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Decentralized Stabilization of Linear Systems via Limited Capacity Communication Networks

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Abstract— **The paper addresses feedback stabilization of discrete-time linear systems with multiple sensors, controllers, and actuators. The information is transmitted between them via a limited capacity deterministic communication network with arbitrary topology, which may be dynamically changed by nodes authorized to switch channels. The transferred messages may be delayed, corrupted or even lost; they may interfere and collide with each other. It is shown that the criterion for stabilizability is given by the network rate region, which answers to the question: how much information can be reliably transmitted from one set of points to another set of points. The system is stabiliazible if and only if a certain vector characterizing its rate of instability in the open-loop lies in the interior of the rate domain. In general, the rate domain of the primal network is not relevant here and a certain extension of the network should be employed. Furthermore, the relevant rate domain corresponds to a special subclass of decoders. In some cases, all decoders can be considered and this domain equals the standard one.**

I. Introduction

Current interest in networked control systems has motivated advances in determining the minimum data rates required for stabilization (see [5], [7], [9]–[11], [13], [14], [16] and the literature therein). The focus in these works was on the simplest networks with only one channel. However many modern control systems are implemented in a distributed fashion and involve multiple sensors, controllers, and actuators communicating over a serial network with a complex topology. Collisions between messages and delays are typical for them.

Several recent researches attempted to extend the fundamental stabilization data-rate limits on nontrivial networks. However, only rather simple networks composed of independent channels were studied. The works [6], [8] deal with networks of channels each connecting one of the sensors with the single common controller. A necessary and sufficient criterion for stabilizability was obtained in the case of delayed lossy channels and not necessarily diagonalizable systems. The case of multiple both sensors and actuators, where each sensor is directly linked with every actuator by a separate perfect channel, was studied in [12] for real-diagonalizable systems. Separate necessary and sufficient conditions for stabilizability were obtained. In general, they are not tight [12]. In the case where the system is stabiliazible by every actuator and detectable by every sensor, a single necessary and sufficient criterion was established in [12].

In this paper, we also study stabilization of discrete time unstable linear systems via limited capacity networks. Unlike the previous researches, we examine general networks, within which multiple sensors, controllers, and actuators may occupy arbitrary positions. The topology of the network is also arbitrary. Moreover, it may be dynamically altered by switching channels on the basis of information available at some node or nodes. The protocol of switching may be a part of the designable control strategy for some such nodes, whereas it may be given a priory for other ones. The messages transmitted across the network incur delays, may be lost and corrupted; they may interfere with each other. So messages arriving at a given node may depend on the packets dispatched from many (up to all) nodes, including one at hand. The consideration is focused on deterministic stationary networks with limited memories.

We give an evidence that in a certain sense, the problem of finding the stabilizability domain is reduced to the long-standing standard problem of the information theory: finding the rate (capacity) domain of the network. The latter domain describes how much information can be transmitted from a node or set of nodes to another node or set of nodes. It is shown that the system is stabiliazible if and only if a certain vector characterizing its rate of instability in the open-loop belongs to the interior of the rate domain. However, the domain of the primal network is not relevant here and another network should be put in use. It is obtained by including several infinite capacity delayed channels in the original network. They link each actuator with every sensor influenced by this actuator. Furthermore, the rate domain associated with a certain subclass (related to linear systems) of decoders (serving actuators) is employed.

The new channels included in the network explicitly express the view of the control loop as a link transmitting information. A posteriory, this means that the control loop does transmit information, though its contents may not be clearly specified a priory [1], which is illuminative for necessary conditions. The "constructive" part of this view is the idea that the control signals can be employed as carriers of a priory pre-specified information. The results of this paper are based on a scheme of information transmission by means of control, which does not rely on special circumstances like sensor's access to control, deals with a general network, and ensures transmission of as much information as desired without violating the main objective of stabilization.

The paper is organized as follows. We first illustrate the

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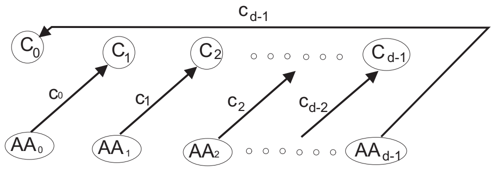

Fig. 1. Stabilization of autonomous agents

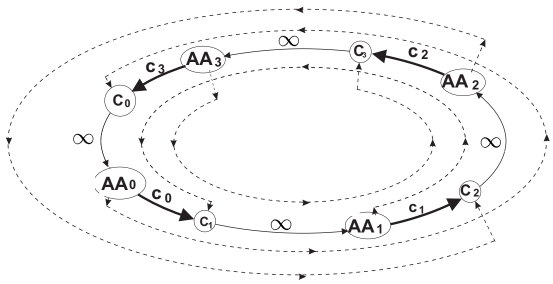

Fig. 2. The information exchange in the system

main result by an example in Section II. The network model introduced in Section III is illustrated by examples in Section IV. Sections V and VII present the problem statement and assumptions, respectively. Section VI introduces the capacity domain employed in the formulations of the main results, which are presented in Section VIII. Their proofs will be given in the full version of the paper and are available upon request. Section IX gives the proof of a result from Section II.

II. Example

We consider a platoon of d unstable controllable agents $\mathbf{AA}_i$ with independent dynamics $\boldsymbol{x}_i(t+1) = A_i \boldsymbol{x}_i(t) + B_i \boldsymbol{u}_i(t)$, each equipped by a separate controller $\mathbf{C}_i$ with an unlimited memory. So the agents could stabilize their motions about required trajectories, provided each of them observes its own state. However, any agent $\mathbf{AA}_i$ gets data only about the state of its "cyclic follower" $\mathbf{AA}_{i^-}, i^- := i-1(\mod d)$. These data are sent via a digital channel with capacity c_{i^-} (see Fig. 1). The agents do not communicate with each other. *Is it possible to stabilize the platoon motion?*

Since none of the agents observes its own state, the answer seems to be in the negative at the first sight. However, the following fact will be proved in Section IX.

Proposition 1: The platoon is stabiliazible if and only if

$$\log_2 |\det A_0^+| + \cdots + \log_2 |\det A_{d-1}^+| < \min_{i=0,\dots,d-1} c_i.$$

Here A_i^+ is the unstable part of the matrix A_i.

This proposition is underlaid by the fact that data can be transmitted by means of controls, as was discussed in Introduction. So the information streams may in fact go through d additional channels $\mathbf{C}_i \mapsto \mathbf{AA}_i$, which are displayed as thin arrows in Fig. 2. The dotted lines show the routs over which data about the ith agent can be delivered to its controller $\mathbf{C}_i$.

This rather special example highlights typical phenomena encountered in networked stabilization. They concern ways by which data about an unstable mode of the system may reach actuators affecting this mode.

III. The network model

We consider a *network* with *nodes*, labeled $1, \dots, n$. It is fed by data from k external *sources*. These data are transmitted between and processed at the nodes, and ultimately go to l external *destinations*. The data from the jth source enters the network at *input nodes* $h \in \mathfrak{N}_{\leftarrow j}$. The message to the ith destination is generated by *output nodes* $h \in \mathfrak{N}_{\rightarrow i}$. These relations are given by the *source-destination scheme*:

$$\mathbf{SDS} = \left[\left\{ \mathfrak{N}_{\leftarrow j} \right\}_{j=1}^{k} ; \left\{ \mathfrak{N}_{\rightarrow i} \right\}_{i=1}^{l} \right]. \tag{1}$$

In the stabilization problem, sources are sensors, nodes are controllers, and destinations are actuators. In the problem of reliable communication, sources are informants, destinations are recipients, and nodes are intermediate processors incorporated into the communication network.

Relations between sources and the network. Each source j produces a time sequence of symbols $s_j(t)$ from the *source alphabet* $\mathfrak{S}_j$. The symbol $s_j(t)$ is accessible at any node $h \in \mathfrak{N}_{\leftarrow j}$ at time t. All such symbols accessible at a given node h constitute its *external input*:

$$S_h(t) := \left\{ s_j(t) \right\}_{j : h \in \mathfrak{N}_{\leftarrow j}}. \tag{2}$$

Operation of the network consists in carrying messages between nodes. At any time t, each node receives an *internal input* message $r_h(t) \in \mathfrak{R}_h$ from other nodes and emits an *internal output* message $e_h(t) \in \mathfrak{E}_h$ to them. The *input* $\mathfrak{R}_h$ (*output* $\mathfrak{E}_h$) alphabet contains (may contain) a "void" element $\circledast$, which symbolizes that nothing is received or sent.

During transmission, messages may loose their contents and be corrupted by mutual interference and noise. So the transmission result may be somewhat different from the dispatched messages. We consider deterministic time-invariant networks and suppose that transmissions between nodes are completed for the unit time. Then the data transfer within the network is described by equations of the form:

$$r_h(t) = \mathcal{R}_h \big[e_1(t-1), \dots, e_n(t-1) \big], \ h \in [1:n]. \tag{3}$$

Here the *transmission map* $\mathcal{R}_h[e_1, \dots, e_n]$ may be in fact independent of some variables $e_{h'}$ and has the property $\circledast = \mathcal{R}_h \big[\circledast, \dots, \circledast \big]$: nothing is received if nothing is sent.

Operation of nodes. Each node acts as a *processor*: it emits a signal $e_h(t)$ on the basis of the currently and previously received data by executing a *processing algorithm* (**PA**). The information about the past messages is taken from the *memory*. Its contents is represented by a symbol $m_h(t)$ from the *memory alphabet* $\mathfrak{M}_h$ with a "void state" $\circledast \in \mathfrak{M}_h$ symbolizing that the memory is empty. Thus

$$e_h(t) = \mathcal{E}_h[t, \overline{r}_h(t), m_h(t)], \text{ where } \overline{r}_h(t) := \begin{cases} [r_h(t), S_h(t)] & \text{if } h \text{ has an external input (2)} \\ r_h(t) & \text{otherwise} \end{cases} \tag{4}$$

is the data received at node h at time t. The memory is updated in accordance with a rule of the form:

$$m_h(t+1) = \mathcal{M}_h[t, \overline{r}_h(t), m_h(t)]. \tag{5}$$

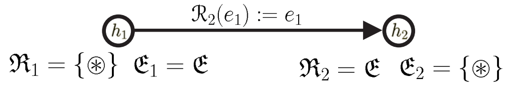

Fig. 3. A perfect channel

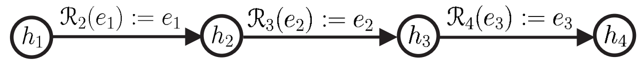

Fig. 4. 3-retarding chain

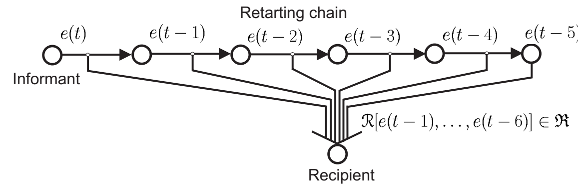

Fig. 5. Channel with collisions between delayed messages

Node h with no memory (internal input, internal output) is that with $\mathfrak{M}_h = \{\circledast\}$ ($\mathfrak{R}_h = \{\circledast\}, \mathfrak{E}_h = \{\circledast\}$).

Fixed nodes represent the memory of the communication medium. Formally, there is a set (maybe, empty) $\mathfrak{F} \subset [1:n]$ of *fixed* nodes h for which **PA** (i.e., $\mathcal{E}_h, \mathcal{M}_h, \mathfrak{M}_h$) is given. For the *designable* nodes $h \in \mathfrak{H} := [1:n] \setminus \mathfrak{F}$, **PA** is to be designed to archive a certain objective. (We suppose that $\circledast \in \mathfrak{E}_h$ for any designable node h.)

Relations between the network and destinations are established by *decoders*. On the basis of the data from all nodes accessible by the ith destination, the ith decoder $\mathcal{D}_i$ generates a message $d_i(t) \in \mathfrak{D}_i$ to this destination. Here $\mathfrak{D}_i$ is the *destination alphabet*. In brief,

$$d_i(t) = \mathcal{D}_i\Big[t, \big\{[\overline{r}_h(t), m_h(t)]\big\}_{h \in \mathfrak{N}_{\to i}}\Big]. \qquad (6)$$

Initial state. Initially, all designable nodes are in the "void" state and silent: $m_h(0) = \circledast, e_h(-1) = \circledast$ for all $h \in \mathfrak{H}$. Given the initial states $m_h(0) = m_h^0, e_h(-1) = e_h^0$ of the fixed nodes $h \in \mathfrak{F}$, the further evolution of the network is determined by the sequences of messages from the sources.

Data processing strategy is constituted by the decoders and processing algorithms at the designable nodes:

$$\textbf{DPS} := \Big[\big\{\mathcal{E}_h, \mathcal{M}_h, \mathfrak{M}_h\big\}_{h \in \mathfrak{H}}; \big\{\mathcal{D}_i\big\}_{i=1}^{l}\Big]. \qquad (7)$$

The sizes $|\mathfrak{M}_h|$ of the memory alphabets must meet given bounds $|\mathfrak{M}_h| \leq \mu_h$. Here $\mu_h = 1, 2, \dots$ or $\mu_h = \infty$.*

Network represents the communication medium and the memory capabilities of the nodes. In other words, it is constituted by nodes, their input and output alphabets and memory limits, transmission maps, and **PA** at the fixed nodes:

$$\textbf{NW} := \Big[\big\{\mathcal{R}_h, \mathfrak{E}_h, \mathfrak{R}_h\big\}_{h=1}^{n}; \big\{\mu_h\big\}_{h \in \mathfrak{H}}; \big\{\mathcal{E}_h, \mathcal{M}_h, \mathfrak{M}_h\big\}_{h \in \mathfrak{F}}\Big].$$

IV. EXAMPLES

Now we offer examples that fit the above framework and can be used as components to assemble complex networks.

Perfect channel is a network correctly transmitting a message $e \in \mathfrak{E}(\ni \circledast)$ from an *informant* to a *recipient* for the unit time (see Fig. 3).

d-retarding chain does the same with the delay d. It is assembled of d copies of the perfect channel, where all intermediate nodes are memoryless, fixed and merely forward the received message further (see Fig 4, where $d = 3$).

Channel with collisions between delayed messages. In this channel, the transmission result $r(t) = \mathcal{R}[e(t-1), \dots, e(t-d)] \in \mathfrak{R}$, where d is the depth of the channel memory (see Fig. 5). A particular case of the example at hand is the channel with a message-dependent transmission time: $r(t) = e(t-\tau)$ if $\tau = \mathcal{T}[e(t-1), \dots, e(t-d)] \leq \tau_{\max}$ and $r(t) = \circledast$ otherwise, where $\tau_{\max}$ is a given constant.

Perfect channel with a terminal queue. Any message e is characterized by a *size* $\sigma(e)$ (e.g., the number of bits in e). The recipient needs $\sigma(e)/\varsigma_d$ time units to process and accept the message. To await service, messages are put in the buffer of size b. If the size of the buffer is insufficient to add the arriving message to the current content, this message is dropped.

The channel can be modelled as a network with three nodes: the informant h_i, the recipient h_r, and the buffer h_b. The buffer is a fixed node with the memory $\mathfrak{M}_b = \{m_b = [(e_1, \dots, e_s); \sigma_-]\} := \mathfrak{M}_{\mathfrak{b}}{}' \times [0, +\infty) \cup \{\circledast\}$, where $\sigma_- \geq 0$ is the size of the remainder of the currently processed message e_s, $\mathfrak{M}_{\mathfrak{b}}{}' := \bigcup_{j=1}^{d} \mathfrak{E}^j$, and $d := \lceil b/\varsigma_d \rceil + 1$. (Evidently, the buffer accommodates no more than d messages.) The *memory size* is $\sigma(m_b) := \sigma(e_1) + \ldots + \sigma(e_{s-1}) + \sigma_-$ if $m_b \neq \circledast \wedge s \geq 2$; $\sigma(m_b) := \sigma_-$ if $m_b \neq \circledast \wedge s = 1$; $\sigma(m_b) := 0$ if $m_b = \circledast$. The informant h_i sends the message $e(t)$ to h_b over a perfect channel. The memory of h_b is updated by the following rule of the form (5). Firstly, the new message $e(t-1)$ is added to the stock $m_b(t)$ from the left, $s := s+1$, and $m_b(t) = \circledast \Rightarrow \sigma_- := 0$, provided $\sigma[e(t-1)] + \sigma[m_b(t)] \leq b$; otherwise, $m_b(t)$ is kept unchanged. Secondly, if $m_b(t) = \circledast$, then $m_b(t+1) := \circledast$; otherwise, the messages e_j with $j > s'$ are removed from the memory. Here $s' := \max\{j = 0, \dots, s : \varphi(j) > 0\}$ and $\varphi(0) := 1$, $\varphi(j) := \sigma(e_j) + \cdots + \sigma(e_{s-1}) + \sigma_- - \varsigma_d$ for $j \in [1:s]$. Thirdly, $\sigma_- := \varphi(s')$ if $s' > 0$; otherwise, the memory is emptied. The node h_b sends the message $(e_{s'+1}, \dots, e_s)$ ($\circledast$ if $s' = s$) to the recipient h_r over another perfect channel. So $\mathfrak{E}_{h_b} = \mathfrak{R}_{h_r} := \mathfrak{M}_{\mathfrak{b}}{}' \cup \{\circledast\}$.

Multiple access channels with broadcasting capabilities. Consider first the situation where several informants $h_1^i, \dots, h_p^i$ send messages $e_1 \in \mathfrak{E}_1, \dots, e_p \in \mathfrak{E}_p$ to a common recipient h^r over a common communication medium. Due to a mutual interference, h^r receives the message $r(t) = \mathcal{R}[e_1(t-1), \dots, e_p(t-1)] \in \mathfrak{R}$ (see Fig. 6). By merging this situation, we arrive at the case where each informant has its own intended recipient, but the message to it is corrupted by interference with signals from concurrent informants. This network also models the case where there are informants with

*We consider only infinite sets with the cardinality of the continuum.

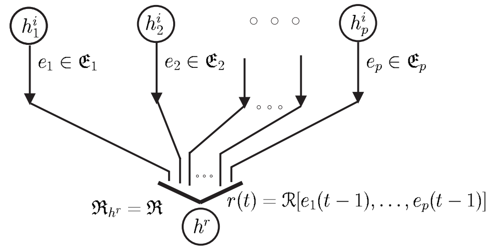

Fig. 6. Multiple access channel

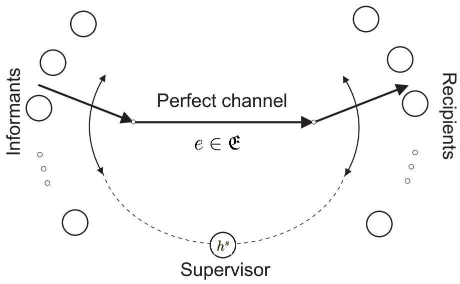

Fig. 7. Switching channel

broadcasting capabilities each sending a message to several intended recipients.

Switching channel. Several informant-recipient pairs $(h_j^i, h_j^r), j \in [1 : p]$ use a common perfect channel. At any time, the channel serves only one pair selected by the *supervisor* node h^s (see Fig. 7). Its output alphabet $\mathfrak{E}^{[s]} = \{\sigma\}$ is $\{1, \ldots, p, \circledast\}$ (the signal $\circledast$ causes disconnection of all pairs). The transmission map to the jth recipient is $\mathcal{R}_j[e_1^i, \ldots, e_p^i, \sigma] = e_j^i$ if $j = \sigma$ and $= \circledast$ otherwise. The particular case $p = 2, h_2^r = h_1^i, h_1^r = h_2^i$ of the example at hand is the **two-way channel**. It transmits messages between two nodes in both directions. However at any time, only one direction can be activated by the supervisor. A modification of the situation from Fig. 7 is that where at any time, arbitrary pair or $q < p$ pairs can be connected. Note also that in the above examples, there may be repeating nodes among h_j^i, h_j^r, e.g., only one informant sending messages to several recipients. The supervisor may be identical to one of the informants or recipients.

Switching channel with a given protocol. The above model encompasses both cases where the protocol of the channel use is and is not given, respectively. For example, fieldbuses are often endowed with τ-periodic protocols [4]. Then the supervisor is a fixed node with the memory $\{m\} = \mathfrak{M} := [0 : \tau - 1] \cup \{\circledast\}$ updated by the rule $m(t+1) := m(t) + 1(\mod \tau)$ if $m(t) \neq \circledast$ and $m(t+1) := \circledast$ otherwise. This node emits the signal $m(t)$. A partition $[0 : \tau - 1] = T_1 \cup \ldots \cup T_p$ is given. Whenever $m(t-1) \in T_j$, the channel serves the jth pair: $\mathcal{R}_j[e_1^i, \ldots, e_p^i, m] = e_j^i$ if $m \in T_j$ and $= \circledast$ otherwise.

By the following operations complex networks fitting the model from Section III can be assembled of primitive ones.

Free union is interpreting several mutually independent networks as a single network.

Welding of designable nodes $h^1, \ldots, h^p$ is interpreting them as various parts of a single node h^w. It receives signals $[r_{h^1}(t), \ldots, r_{h^p}(t)]$ and emits messages $[e_{h^1}(t), \ldots, e_{h^p}(t)]$. So up to formalities, the transmission maps remain unchanged. The memory size of h^w is $\mu_{h^w} := \mu_{h^1} \times \cdots \times \mu_{h^p}$. The new node h^w is able to generate each of the output messages e_{h^j} on the basis of the entire set of inputs $r_{h^1}, \ldots, r_{h^p}$ and the memories contents $m_{h^1}, \ldots, m_{h^p}$. This is in contrast with the initial network, where only the matching elements r_{h^j} and m_{h^j} can be used.

V. Statement of the Main Problem

We consider linear time-invariant discrete time multiple sensor and actuator systems of the form:

$$\boldsymbol{x}(t+1) = A\boldsymbol{x}(t) + \sum_{i=1}^{l} B_i \boldsymbol{u}_i(t); \qquad \boldsymbol{x}(0) = \boldsymbol{x}^0; \quad (8)$$

$$\boldsymbol{y}_j(t) = C_j \boldsymbol{x}(t), \qquad j = 1, \ldots, k. \quad (9)$$

Here $\boldsymbol{x} \in \mathbb{R}^{n_x}$ is the state, $\boldsymbol{u}_i \in \mathbb{R}^{n_{u,i}}$ and $\boldsymbol{y}_j \in \mathbb{R}^{n_{y,j}}$ are the outputs of the ith actuator and jth sensor, respectively. The system is unstable: it has an eigenvalue λ with $|\lambda| \geq 1$. The objective is to stabilize the plant: $\boldsymbol{x}(t) \xrightarrow{t \to \infty} 0$.

We consider a decentralized control setup. The sensors, controllers, and actuators are spatially distributed and linked via a communication network. It is endowed with a source-destination scheme (1). Sensors are sources: $s_j(t) = \boldsymbol{y}_j(t), \mathfrak{S}_j = \mathbb{R}^{n_{y,j}}$, the actuators are destinations: $d_i(t) = \boldsymbol{u}_i(t), \mathfrak{D}_i = \mathbb{R}^{n_{u,i}}$, the nodes are controllers. Each actuator i is equipped with its own *local* controller $h_{\to i}$, which constitutes the set $\mathfrak{N}_{\to i} = \{h_{\to i}\}$ in (1). The decoder (6) is interpreted as a part of this controller. Apart from local controllers, there may be *intermediate* ones $h \neq h_{\to i}$. In this case, the strategy (7) will be also called the *decentralized control strategy*. This strategy, the initial data $\boldsymbol{x}^0$ from (8), and the initial states of the fixed nodes uniquely determine the process in the closed-loop system.

Definition 1: The system (8) is said to be *exponentially stabilizable via the network* if some decentralized control strategy makes all trajectories uniformly exponentially converging to zero: there exists $\rho \in (0, 1)$ such that

$$\sup \sup_{|x^0| \leq R} \rho^{-t} \Big[|\boldsymbol{x}(t)| + \sum_{i=1}^{l} |\boldsymbol{u}_i(t)| \Big] \xrightarrow{t \to \infty} 0 \; \forall R > 0.$$

Here the first sup is over all initial states of the fixed nodes. *What is the minimum rate of the information exchange in the network for which stabilization is possible?* In other words, we look for criteria of stabilizability expressed in terms of the plant parameters and relevant characteristics of the network, which describe its capability to transmit information.

VI. Capacity Domain of a Network

Those characteristics are related to the question: How much data can be communicated from one point, or set of points, to another point, or set of points? The answer is given in terms of a fundamental concept of the information theory: the rate (capacity) domain. Now we recall this notion.

Let a network from Section III be used to transmit data from k independent transmitters. For each of them, given are the nodes where the data from this transmitter enter the network. Another given set of nodes indicates the points where the data can be taken to make a decision about the message from this transmitter. Briefly, we are given a **SDS** (1) with a common number of sources and destinations $k = l$: the data from the ith source should be transmitted to the ith destination. Such a scheme is said to be *coordinated*.

Definition 2: Let a network and a coordinated source-destination scheme be given. A *networked block code* with block length N is a **DPS** that is defined on the time interval $[0 : N-1]$ and serves destinations with the alphabets of the form $[1 : M_i] \cup \{\circledast\}, i \in [1 : k]$ and sources producing constant message sequences $s_j(t) \equiv \nu_j \; \forall t$ from the alphabets $\nu_j \in [1 : M_j]$. The *rate vector* of this code is defined to be

$$\left(\frac{\log_2 M_1}{N}, \dots, \frac{\log_2 M_k}{N} \right). \tag{10}$$

The decoder output $d_i = \circledast$ means that no decision is made. Note that the block code serves simultaneous transmission of the data from all sources to their destinations.

Definition 3: A networked block code is *errorless* if at the terminal time it correctly recognizes the messages from all sources $d_i(N-1) = \nu_i \; \forall i$ irrespective of both the initial states of the fixed nodes and which messages $\nu_i \in [1 : M_i]$ are sent.

Definition 4: For a network **NW** and a coordinated **SDS**, consider a networked block code. Let this code ranges over all errorless codes with arbitrary block lengths. Then the rate vector (10) runs over some subset of $\mathbb{R}^k$. Its closure is called the *rate (capacity) domain* **CD[NW,SDS]**.

Finding the rate domain is a long-standing problem in the information theory. We refer the reader to [2], [15] for a survey of achievements and troubles in this field.

Almost additive capacity domain. If some unstable mode of the system is affected by several independent actuators, decoders of special kind are of interest.

Definition 5: The decoder (6) is said to be *ε-additive* ($\varepsilon \in (0, 1/2)$) if each node $h \in \mathfrak{N}_{\to i}$ feeding the ith destination can be endowed with a real function $\mathfrak{d}_{i,h} = \mathfrak{d}_{i,h}[t, \overline{r}_h(t), m_h(t)]$ of the data available at h so that the decoder output $d_i(t)$ from (6) equals the number $\nu_i \in [1 : M_i]$ of the interval $(\nu_i - \varepsilon, \nu_i + \varepsilon)$ containing $\sum_{h \in \mathfrak{N}_{\to i}} \mathfrak{d}_{i,h}$ if such a ν_i exists, and $d_i(t) := \circledast$ otherwise.
Such decoders in fact act by separate encoding the data at each node $h \in \mathfrak{N}_{\to i}$ for transmission over an imaginary real additive channel and projecting the transmission result on the integer grid with accuracy ε.

Definition 6: Let in Definition 4, only networked block codes with ε-additive decoders are considered. This gives rise to the *ε-additive capacity domain* **CD$^{\varepsilon}$[NW,SDS]**. Their intersection over $\varepsilon \in (0, 1/2)$ is called the **aa-(almost additive) capacity domain CD^{aa}[NW,SDS]**.

Existence of an errorless block code with block length N does not ensure that there is such a code with ε-additive decoders. Indeed, consider the network from Fig. 8a. Nodes 2 and 3 are memoryless, both with the input alphabet $\{a, b, c\}$. The messages ν are transformed by the channel into the pairs $r = (r_1, r_2), r_i = a, b, c$ depicted as dots in Fig. 8b. So an errorless decoder with block length 1 can be offered. (Such a decoder also exists if the destination from Fig. 8a is interpreted as a recipient linked with nodes 2 and 3 by a real additive channel, provided that data is properly encoded at nodes 2 and 3.) However, there is no such an ε-additive decoder with $\varepsilon < 1/6$. Indeed otherwise there would be two functions $\mathfrak{d}_1(r_1), \mathfrak{d}_2(r_2)$ such that $|\mathfrak{d}(r) - \nu(r)| < 1/6$ for any "dot" r, where $\mathfrak{d}(r) := \mathfrak{d}_1(r_1) + \mathfrak{d}_2(r_2)$ and $\nu(r)$ is the true decoded value of r. For the function $\sigma(r)$ introduced in Fig. 8b, one has $\left| \sum_r \sigma(r) \left[\mathfrak{d}(r) - \nu(r) \right] \right| < 6\varepsilon < 1$. However $\sum_r \sigma(r) \mathfrak{d}(r) = 0$ and $\sum_r \sigma(r) \nu(r) = 3$. This contradiction proves that there is no ε-additive errorless decoder with block length 1.

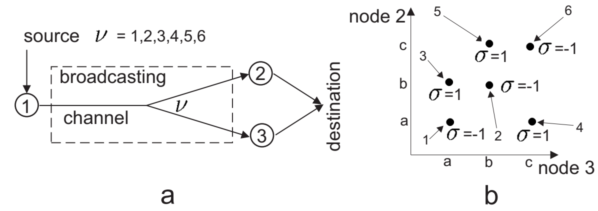

Fig. 8. Network with three nodes

Proposition 2: Let any destination be fed by only one node. Then the capacity and aa-capacity domains coincide.

VII. Assumptions about the main problem

The first two assumptions are of minor importance and are adopted to simplify the matters and meet the paper length limitations. Modulo the arguments from [6], [8], extensions on the general case are rather straightforward.

Assumption 1: The unstable $|\lambda_\nu| \geq 1$ eigenvalues $\lambda_1, \dots, \lambda_g$ of A are real and distinct.

Assumption 2: Input nodes $h \in \bigcup_j \mathfrak{N}_{\leftarrow j}$ differ from output ones $h \in \bigcup_i \mathfrak{N}_{\to i}$, i.e., $\mathfrak{N}_{\leftarrow j} \cap \mathfrak{N}_{\to i} = \emptyset \; \forall i, j$.

Assumption 3: All input and output nodes have infinite memories and are not fixed.

Assumption 4: Either there are no fixed nodes or any of them h is time invariant: in (4) and (5), the functions $\mathcal{E}_h(\cdot), \mathcal{M}_h(\cdot)$ do not depend on time t.

We close the section with useful technical facts.

Lemma 1: Let Assumptions 2—4 hold for a coordinated **SDS**. Then the following claims are true:

- **i)** The capacity and aa-capacity domains are convex closed sets containing the origin;
- **ii)** Consider the question: How much data can be transmitted from each source i to every relevant output node $h \in \mathfrak{N}_{\to i}$? It is associated with **SDS$^{s \to o}$** whose sources $s_{i,h}$ and destinations $d_{i,h}$ are related to $(i, h) \in \Xi := \{(i, h) : h \in \mathfrak{N}_{\to i}\}$, with $d_{i,h}$ fed by h and $s_{i,h}$ feeding

all $h' \in \mathfrak{N}_{\leftarrow i}$. Then

$$\Big\{\Big(\sum_{h \in \mathfrak{N}_{\to i}} r_{i,h}\Big)_{i=1}^{k} : (r_{i,h})_{(i,h)\in\Xi} \in \textbf{CD[NW,SDS}^{\textbf{s}\to\textbf{o}}\textbf{]}\Big\}$$
$$\subset \textbf{CD}^{\textbf{aa}}\textbf{[NW,SDS]} \subset \textbf{CD[NW,SDS]};$$

iii) Suppose that **SDS** contains repeating entries: $\mathfrak{N}_{\leftarrow i'} = \mathfrak{N}_{\leftarrow i''} \wedge \mathfrak{N}_{\to i'} = \mathfrak{N}_{\to i''}$ if $i', i'' \in \Delta_\alpha$ for some $\alpha \in [1:p]$. Here $\Delta_1, \ldots, \Delta_p$ is a partition of $[1:k]$ into nonempty sets. Consider the reduced scheme obtained by picking only one sample from any group of repeating entries: $\textbf{SDS}^{\textbf{red}} := \big[\{\mathfrak{N}_{\leftarrow i^\alpha}\}_{\alpha=1}^{p}; \{\mathfrak{N}_{\to i^\alpha}\}_{\alpha=1}^{p}\big]$, $i_\alpha \in \Delta_\alpha \, \forall \alpha$. Then

$$\textbf{CD[NW;SDS]} = \Big\{(r_i)_{i=1}^{k} : r_i \geq 0,$$
$$\Big(\sum_{i\in\Delta_1} r_i, \ldots, \sum_{i\in\Delta_p} r_i\Big) \in \textbf{CD[NW;SDS}^{\textbf{red}}\textbf{]}.\Big\}$$

Remark 1: In general, here the set on the left is covered by one on the right for ordinary and aa-capacity domains if

$$\textbf{SDS}^{\textbf{red}} := \Big[\Big\{\bigcup_{i\in\Delta_\alpha} \mathfrak{N}_{\leftarrow i}\Big\}_{\alpha=1}^{p} \textbf{;} \Big\{\bigcup_{i\in\Delta_\alpha} \mathfrak{N}_{\to i}\Big\}_{\alpha=1}^{p}\Big].$$

VIII. Main results

A certain network **NW** is given by the stabilization problem setup. However the criterion for stabilizability employs the rate domain of an extended network. To describe it, we introduce the subspaces that are not observed by a given sensor and are controllable by a given actuator, respectively:

$$L_j^{-o} := \bigcap_{\nu=0}^{n_x-1} \ker C_j A^\nu, \; L_i^{+c} := \text{Lin}\left[\bigcup_{\nu=0}^{n_x-1} \text{Im}\,(A^\nu B_i)\right].$$

As can be shown, there exists a method to transmit as much information as desired from any actuator i to any sensor j affected by this actuator: $L_i^{+c} \not\subset L_j^{-o}$. To this end, data is encoded into a sequence of controls $\boldsymbol{u}_i$. The sensor observes the resultant motion and restores the data by decoding the measurements $\boldsymbol{y}_j$. Since $\boldsymbol{u}_i$ influences $\boldsymbol{y}_j$ with the delay of

$$f_{i\to j} := \min\{f = 0, 1, \ldots : C_j A^f B_i \neq 0\} + 1 \qquad (11)$$

units of time, transmitted data incur the same delay. The extended network results from explicit incorporation of all these communication channels into the model. The node $h_{\to i}$ producing $\boldsymbol{u}_i$ acts as a transmitter for such a channel. Nodes h fed by the jth sensor $h \in \mathfrak{N}_{\leftarrow j}$ are receivers each served by a separate channel. Formally, for every triplet (i, j, h) such that $L_i^{+c} \not\subset L_j^{-o}$ and $h \in \mathfrak{N}_{\leftarrow j}$, one should introduce a $f_{i\to j}$-retarding chain $R_{i,j,h}$ from $h_{\to i}$ to h with an infinite alphabet $\mathfrak{A}_{i,j,h}$. This results in the *control based extension* **CBE(NW)** of the original network **NW**. **CBE** of the network from Fig. 1 is illustrated by Fig. 2.

If a node h is fed by several sensors affected by a common $\boldsymbol{u}_i$, there are several chains $R_{i,j,h}$ departing from $h_{\to i}$ and arriving at h. Since the memory of h and the capacities of these chains are infinite, all of them except for one providing the least delay could be discarded. However we do not do so, partially in order to make **CBE(NW)** uniquely defined.

The interest to **CBE** is illuminated by the following fact.

Theorem 1: Let Assumptions 2—4 hold. Then the system is exponentially stabiliazible via the network **NW** and its control-based extension **CBE[NW]** only simultaneously.

To state the criterion for stabilizability, we need one more construction. Any state $\boldsymbol{x}$ can be represented as the sum

$$\boldsymbol{x} = \boldsymbol{x}_{st} + \boldsymbol{x}_1 + \cdots + \boldsymbol{x}_g, \; \boldsymbol{x}_{st} \in L_{st} \; \boldsymbol{x}_\nu \in L_\nu \; \forall \nu,$$

where $L_1, \ldots, L_g$ are the eigenspaces related to the unstable eigenvalues $\lambda_1, \ldots, \lambda_g$ and L_{st} is the complementary stable subspace of A. Now we pose the question: How much data about each unstable *mode* $\boldsymbol{x}_\nu$ can be communicated across the network from the sensors to the nodes h that can control this mode? These are the nodes

$$h \in \mathfrak{N}_{\to\nu}^{mw} := \{h_{\to i}\}_{i: L_\nu \subset L_i^{+c}}, \qquad (12)$$

and the summary information available at them is meant. Data about different modes should be transmitted via the network in parallel. Data about $\boldsymbol{x}_\nu$ enters the network at nodes h fed by sensors observing this mode

$$h \in \mathfrak{N}_{\leftarrow\nu}^{mw} := \bigcup_{j: L_\nu \cap L_j^{-o} = \{0\}} \mathfrak{N}_{\leftarrow j}. \qquad (13)$$

The answer is clearly given by the capacity domain related to the following *mode-wise source-destination scheme*

$$\textbf{SDS}^{\textbf{mw}} := \Big[\{\mathfrak{N}_{\leftarrow\nu}^{mw}\}_{\nu=1}^{g}; \{\mathfrak{N}_{\to\nu}^{mw}\}_{\nu=1}^{g}\Big]. \qquad (14)$$

Now we are in a position to state the main result.

Theorem 2: Suppose that Assumptions 1—4 hold. We supply the control based extension **CBE(NW)** of the primal network **NW** with the mode-wise source-destination scheme $\textbf{SDS}^{\textbf{mw}}$ and consider the corresponding almost additive capacity domain $\textbf{CD}^{\textbf{aa}} := \textbf{CD}^{\textbf{aa}}\textbf{[CBE(NW);SDS}^{\textbf{mw}}\textbf{]}$. Then the following two statements are equivalent:

a) The system (8), (9) is exponentially stabiliazible via the original network **NW** (see Definition 1);

b) $\Big(\log_2|\lambda_1|, \ldots, \log_2|\lambda_g|\Big) \in \text{int}\big[\textbf{CD}^{\textbf{aa}}\big]$, where $\lambda_1, \ldots, \lambda_g$ are the unstable eigenvalues of the system.

Proposition 2 and **ii)** of Lemma 1 imply the following.

Remark 2: Suppose that **c)** different actuators influence no common unstable modes: $L_{i'}^{+c} \cap L_{i''}^{+c} \subset L_{st} \; \forall i' \neq i''$. Then in **b)**, the aa-capacity domain $\textbf{CD}^{\textbf{aa}}$ can be replaced by the standard rate domain **CD**. In the general case, the inclusion so obtained is necessary for **a)** to be true.

Note that **c)** holds for e.g., platoons of independent agents.

The sets of nodes observing and affecting, respectively, a given mode may be common for several modes. Then these modes give rise to repeating entries in (14) and by **iii)** of Lemma 1, the capacity domain can be found from a domain of lesser dimension. The sums $\sum_{i\in\Delta_\alpha} r_i$ from **iii)** shape into $\sum_{i\in\Delta_\alpha} \log_2|\lambda_i| = \log_2|\det A_{\widehat{L}_\alpha}|$, where $\widehat{L}_\alpha$ is the linear hull of $\{\boldsymbol{x}_i\}_{i\in\Delta_\alpha}$. The distinguishing feature of the subspaces $\widehat{L}_\alpha$ is that they are A-invariant, decompose the unstable subspace L_{unst} of A into a direct sum, and

$$\text{either } \widehat{L}_\alpha \subset L_i^{+c} \text{ or } \widehat{L}_\alpha \cap L_i^{+c} = \{0\}\ \forall i \quad \text{and} \quad \text{either } \widehat{L}_\alpha \subset L_j^{-o} \text{ or } \widehat{L}_\alpha \cap L_j^{-o} = \{0\}\ \forall j. \quad (15)$$

Summarizing, we arrive at the following.

Corollary 1: Suppose that the hypotheses of Theorem 2 hold and the unstable space of the system (8) is decomposed into a direct sum $L_{unst} = \widehat{L}_1 \oplus \cdots \oplus \widehat{L}_p$ of A-invariant subspaces with the properties (15). Introduce the aggregated source-destination scheme:

$$\widehat{\mathbf{SDS}}^{\mathbf{mw}} := \left[\left\{\widehat{\mathfrak{N}}_{\leftarrow\alpha}\right\}_{\alpha=1}^p; \left\{\widehat{\mathfrak{N}}_{\rightarrow\alpha}\right\}_{\alpha=1}^p\right], \ \widehat{\mathfrak{N}}_{\leftarrow\alpha} := \\ := \bigcup_{j:\widehat{L}_\alpha \cap L_j^{-o} = \{0\}} \mathfrak{N}_{\leftarrow j}, \ \widehat{\mathfrak{N}}_{\rightarrow\alpha} := \left\{h = h_{\rightarrow i} : \widehat{L}_\alpha \subset L_i^{+c}\right\}.$$

Then **a)** implies and, under the condition **c)** of Remark 2, is equivalent to the inclusion

$$\left(\log_2 |\det A|_{\widehat{L}_\alpha}|\right)_{\alpha=1}^p \in \text{int}\,\mathbf{CD[CBE(NW);}\widehat{\mathbf{SDS}}^{\mathbf{mw}}\mathbf{]}. \quad (16)$$

A particular instance of this claim deals with the total capacity of the network. To introduce it, we consider **SDS** with one source sending signals to all input nodes and one destination receiving messages from all output ones. By Proposition 1, the one-dimensional rate domain related to this scheme has the form $\{r \in [0, \infty) : r \leq \mathfrak{c}\}$, where $\mathfrak{c} \in [0, \infty]$ is the network *total capacity*.

Corollary 2: Let the system (8) be detectable from every input node and stabiliazible by each actuator. Then the conclusion of Corollary 1 remains true with (16) replaced by $\mathfrak{c} > \log_2 \left|\det A|_{L_{unst}}\right|$.

IX. Proof of Proposition 1

By Corollary 1, it suffices to show that $r = (r_i)_{i=0}^{d-1}$ $\in$**CD[CBE(NW),SDS]** $\Leftrightarrow r_i \geq 0 \& \sum_i r_i \leq \min_i c_i$. **CBE** of the network **NW** from Fig. 1 is illustrated by Fig. 2, and **SDS** has d both sources and destinations, each feeding and fed by particular nodes $\mathbf{AA}_i$ and $\mathbf{C}_i$, respectively.

Sufficiency. If $\sum_i r_i \leq \min_i c_i$, **CBE** evidently can accommodate a continuous fluid flow composed by d sub-flows along the dotted directed arcs from Fig. 2, each of intensitivity r_i. Then the arguments from the first part of the proof of Theorem 3.1 [3] show that $r \in$**CD[CBE(NW),SDS]**.

Necessity. The above "fluid" consideration is not relevant any longer.† Let $r \in$**CD[CBE(NW),SDS]**. Given $\varepsilon > 0$, there is an errorless networked block code with block length N and input alphabets $[1 : M_i] = \{\nu_i\}$ such that $|r_i - N^{-1} \log_2 M_i| < \varepsilon\,\forall i$. Let A_i and C_i symbolize the summary information received at nodes $\mathbf{AA}_i$ and $\mathbf{C}_i$, respectively, by time $N-1$. As follows from Fig. 2, $C_{i^+} = \Phi_i(A_i), i^+ := i+1(\mod d), A_i = \Psi_i(C_i, \nu_i)\,\forall i$. Since the code is errorless, $\nu_i = \Xi_i(C_i)$. Thus $C_{i+} = \Phi_i\{\Psi_i[C_i, \Xi_i(C_i)]\}\,\forall i$, i.e., $(C_j)_{j=0}^{d-1} = \Lambda_i(C_i)\,\forall i$ and so $(\nu_j)_{j=0}^{d-1} = \Gamma_i(C_i)$ for all i and $\nu_j \in [1 : M_j]$. Since C_{i+} is received over a channel of capacity c_i, it follows that $\sum_j \log_2 M_j \leq Nc_i\,\forall i$. Hence $\sum_i r_i \leq d\varepsilon + \sum_j N^{-1} \log_2 M_j \leq d\varepsilon + \min_i c_i$. It remains to let $\varepsilon \to 0$.

†In [3], existence of a coding-decoding scheme ensuring data rates coherent with the intensitivities of the continuous fluid flow at the input nodes was justified only for the case of one recipient. The authors are unaware of researches, where the fluid-like treatment of information streams was justified for several informants and recipients, as is required now.

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