I}([a,b])$ is the set of all subintervals of $[a,b]$), defined by $\langle \hat{\mu}(I), y \rangle = \mu(\chi_I^y)$ for $y \in X^\dagger$, where $\chi_I^y(t) = 0$ if $t \notin I$ and $\chi_I^y(t) = y$ if $t \in I$. We then associate to μ its ***cumulative distribution*** cd_μ, defined by $cd_\mu(t) = -\hat{\mu}([t,b])$ for $t \in [a,b]$. Then cd_μ belongs to the space $bvfn^{0;b}([a,b];X)$ of all functions $\varphi : [a,b] \mapsto X$ that are of bounded variation and such that $\varphi(b) = 0$. (We call φ ***of bounded variation*** if $\|\varphi\|_{bv} < \infty$, where $\|\varphi\|_{bv}$ is the supremum of all the sums $\sum_{j=1}^m \|\varphi(t_j) - \varphi(s_j)\|$, for all $m \in \mathbb{N}$ and $\{s_j\}_{j=1}^m$, $\{t_j\}_{j=1}^m$ such that $a \leq s_1 \leq t_1 \leq s_2 \leq t_2 \leq \cdots \leq s_m \leq t_m \leq b$.) The map $bvadd([a,b];X) \ni \mu \mapsto cd_\mu \in bvfn^{0;b}([a,b];X)$ is a bijection. The dual Banach space norm $\|\mu\|$ of a $\mu \in bvadd([a,b];X)$ coincides with $\|cd_\mu\|_{bv}$.

A $\mu \in bvadd([a,b];X)$ is a ***left*** (resp.***right***) ***delta function*** if there exist an $x \in X$ and a $t \in]a,b]$ (resp. a $t \in [a,b[$) such that $\mu(F) = \langle F(t-), x\rangle$ (resp. $\mu(F) = \langle F(t+), x\rangle$) for all $F \in pc([a,b],X)$. We call μ ***left-atomic*** (resp. ***right-atomic***) if it is the sum of a convergent series of left (resp. right) delta functions.

A $\mu \in bvadd([a,b];X)$ is ***continuous*** if the function cd_μ is continuous. Every $\mu \in bvadd([a,b];X)$ has a unique decomposition into the sum of a continous part μ_{co}, a left-atomic part $\mu_{at,-}$ and a right-atomic part $\mu_{at,+}$. (This resembles the usual decomposition of a countably additive measure into the sum of a continuous part and an atomic part. The only difference is that in the finitely additive setting there are left and right atoms rather than just atoms.)

If Y is a FDNRLS, a ***bounded*** Y***-valued measurable pair on*** $[a,b]$ is a pair (γ_-, γ_+) of bounded Borel measurable functions from $[a,b]$ to Y such that $\gamma_- \equiv \gamma_+$ outside a countable set. If X, Y, Z are FDNRLSs, $Y \times X \ni (y,x) \mapsto \langle y, x\rangle \in Z$ is a bilinear map, $\mu \in bvadd([a,b],X)$, nand (γ_-, γ_+) is a bounded Y-valued measurable pair on $[a,b]$, then the product measure $\gamma \cdot \mu$ is a member of $bvadd([a,b],Z)$ defined by multiplying the continuous part μ_{co} by γ_- or γ_+, the left-atomic part by γ_-, and the right-atomic part by γ_+. In particular, the product $\gamma \cdot \mu$ is a well defined member of $bvadd([a,b],X)$ whenever $\mu \in bvadd([a,b],\mathbb{R})$ and γ is a bounded X-valued measurable pair on $[a,b]$.

Finally, we need to study the solutions of an "adjoint" Cauchy problem represented formally as

$$dy(t) = -y(t) \cdot L(t) \cdot dt + d\mu(t), \quad y(b) = \bar{y}, \qquad (2)$$

where $\mu \in bvadd([a,b],X^\dagger)$ and $L \in L^1([a,b], Lin(X,X))$. We do this by rewriting our Cauchy problem as the integral equation $y(t) - V(t) = \int_t^b y(s) \cdot L(s) \cdot ds$, where $V = cd_\mu$. This is easily seen to have a unique solution π, given by

$$\pi(t) = \bar{y} \cdot M_L(b,t) - \int_{[t,b]} d\mu(s) \cdot M_L(s,t), \qquad (3)$$

where $M_L : [a,b] \times [a,b] \mapsto Lin(X,X)$ is the fundamental solution of $\dot{M} = M \cdot L$, characterized by the identity $M_L(\tau, t) = \mathbb{I}_X + \int_t^\tau L(r) \cdot M_L(r,t)\, dr$.

III. Generalized differential quotients (GDQs) and variational generators

Cellina continuosly approximable maps. If K, Y are metric spaces and K is compact, then $SVM_{comp}(K,Y)$ will denote the subset of $SVM(K,Y)$ whose members are the set-valued maps from K to Y that have a compact graph. We say that a sequence $\{F_j\}_{j\in\mathbb{N}}$ of members of $SVM_{comp}(K,Y)$ ***inward graph-converges*** to an $F \in SVM_{comp}(K,Y)$—and write $F_j \xrightarrow{\text{igr}} F$—if for every open subset Ω of $K \times Y$ such that $\text{Gr}(F) \subseteq \Omega$ there exists a $j_\Omega \in \mathbb{N}$ such that $\text{Gr}(F_j) \subseteq \Omega$ whenever $j \geq j_\Omega$.

If X and Y are metric spaces and $F : X \mapsto\!\!\!\!\to Y$, then F is ***compactly graphed*** if for every compact subset K of X the restriction $F\lceil K$ of F to K has a compact graph.

Definition 3.1: Assume that X, Y are metric spaces. A ***Cellina continuously approximable set-valued map*** (abbr. "CCA map") from X to Y is a compactly graphed set-valued map $F : X \mapsto\!\!\!\!\to Y$ such that

- *for every compact subset K of X, $F\lceil K$ is a limit—in the sense of inward graph-convergence—of a sequence of continuous single-valued maps from K to Y.*

We use $\text{CCA}(X;Y)$ to denote the set of all CCA set-valued maps from X to Y. ■

GDQs. The precise definition of "generalized differential quotient" is as follows. Let us assume that (i) X and Y are FDNRLSs, (ii) $F : X \mapsto\!\!\!\!\to Y$ is a set-valued map, (iii) $\bar{x}_* \in X$, (iv) $\bar{y}_* \in Y$, and (v) $S \subseteq X$. We say that Λ is a ***generalized differential quotient*** (abbreviated "GDQ") ***of*** F ***at*** $(\bar{x}_*, \bar{y}_*)$ ***in the direction of*** S, and write $\Lambda \in GDQ(F; \bar{x}_*, \bar{y}_*; S)$, if (I) Λ is a compact subset of $Lin(X,Y)$, (II) for every neighborhood $\hat{\Lambda}$ of Λ in $Lin(X,Y)$ there exist U, G such that (II.1) U is a neighborhood of $\bar{x}_*$ in X; (II.2) $\bar{y}_* + G(x)\cdot(x - \bar{x}_*) \subseteq F(x)$ for every $x \in U \cap S$; and (III.3) G is a CCA set-valued map from $U \cap S$ to $\hat{\Lambda}$.

Variational generators. It will be convenient to define two types of "variational generators." We will assume that

(VGA) *X and Y are FDNRLSs, $a, b \in \mathbb{R}$, $a \leq b$ $\xi_* \in C^0([a,b]; X)$, $\sigma_* : [a,b] \hookrightarrow Y$, $S \subseteq X \times \mathbb{R}$, and $F : X \times \mathbb{R} \mapsto\!\!\!\!\to Y$.*

We recall that the ***distance*** $\text{dist}(S, S')$ between two subsets S, S' of a metric space M with distance function d_M is defined by $\text{dist}(S,S') = \inf\{d_M(s,s') : s \in S,\ s' \in S'\}$.

Definition 3.2: Assume that (VGA) holds. An L^1 ***fixed-time GDQ variational generator of*** F ***along*** (ξ_*, σ_*) ***in the direction of*** S is a set-valued map $\Lambda : [a,b] \mapsto\!\!\!\!\to Lin(X,Y)$ such that,

- there exist a positive number $\bar{\delta}$ and a family $\{\kappa^\delta\}_{0<\delta\leq\bar{\delta}}$ of measurable functions $\kappa^\delta : [a,b] \mapsto [0,+\infty]$ such that $\lim_{\delta\downarrow 0}\int_a^b \kappa^\delta(t)\,dt = 0$ and, in addition,
$$\text{dist}(\sigma_*(t) + \Lambda(t)\cdot h, F(\xi_*(t)+h,t)) \leq \delta\kappa^\delta(t) \quad (4)$$
if $h \in X$, $t \in [a,b]$, $(\xi_*(t)+h,t) \in S$, and $\|h\| \leq \delta$. ■

We will use the expression $VG^{L^1,ft}_{GDQ}(F;\xi_*,\sigma_*;S)$ to denote the set of all L^1 fixed-time GDQ variational generators of F along (ξ_*,σ_*) in the direction of S.

Definition 3.3: Assume that (VGA) holds. A ***pointwise robust GDQ variational generator of*** F ***along*** (ξ_*, σ_*) ***in the direction of*** S is a set-valued map $\Lambda : [a,b] \mapsto\!\!\!\!\to Lin(X,Y)$ such that,

- there exist positive numbers $\bar{\delta}$, $\bar{s}$, and a family $\{\kappa^{\delta,s}\}_{0<\delta\leq\bar{\delta}, 0<s\leq\bar{s}}$ of functions $\kappa^{\delta,s} : [a,b] \mapsto [0,+\infty]$ such that
$$\lim_{\delta\downarrow 0, s\downarrow 0} \kappa^{\delta,s}(t) = 0 \quad \text{for every } t \in [a,b] \quad (5)$$
and, in addition,
$$\begin{aligned} &\text{dist}(\sigma_*(t+s) + \Lambda(t)\cdot h, F(\xi_*(t+s)+h, t+s)) \\ \leq\ & \delta\kappa^{\delta,s}(t) \quad (6) \end{aligned}$$
whenever $h \in X$, $\|h\| \leq \delta$, $t \in [a,b]$, $t+s \in [a,b]$, and $(\xi_*(t+s)+h, t+s) \in S$. ■

We write $VG^{pw,rob}_{GDQ}(F;\xi_*,\sigma_*;S)$ to denote the set of all pointwise robust GDQ variational generators of F along (ξ_*,σ_*) in the direction of S.

Examples of variational generators. We now state four propositions giving important examples of variational generators, omitting the proofs. In their statements, we use $\partial_x f(q,t)$ to denote the Clarke generalized Jacobian at $x = q$ of the map $x \mapsto f(x,t)$, and $\partial_x^o f(q,t)$—if f has scalar values—to denote the Michel-Penot subdifferential of $x \mapsto f(x,t)$ at $x = q$.

We recall that the notions of ***epimap*** and ***constraint indicator map*** were defined in §II.

If $(S,\mathcal{A})$ is a measurable space (that is, S is a set and $\mathcal{A}$ is a σ-algebra of subsets of S), X is a FDNRLS, and $\Lambda : S \mapsto\!\!\!\!\to X$, then Λ is ***measurable*** if $\{s \in S : \Lambda(s) \cap \Omega \neq \emptyset\} \in \mathcal{A}$ for every open subset Ω of X. If $(S,\mathcal{A},\mu)$ is a nonnegative-measure space (that is, $(S,\mathcal{A})$ is a measurable space and $\mu : \mathcal{A} \mapsto [0,+\infty]$ is a nonnegative measure) then Λ is ***integrably bounded*** if there exists a μ-integrable function $k : S \mapsto [0,+\infty]$ such that $\Lambda(s) \subseteq \{x \in X : \|x\| \leq k(s)\}$ for μ-almost all $s \in S$.

In the first three propositions, we will assume that

(#) *X and Y are FDNRLSs, $f : X \times \mathbb{R} \hookrightarrow Y$, $\xi_* \in C^0([a,b],X)$, $\bar{\delta} > 0$, $\mathcal{T}^X(\xi_*,\bar{\delta}) \subseteq \text{Do}(f)$, and each partial map $t \mapsto f(x,t)$ is measurable.*

Proposition 3.4: Assume that (#) holds and each partial map $x \mapsto f(x,t)$ is Lipschitz with a Lipschitz constant $C(t)$ such that the function $C(\cdot)$ is integrable. Let $\Lambda(t) = \partial_x f(\xi_*(t),t)$, and let $\sigma_*(t) = f(\xi_*(t),t)$. Then Λ is an integrably bounded measurable set-valued function with a.e. nonempty compact convex values, and Λ is an L^1 fixed-time variational GDQ of f along (ξ_*,σ_*) in the direction of $X \times [a,b]$. ■

Proposition 3.5: Assume that (#) holds, $Y = \mathbb{R}$, and each partial map $x \mapsto f(x,t)$ is Lipschitz with a Lipschitz constant $C(t)$ such that the function $C(\cdot)$ is integrable. Let

$\Lambda(t) = \partial_x^o f(\xi_*(t), t)$, and let $\sigma_*(t) = f(\xi_*(t), t)$. Let F be the epimap of f. Then Λ is an integrably bounded measurable set-valued function with a.e. nonempty compact convex values, and Λ is an L^1 fixed-time variational GDQ of F along (ξ_*, σ_*) in the direction of $X \times [a, b]$. ■

Proposition 3.6: Assume that (#) holds and each partial map $x \mapsto f(x, t)$ is continuous. Also, assume that (i) for each t the map $x \mapsto f(x, t)$ is differentiable at $\xi_*(t)$, and (ii) there exists a nonnegative integrable function $[a, b] \ni t \mapsto C(t) \in \mathbb{R}$ such that $\|f(\xi_*(t) + h, t) - f(\xi_*(t), t)\| \leq C(t)\|h\|$ whenever $t \in [a, b]$, $h \in X$, and $\|h\| \leq \bar{\delta}$. Let $\Lambda(t) = \{D_x f(\xi_*(t), t)\}$, and let $\sigma_*(t) = f(\xi_*(t), t)$. Then Λ is an L^1 fixed-time variational GDQ of f along (ξ_*, σ_*) in the direction of $X \times [a, b]$. ■

Proposition 3.7: Assume that X is a FDNRLS, $\xi_* \in C^0([a, b], X)$, $\bar{\delta} > 0$, $T = \mathcal{T}^X(\xi_*, \bar{\delta})$, and $g : T \mapsto \mathbb{R}$ is a single-valued everywhere defined function. Assume that (a) $g(\xi_*(t), t) \leq 0$ for all $t \in [a, b]$, (b) each partial map $t \mapsto g(x, t)$ is upper semicontinuous on $\{t \in \mathbb{R} : (x, t) \in T\}$, (c) each partial map $x \mapsto g(x, t)$ is Lipschitz on $\{x \in X : \|x - \xi_*(t)\| \leq \bar{\delta}\}$, with a Lipschitz constant C which is independent of t for $t \in [a, b]$. Let $Av_g = \{(x, t) \in T : g(x, t) > 0\}$ so $Av_g = \mathrm{Do}(\chi_g^{co})$.

For each $t \in [a, b]$, let $\Lambda(t) = \partial_x^> g(\xi_*(t), t)$, where

(*) $\partial_x^> g(\bar{x}, t)$ is the convex hull of the set of all limits $\lim_{j\to\infty} \omega_j$, for all sequences $\{(x_j, t_j, \omega_j)\}_{j\in\mathbb{N}}$ such that $\lim_{j\to\infty}(x_j, t_j) \to (\bar{x}, t)$ and, for all j,

(*.i) (1) $(x_j, t_j) \in Av_g$, (2) the function $x \mapsto g(x, t_j)$ is differentiable at x_j, and (3) $\omega_j = \nabla_x g(x_j, t_j)$,

Let $K = \{t \in [a, b] : (\xi_*(t), t) \in \mathrm{Clos}\, Av_g\}$. Let $\sigma_*(t) = 0$ for $t \in [a, b]$. Then (I) Λ is an upper semicontinuous set-valued map with compact convex values, (II) K is compact, (III) $K = \{t \in [a, b] : \Lambda(t) \neq \emptyset\}$, and (IV) Λ is a pointwise robust GDQ variational generator of χ_g^{co} along (ξ_*, σ_*) in the direction of Av_g. ■

IV. Discontinuous vector fields

Integral boundedness and integral continuity. If X is a FDRLS, $\mathcal{BL}^e(X, \mathbb{R})$ will denote the σ-algebra of subsets of $X \times \mathbb{R}$ generated by (a) all the products $B \times L$, with B a Borel subset of X and L a Lebesgue-measurable subset of $\mathbb{R}$, together with (b) all the subsets S of $X \times \mathbb{R}$ such that the set $\{t \in \mathbb{R} : (\exists x \in X)((x, t) \in S)\}$ is Lebesgue-null.

Let X, Y be FDNRLSs, let f be a ppd map from $X \times \mathbb{R}$ to Y, and let K be a compact subset of $X \times \mathbb{R}$.

1. We say that f is ***essentially Borel×Lebesgue measurable on*** K, or $\mathcal{BL}^e(X, \mathbb{R})$*-measurable on* K, if $K \subseteq \mathrm{Do}(f)$ and $f^{-1}(U) \cap K \in \mathcal{BL}^e(X, \mathbb{R})$ for all open subsets U of Y.
2. An ***integrable bound*** for f on K is an integrable funtion $\mathbb{R} \ni t \to \varphi(t) \in [0, +\infty]$ such that $\|f(x, t)\| \leq \varphi(t)$ for all $(x, t) \in K$.
3. If $Y = \mathbb{R}$, an ***integrable lower bound*** for f on K is an integrable funtion $\mathbb{R} \ni t \to \varphi(t) \in [0, +\infty]$ such that $f(x, t) \geq -\varphi(t)$ for all $(x, t) \in K$.
4. We call f ***integrably bounded*** (IB)—resp. ***integrably lower bounded*** (ILB)—on K if f is $\mathcal{BL}^e(X, \mathbb{R})$-measurable on K and there exists an integrable bound—resp. an integrable lower bound—for f on K.
5. We write $\mathcal{IB}(X \times \mathbb{R}, K; Y)$, $\mathcal{ILB}(X \times \mathbb{R}, K; \mathbb{R})$ to denote, respectively, the sets of (i) all ppd maps from $X \times \mathbb{R}$ to Y that are IB on K, and (ii) all ppd maps from $X \times \mathbb{R}$ to $\mathbb{R}$ that are ILB on K. ■

If $S \subseteq X \times \mathbb{R}$, we write $\mathrm{Arc}\,(S)$ to denote the set of all ξ such that, for some nonempty compact interval I_ξ, (i) $\xi \in C^0(\,I_\xi\,;\,X\,)$, and (ii) $(\xi(t), t) \in S$ for all $t \in I_\xi$. If $k : \mathbb{R} \mapsto [0, +\infty]$ is a locally integrable function, then $\mathrm{Arc}_{\,k}(S)$ denotes the set of all $\xi \in \mathrm{Arc}\,(S)$ such that ξ is absolutely continuous and $\|\dot{\xi}(t)\| \leq k(t)$ for a. e. $t \in I_\xi$.

The sets $\mathrm{Arc}\,(S)$ are metric spaces, with the distance $d(\xi, \xi')$ of two members $\xi : [a, b] \mapsto X$, $\xi' : [a', b'] \mapsto X$ of $\mathrm{Arc}\,(S)$ defined by

$$d(\xi, \xi') = |a - a'| + |b - b'| + \sup\{\|\tilde{\xi}(t) - \tilde{\xi}'(t)\| : t \in \mathbb{R}\}$$

where, for any continuous map $\gamma : [\alpha, \beta] \mapsto X$, $\tilde{\gamma}$ is the extension of γ to $\mathbb{R}$ which is identically equal to $\gamma(\alpha)$ on $]-\infty, \alpha]$ and to $\gamma(\beta)$ on $[\beta, +\infty[$.

If X, Y are FDNRLSs, $K \subseteq X \times \mathbb{R}$ is compact, and $f \in \mathcal{IB}(X \times \mathbb{R}, K; Y)$, then we define a real-valued ***integral map*** $\mathcal{I}_{f,K} : \mathrm{Arc}\,(K) \mapsto \mathbb{R}$, by letting $\mathcal{I}_{f,K}(\xi) = \int_{\mathrm{Do}(\xi)} f(\xi(s), s)\, ds$ for every $\xi \in \mathrm{Arc}\,(K)$. If $\mathcal{S} \subseteq \mathrm{Arc}\,(K)$, we call f ***integrally continuous*** (abbr. IC) ***on*** $\mathcal{S}$ if $\mathcal{I}_{f,K} \lceil \mathcal{S}$ is continuous. If $f \in \mathcal{ILB}(X \times \mathbb{R}, K; \mathbb{R})$, then $\mathcal{I}_{f,K}$ is still well defined as a map into $\mathbb{R} \cup \{+\infty\}$, and we call f ***integrally lower semicontinuous*** (abbr. ILSC) ***on*** $\mathcal{S}$ if $\mathcal{I}_{f,K} \lceil \mathcal{S}$ is lower semicontinuous.

We will be particularly interested in maps f that, for some integrable function k, are both integrably bounded with integral bound k and integrally continuous on $\mathrm{Arc}_{\,k}(K)$.

Definition 4.1: If X, Y are FDNRLSs, K is a compact subset of $X \times \mathbb{R}$, and $f : X \times \mathbb{R} \hookrightarrow Y$, we call f ***co-IBIC*** ("co-integrably bounded and integrally continuous") ***on*** K if $f \in \mathcal{IB}(X \times \mathbb{R}, K; Y)$ and there exists an integrable bound $k : \mathbb{R} \mapsto [0, +\infty]$ for f on K such that f is IC on $\mathrm{Arc}_{\,k}(K)$. If $f : X \times \mathbb{R} \hookrightarrow \mathbb{R}$, we call f ***co-ILBILSC*** ("co-integrably bounded and integrally lower semicontinuous") ***on*** K if $f \in \mathcal{ILB}(X \times \mathbb{R}, K; \mathbb{R})$ and there exists an integrable lower bound $k : \mathbb{R} \mapsto [0, +\infty]$ for f on K such that f is ILSC on $\mathrm{Arc}_{\,k}(K)$. ■

Points of approximate continuity. Suppose that X and Y are FDNRLSs, $f : X \times \mathbb{R} \hookrightarrow Y$, and $(\bar{x}_*, \bar{t}_*) \in X \times \mathbb{R}$. A ***modulus of approximate continuity*** (abbr. MAC) for f near $(\bar{x}_*, \bar{t}_*)$ is a function $]\,0, +\infty\,[\, \times \mathbb{R} \ni (\beta, r) \mapsto \psi(\beta, r) \in\,]\,0, +\infty\,]$ such that

(MAC.1) *the function* $\mathbb{R} \ni r \mapsto \psi(\beta, r) \in\,]\,0, +\infty\,]$ *is measurable for each* $\beta \in\,]\,0, +\infty\,[$,

(MAC.2) $\lim_{(\beta,\rho)\to(0,0),\beta>0,\rho>0} \frac{1}{\rho}\int_{-\rho}^{\rho} \psi(\beta, r)\, dr = 0$, (MAC.3) *there exist positive numbers* β_*, ρ_*, *such that*

(MAC.3.a) $f(x,t)$ *is defined whenever* $\|x-\bar{x}_*\| \leq \beta_*$ *and* $|t-\bar{t}_*| \leq \rho_*$,

(MAC.3.b) *whenever* $\beta \in \mathbb{R}$, $x \in X$, $t \in \mathbb{R}$, $|t-\bar{t}_*| \leq \rho_*$, *and* $\|x-\bar{x}_*\| \leq \beta \leq \beta_*$, *it follows that* $\|f(x,t)-f(\bar{x}_*,\bar{t}_*)\| \leq \psi(\beta, t-\bar{t}_*)$.

Definition 4.2: A ***point of approximate continuity*** (abbr. PAC) for f is a point $(\bar{x}_*, \bar{t}_*) \in X \times \mathbb{R}$ such that there exists a MAC for f near $(\bar{x}_*, \bar{t}_*)$. ■

An important example of a class of maps with many points of approximate continuity is given by the following corollary of the well-known Scorza-Dragoni theorem.

Proposition 4.3: Suppose X, Y are FDNRLSs, Ω is open in X, $a, b \in \mathbb{R}$, $a < b$, and $f : \Omega \times [a,b] \mapsto Y$ is such that (a) the partial map $[a,b] \ni t \mapsto f(x,t) \in Y$ is measurable for every $x \in \Omega$, (b) the partial map $\Omega \ni x \mapsto f(x,t) \in Y$ is continuous for every $t \in [a,b]$, and (c) there exists an integrable function $[a,b] \ni t \mapsto k(t) \in [0,+\infty]$ such that the bound $\|f(x,t)\| \leq k(t)$ holds whenever $(x,t) \in \Omega \times [a,b]$. Then there exists a subset G of $[a,b]$ such that $\mathrm{meas}([a,b] \backslash G) = 0$, having the property that every $(\bar{x}_*, \bar{t}) \in \Omega \times G$ is a PAC of f. ■

Another important example of maps with many PACs is given by the following result, proved in [5].

Proposition 4.4: Suppose X, Y are FDNRLSs, $a, b \in \mathbb{R}$, $a < b$, and $F : X \times [a,b] \mapsto\!\!\!\to Y$ is an almost lower semicontinuous set-valued map with closed nonempty values such that for every compact subset K of X the function $[a,b] \ni t \mapsto \sup\{\min\{\|y\| : y \in F(x,t)\} : x \in K\}$ is integrable. Then there exists a subset G of $[a,b]$ such that $\mathrm{meas}([a,b] \backslash G) = 0$, having the property that, whenever $x_* \in X$, $t_* \in G$, $v_* \in F(x_*, t_*)$, and $K \subseteq X$ is compact, there exists a map $K \times [a,b] \ni (x,t) \mapsto f(x,t) \in F(x,t)$ which is co-IBIC on $K \times [a,b]$ and such that (x_*, t_*) is a PAC of f and $f(x_*, t_*) = v_*$. ■

V. THE MAXIMUM PRINCIPLE

We consider a *fixed time-interval optimal control problem with state space constraints*, of the form

$$\text{minimize} \quad \varphi(\xi(b)) + \int_a^b f_0(\xi(t), \eta(t), t)\, dt$$

subject to the conditions: (i) $\xi(\cdot) \in W^{1,1}([a,b], X)$, (ii) $\dot{\xi}(t) = f(\xi(t), \eta(t), t)$ for a.e. t, (iii) $\xi(a) = \bar{x}_*$, (iv) $g_i(\xi(t), t) \leq 0$ for $t \in [a,b]$, $i = 1, \ldots, m$, (v) $\xi(b) \in S$, (vi) $h_j(\xi(b)) = 0$ for $j = 1, \ldots, \tilde{m}$, (vii) $\eta(t) \in U$ for all $t \in [a,b]$, (viii) $\eta(\cdot) \in \mathcal{U}$

and a *reference trajectory-control pair* (ξ_*, η_*).

The technical hypotheses. We assume that the data 14-tuple $\mathcal{D} = (X, m, \tilde{m}, U, a, b, \varphi, f_0, f, \bar{x}_*, \mathbf{g}, \mathbf{h}, S, \mathcal{U})$ satisfies:

(H1) X *is a FDNRLS,* $m \in \mathbb{Z}_+$, $\tilde{m} \in \mathbb{Z}_+$*;* U *is a set,* $a, b \in \mathbb{R}$, $a < b$, $\bar{x}_* \in X$ *and* $S \subseteq X$;

(H2) f_0 *is a ppd function from* $X \times U \times \mathbb{R}$ *to* $\mathbb{R}$;

(H3) f *is a ppd function from* $X \times U \times \mathbb{R}$ *to* X;

(H4) $\mathbf{g} = (g_1, \ldots, g_m)$ *is an* m*-tuple of ppd functions from* $X \times \mathbb{R}$ *to* $\mathbb{R}$;

(H5) $\mathbf{h} = (h_1, \ldots, h_{\tilde{m}})$ *is an* $\tilde{m}$*-tuple of ppd functions from* X *to* $\mathbb{R}$;

(H6) φ *is a ppd function from* X *to* $\mathbb{R}$;

(H7) $\mathcal{U}$ *is a set of ppd functions from* $\mathbb{R}$ *to* U *such that the domain of every* $\eta \in \mathcal{U}$ *is a nonempty compact interval.*

Given such a $\mathcal{D}$, a ***controller*** is a ppd function $\eta : \mathbb{R} \hookrightarrow U$ whose domain is a nonempty compact interval. (Hence (H7) says that $\mathcal{U}$ is a set of controllers.) An ***admissible controller*** is a member of $\mathcal{U}$. If $\alpha, \beta \in \mathbb{R}$ and $\alpha \leq \beta$, then we use $W^{1,1}([\alpha,\beta], X)$ to denote the space of all absolutely continuous maps $\xi : [\alpha,\beta] \mapsto X$. A ***trajectory*** for a controller $\eta : [\alpha,\beta] \mapsto U$ is a map $\xi \in W^{1,1}([\alpha,\beta], X)$ such that, for almost every $t \in [\alpha,\beta]$, $(\xi(t), \eta(t), t)$ belongs to $\mathrm{Do}(f)$ and $\dot{\xi}(t) = f(\xi(t), \eta(t), t)$. A ***trajectory-control pair*** (abbr. TCP) is a pair (ξ, η) such that η is a controller and ξ is a trajectory for η. The ***domain*** of a TCP (ξ, η) is the domain of η, which is, by definition, the same as domain of ξ. A TCP (ξ, η) is ***admissible*** if $\eta \in \mathcal{U}$.

A TCP (ξ, η) with domain $[\alpha, \beta]$ is ***cost-admissible*** if (i) (ξ, η) is admissible, (ii) the function $[\alpha,\beta] \ni t \mapsto f_0(\xi(t), \eta(t), t)$ is a. e. defined and measurable, (iii) $\int_\alpha^\beta \min\Big(0, f_0(\xi(t), \eta(t), t)\Big)\, dt > -\infty$, and (iv) $\xi(\beta) \in \mathrm{Do}(\varphi)$.

It follows that if (ξ, η) is cost-admissible then the number $J(\xi, \eta) = \varphi(\xi(\beta)) + \int_\alpha^\beta f_0(\xi(t), \eta(t), t)\, dt$ (called the ***cost*** of (ξ, η)) is well defined and belongs to $]-\infty, +\infty]$.

A TCP (ξ, η) with domain $[\alpha, \beta]$ is ***constraint-admissible*** if it satisfies all our state space constraints, that is, if

(CA1) $\xi(\alpha) = \bar{x}_*$,

(CA2) $(\xi(t), t) \in \mathrm{Do}(g_i)$ and $g_i(\xi(t), t) \leq 0$ *for all* $t \in [\alpha, \beta]$, *and all* $i \in \{1, \ldots, m\}$,

(CA3) $\xi(\beta) \in S \cap \Big(\cap_{j=1}^{\tilde{m}} \mathrm{Do}(h_j)\Big)$

(CA4) $h_j(\xi(\beta)) = 0$ *for* $j = 1, \ldots, \tilde{m}$.

We use $ADM(\mathcal{D})$ and $ADM_{[a,b]}(\mathcal{D})$ to denote the sets of (i) all cost-admissible, constraint-admissible TCPs (ξ, η), and (ii) all $(\xi, \eta) \in ADM(\mathcal{D})$ whose domain is $[a,b]$.

The hypothesis on the reference TCP (ξ_*, η_*) is that it is a cost-minimizer in $ADM_{[a,b]}(\mathcal{D})$. In other words,

(H8) $(\xi_*, \eta_*) \in ADM_{[a,b]}(\mathcal{D})$, $J(\xi_*, \eta_*) < +\infty$, *and* $J(\xi_*, \eta_*) \leq J(\xi, \eta)$ *for all* $(\xi, \eta) \in ADM_{[a,b]}(\mathcal{D})$.

The "cost-augmented dynamics" $\mathbf{f} : X \times U \times \mathbb{R} \hookrightarrow \mathbb{R} \times X$ and the "epi-augmented dynamics" $\check{\mathbf{f}} : X \times U \times \mathbb{R} \mapsto\!\!\!\to \mathbb{R} \times X$ are defined by taking $\mathrm{Do}(\mathbf{f}) = \mathrm{Do}(\check{\mathbf{f}}) = \mathrm{Do}(f_0) \cap \mathrm{Do}(f)$, and then letting, for $z = (x, u, t) \in X \times U \times \mathbb{R}$,

$$\mathbf{f}(z) = (f_0(z), f(z)) \quad \text{and} \quad \check{\mathbf{f}}(z) = [f_0(z), +\infty[\times \{f(z)\}\,.$$

We will also use the constraint indicator maps $\chi_{g_i}^{co} : X \times \mathbb{R} \mapsto\!\!\!\to \mathbb{R}$, for $i = 1, \ldots, m$, and the epimap $\check{\varphi} : X \mapsto\!\!\!\to \mathbb{R}$. (These two notions were defined in §II.)

For $i \in \{1, \ldots, m\}$, we let

$$\sigma_*^{\mathbf{f}}(t) = \mathbf{f}(\xi_*(t), \eta_*(t), t) \ \text{ and } \ \sigma_*^{g_i}(t) = 0 \quad \text{if } t \in [a,b],$$

$$Av_{g_i} = \{(x,t) \in X \times [a,b] : g_i(x,t) > 0\}\,,$$

(so the Av_{g_i} are the "sets to be avoided"). We then define K_i to be the set of all $t \in [a,b]$ such that $(\xi_*(t),t)$ belongs to the closure of Av_{g_i}. Then K_i is obviously compact..

We now make technical hypotheses on $\mathcal{D}$, ξ_*, η_*, and five new objects called $\Lambda^{\mathbf{f}}$, $\Lambda^{\mathbf{g}}$, $\Lambda^{\mathbf{h}}$, Λ^{φ}, and C. To state these hypotheses, we let $\mathcal{U}_{c;[a,b]}$ denote the set of all constant U-valued functions defined on $[a,b]$, and define $\mathcal{U}_{c;[a,b];*} = \mathcal{U}_{c;[a,b]} \cup \{\eta_*\}$.

The technical hypotheses are as follows.

(H9) *For each $\eta \in \mathcal{U}_{c;[a,b];*}$. there exist a positive number δ_η such that*

(H9.a) *$\mathbf{f}(x,\eta(t),t)$ is defined whenever (x,t) belongs to $\mathcal{T}^X(\xi_*,\delta_\eta)$,*

(H9.b) *the map $\mathcal{T}^X(\xi_*,\delta_\eta) \ni (x,t) \mapsto f(x,\eta(t),t)$ is co-IBIC on $\mathcal{T}^X(\xi_*,\delta_\eta)$, and the function $\mathcal{T}^X(\xi_*,\delta_\eta) \ni (x,t) \mapsto f_0(x,\eta(t),t) \in \mathbb{R}$ is co-ILBILSC on $\mathcal{T}^X(\xi_*,\delta_\eta)$,*

(H10) *The number δ_{η_*} can be chosen so that (i) each function g_i is defined on $\mathcal{T}^X(\xi_*,\delta_{\eta_*})$, and (ii) for each $i \in \{1,\ldots,m\}$, $t \in [a,b]$, the set $\{x \in X : g_i(x,t) > 0, \|x - \xi_*(t)\| \le \delta_{\eta_*}\}$ is relatively open in the ball $\{x \in X : \|x - \xi_*(t)\| \le \delta_{\eta_*}\}$,*

(H11) *$\Lambda^{\mathbf{f}}$ is a measurable integrably bounded set-valued map from $[a,b]$ to $X^\dagger \times Lin(X,X)$ with compact convex values such that $\Lambda^{\mathbf{f}} \in VG^{L^1,ft}_{GDQ}(\check{\mathbf{f}};[a,b];\xi_*,\sigma^{\mathbf{f}}_*;X\times\mathbb{R})$,*

(H12) *$\Lambda^{\mathbf{g}}$ is an m-tuple $(\Lambda^{g_1},\ldots,\Lambda^{g_m})$ such that, for each $i \in \{1,\ldots,m\}$, Λ^{g_i} is an upper semicontinous set-valued map from $[a,b]$ to $X^\dagger$ with compact convex values, such that $\Lambda^{g_i} \in VG^{pw,rob}_{GDQ}(\chi^{co}_{g_i};\xi_*,\sigma^{g_i}_*,Av_{g_i})$,*

(H13) $\Lambda^{\mathbf{h}} \in GDQ\Big(\mathbf{h};\Big(\xi_*(b),\mathbf{h}(\xi_*(b))\Big);X\Big)$,

(H14) $\Lambda^{\varphi} \in GDQ\Big(\check{\varphi};\Big(\xi_*(b),\varphi(\xi_*(b))\Big);X\Big)$.

(H15) *C is a limiting Boltyanskii approximating cone of S at $\xi_*(b)$.*

Our last hypothesis requires the concept of an "equal-time interval-variational neighborhood" (abbr. ETIVN) of a controller η. We say that a set $\mathcal{V}$ of controllers is an ETIVN of a controller η if

- *for every $n \in \mathbb{Z}_+$ and every n-tuple $\mathbf{u} = (u_1,\ldots,u_n)$ of members of U, there exists a positive number $\varepsilon = \varepsilon(n,\mathbf{u})$ such that whenever $\eta' : \mathrm{Do}(\eta) \mapsto U$ is a map obtained from η by selecting an n-tuple $\mathbf{I} = (I_1,\ldots,I_n)$ of pairwise disjoint subintervals of $\mathrm{Do}(\eta)$ such that $\sum_{j=1}^n \mathrm{meas}(I_j) \le \varepsilon$, and substituting the constant value u_j for the value $\eta(t)$ for every $t \in I_j$, $j = 1,\ldots,n$, it follows that $\eta \in \mathcal{U}$.*

We will then assume

(H16) *The class $\mathcal{U}$ is an equal-time interval-variational neighborhood of η_*.*

We are now ready to state our version of the maximum principle. First, we define the ***Hamiltonian*** to be the function $H_\alpha : X \times U \times X^\dagger \times \mathbb{R} \hookrightarrow \mathbb{R}$ (depending on $\alpha \in \mathbb{R}$) given by $H_\alpha(x,u,p,t) = p \cdot f(x,u,t) - \alpha f_0(x,u,t)$.

Theorem 5.1: Assume that (H1-16) hold, and let $I = \{i \in \{1,\ldots,m\} : K_i \neq \emptyset\}$. Then there exist

1. a covector $\bar{\pi} \in X^\dagger$, a nonnegative real number π_0, and an $\tilde{m}$-tuple $\boldsymbol{\lambda} = (\lambda_1,\ldots,\lambda_{\tilde{m}})$ of real numbers,
2. a measurable map $[a,b] \ni t \mapsto (L_0(t),L(t)) \in \Lambda(t)$,
3. measurable pairs (cf. §II) $\gamma^i = (\gamma^i_-,\gamma^i_+)$ of selections of the set-valued maps Λ^{g_i}, defined on K_i, for $i \in I$,
4. a member $L^{\mathbf{h}} = (L^{h_1},\ldots,L^{h_{\tilde{m}}}) \in (X^\dagger)^{\tilde{m}}$ of $\Lambda^{\mathbf{h}}$,
5. a member L^φ of Λ^φ,
6. a family $\{\nu_i\}_{i\in I}$ of nonnegative additive measures $\nu_i \in bvadd([a,b];\mathbb{R})$ such that $\mathrm{support}(\nu_i) \subseteq K_i$,

such that the following three conditions are satisfied:

I. ***Hamiltonian maximization:*** the inequality $H_{\pi_0}(\xi_*(\bar{t}),\eta_*(\bar{t}),\pi(\bar{t})) \ge H_{\pi_0}(\xi_*(\bar{t}),u,\pi(\bar{t}))$ holds whenever $u \in U$, $\bar{t} \in [a,b]$ are such that $(\xi_*(\bar{t}),\bar{t})$ is a point of approximate continuity of both augmented vector fields $(x,t) \mapsto \mathbf{f}(x,u,t)$ and $(x,t) \mapsto \mathbf{f}(x,\eta_*(t),t)$,

II. ***transversality:*** $-\bar{\pi} \in C^\dagger$,

III. ***nontriviality:*** $\|\bar{\pi}\| + \pi_0 + \sum_{j=1}^{\tilde{m}} |\lambda_j| + \sum_{i\in I} \|\nu_i\| > 0$,

where $\pi : [a,b] \mapsto X^\dagger$ is the unique solution of

$$\begin{cases} d\pi(t) = (-\pi(t)\cdot L(t) + \pi_0 L_0(t))dt + \sum_{i\in I} \gamma^i(t) d\nu_i(t) \\ \pi(b) = \bar{\pi} - \sum_{j=1}^{\tilde{m}} \lambda_j L^{\mathbf{h}}_j - \pi_0 L^\varphi \end{cases}$$

Remark 5.2: The adjoint covector π can also be expressed using (3). The result is the formula

$$\pi(t) = \pi(b) - \int_t^b M_L(s,t)^\dagger \Big(\pi_0 L_0(s)\,ds + \sum_{i\in I} d(\gamma^i \cdot \nu_i)(s)\Big),$$

where $\pi(b) = \bar{\pi} - \sum_{j=1}^{\tilde{m}} \lambda_j L^{\mathbf{h}}_j - \pi_0 L^\varphi$, and M_L is the fundamental solution of $\dot{M} = L \cdot M$. ■

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