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Cost-detectability and Stability of Adaptive Control Systems

R. Wang, A. Paul, M. Stefanovic and M. G. Safonov

***Abstract*— Hysteresis switching adaptive control systems designed using certain types of $\mathfrak{L}_{2e}$-gain type cost functions are shown to be robustly stabilizing if and only if certain plant-independent conditions are satisfied by the candidate controllers. These properties ensure closed-loop stability for the switched multi-controller adaptive control (MCAC) system whenever stabilization is feasible. The result is a *safe adaptive control system* that has the property that closing the adaptive loop can never induce instability provided only that at least one of the candidate controllers is capable of stabilizing the plant.**

***Index Terms*— unfalsified control, robust adaptive control, switching control, unfalsification, cost-detectability, stability**

I. INTRODUCTION

Generally, an adaptive control system is defined by three essential elements: goals, information and a set of candidate controllers. An adaptive control algorithm for such systems is the scheme to select/choose/order/tune/switch among candidate controllers by using real-time and prior information to achieve specified goals. Among all these goals, achieving stability is the minimum goal for an adaptive control system. Whenever an active controller does not stabilize the system, a safe algorithm [1], [2] should be able to recognize instability, abandon the active controller, and change to a stabilizing controller if there is one in the candidate controller set. If there is a stabilizing controller for the system in the candidate controller set at any time, *i.e.*, if stabilization by one of the candidate controllers is feasible, then a good adaptive control algorithm should be able to stabilize the system without further assumptions on the plant.

Martensson [3] showed how to achieve adaptive goals using only the feasibility assumption via a pre-routed switching among the candidate controllers until one controller was found which could achieve the control objective. Other pre-routed based switching schemes can be found, for example, in [4], [5] and [6]. However, though pre-routed switching schemes generally require only feasibility for convergence, they give poor transient response and switching to a stabilizing controller can take a long time, especially when the number of candidate controllers is large. Feasibility is the weakest assumption on the plant under which adaptive stabilization can be assured.

Unfortunately, safe adaptive control algorithms are rare. Aside from the impractical pre-routed switching methods, modern adaptive methods have for the most part continued to rely on additional plant modeling assumptions, compromising the robust performance properties that adaptive control is intended to enhance. See, for example, [7] for overview. Recent efforts to partially relax some, but not all, plant assumptions appear in [8]–[10], but these fall short of solving the safe adaptive control problem mentioned above where the only assumption is feasibility. A notable exception is [1], [2], [11], where it was shown that *unfalsified adaptive control* can overcome the poor transient response associated with the earlier pre-routed schemes by doing direct validation of candidate controllers very fast by using experimental data only, without making any assumptions on the plant beyond feasibility, and thus can potentially provide a practical solution to the safe adaptive control problem. But, not all unfalsified adaptive controller are safe. Preliminary results in [1] suggested that, to be safe, the unfalsification criterion used in unfalsified adaptive control must have a property known as *cost detectability*.

In [1], stability of unfalsified adaptive control systems was re-examined from the perspective of the hysteresis switching lemma of Morse, Mayne and Goodwin [12] in order to address the above problem. Sufficient conditions for the stabilization of adaptive control using multiple controllers were provided. It was found that if a system is cost-detectable, it can always be stabilized if the problem is feasible.

In this paper, we expand the results from [1] and give necessary and sufficient conditions for cost-detectability.

The paper is organized as follows. In Section II, the formulation of a safe adaptive control problem is given. In Section III, some important definitions in unfalsified adaptive control method are introduced; in Section IV, some properties of this method are examined. Based on the definitions and properties in Section III, an unfalsified multiple controller adaptive control algorithm is studied in Section V. Conclusion follows in Section VI.

II. SAFE ADAPTIVE CONTROL PROBLEM

Let $\Re_+ = (0, \infty)$. Define the truncation of a signal over a time interval (a, b) as

$$x_{(a,b)}(t) = \begin{cases} x(t), & \text{if } t \in (a,b) \\ 0, & \text{otherwise.} \end{cases}$$

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and x_τ denotes the time truncation over the interval $(0, \tau)$

$$x_\tau(t) = \begin{cases} x(t), & \text{if } t \in (0,\tau) \\ 0, & \text{otherwise.} \end{cases}$$

It is said that $x \in \mathfrak{L}_{2\mathfrak{e}}$ if $\|x_\tau\|$ exists for all $\tau < \infty$ where

$$\|x_\tau\| = \sqrt{\int_0^\tau \|x(t)\|^2 \, dt}$$

For any $\tau \in \Re_+$, a *truncation operator* P_τ is a linear projection operator

$$[P_\tau z](t) \triangleq \begin{cases} z(t), & \text{if } 0 \le t \le \tau \\ \text{not defined otherwise.} \end{cases}$$

The following definition of stability of a system is related to the $\mathfrak{L}_{2\mathfrak{e}}$-gain, which pertains to the relationship of the norm of the output and the norm of the input.

Definition 1: (Stability and Gain) A system G with input v and output w is said to be *stable* if for every input $v \in \mathfrak{L}_{2\mathfrak{e}}$ there exist constants $\beta, \alpha \ge 0$ such that

$$\|w_\tau\| < \beta\|v_\tau\| + \alpha, \forall\, t > 0; \tag{1}$$

otherwise, it is said to be *unstable*. Furthermore, if (1) holds with a single pair $\beta, \alpha \ge 0$ for all $v \in \mathfrak{L}_{2\mathfrak{e}}$, then the system G is said to be *finite-gain stable*, in which case the *gain* of G is the least such β. □

Definition 2: (Incremental Stability and Incremental Gain) The system G is said to be *incrementally stable* if, for every pair of inputs v_1, v_2 and outputs $w_1 = Gv_1, w_2 = Gv_2$, there exist constants $\tilde{\beta}, \tilde{\alpha} \ge 0$ such that

$$\|[w_2 - w_1]_\tau\| < \tilde{\beta}\|[v_2 - v_1]_\tau\| + \tilde{\alpha}, \forall\, t > 0; \tag{2}$$

and the *incremental gain* of G, when it exists, is the least $\tilde{\beta}$ satisfying (2) for some $\tilde{\alpha}$ and all $v_1, v_2 \in \mathfrak{L}_{2\mathfrak{e}}$. □

Following Willems ([13], [14]), our definitions of stability admit non-zero values for the parameters $\alpha, \tilde{\alpha}$. These parameters allow for consideration of systems with non-zero initial state, and would depend on the initial state (e.g., $\alpha = \alpha(x_0)$ for an initial state were x_0). This slight generalization of classic input-output stability definitions of Zames [15] turns out to be useful in analyzing switching controllers, since switching from one controller to another generally leaves a system in a non-zero state.

We are examining stability of a switching adaptive control system in this paper. An *adaptive control system* is a control system with an adaptive controller. An *adaptive controller* is a controller with adjustable parameters/structures and a mechanism for adjusting the parameters/structures [16]. A set composed by time-invariant controllers with any of these possible parameters/structures is called *candidate controller set*.

In this paper we consider a general adaptive control system $\Sigma(P, \widehat{K})$ mapping r into (u, y) whose block diagram depicted is shown in Fig.1. The system is defined on $\mathfrak{L}_{2\mathfrak{e}}$ [15], which is to say that the signals r, u, y are all assumed to be square-integrable over every *bounded* interval

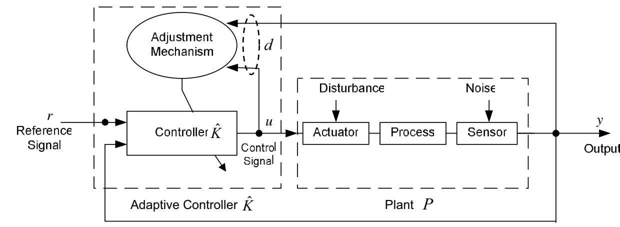

Fig. 1. Adaptive control system $\Sigma(P, \hat{K})$

$[0, \tau]$, $(\tau \in \Re_+)$. The adaptive *adjustment mechanism* has an input signal

$$d \triangleq \begin{bmatrix} u \\ y \end{bmatrix}.$$

and output $K \in \mathbb{K}$ so that the *adaptive control law* has the general form

$$u = \widehat{K}(t, d) \begin{bmatrix} r \\ y \end{bmatrix}.$$

We limit our consideration in this paper to the case in which the *candidate controller set* $\mathbb{K}$ has finitely many elements, where $\widehat{K}(t, d) \in \mathbb{K}$ is piecewise constant. The adaptive control law $\widehat{K}(t, d)$ 'switches' among the controllers K in the candidate set according to the measured data d. If the switching stops, then the final controller and final switching time are denoted $K^f(d)$ and $t^f(d)$, respectively.

The plant P in Figure 1 is a *completely unknown plant*, *i.e.* we have no prior knowledge of P other than that its input and output are in $\mathfrak{L}_{2\mathfrak{e}}$. At time $t = 0$ prior to collecting data d, we know neither the plant structure, its parameters, its order, its operating environment, nor the noises and disturbances. The plant may be unstable, non-minimum phase, non-linear and of infinite order. An adaptive control problem is called *feasible* if at least one stabilizing controller K is available in the candidate controller set $\mathbb{K}$, even though which controllers are stabilizing is not known *à priori*. Given a completely unknown plant and a candidate controller set, a *safe adaptive control* law, introduced in [17] is one that never fails to stabilize whenever adaptive stabilization is feasible.

Problem 1: Safe adaptive control problem [17]. Find, when feasible, an adaptive control law $\widehat{K}(t, d)$ that switches among the candidate controllers $K \in \mathbb{K}$ in such a way that the resultant closed-loop adaptive system $\Sigma(P, \widehat{K}(t, d))$ is stable. □

It should be emphasized that a safe adaptive system is more than just robustly stable for plant uncertainty in a given uncertainty set. A safe adaptive system stabilizes the plant whenever a stabilizing controller exists in its candidate controller set, without regard to other prior knowledge of the plant model or its uncertainty. In this sense, a *safe* adaptive controller is a maximally robust adaptive controller.

III. UNFALSIFIED ADAPTIVE CONTROL

Let $d = (u, y)$ denote one possible experimental plant data over the time 0 to ∞, and $\mathbb{D}$ the set of all possible d.

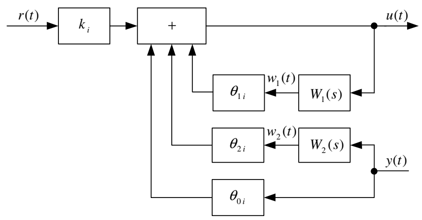

Fig. 2. A controller with a minimum-phase subsystem from r to u

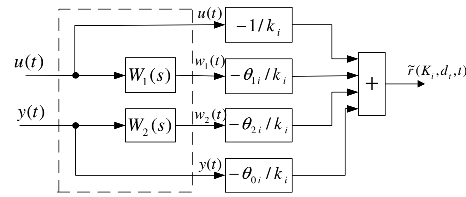

Fig. 3. Fictitious reference signal of the controller in Fig.2

Denote by d_τ the time-truncation of d. Thus, d_τ represents past experimental plant data up to current time τ. Given past data d_τ, we denote by D_τ the set of signals in $\mathcal{R} \times \mathcal{Y} \times \mathcal{U}$ that interpolate (*i.e.*, are consistent with) d_τ:

$$D_\tau \triangleq D(d_\tau) = \left\{ \ (r, y, u) \ \middle| \ (y_\tau, u_\tau) = d_\tau \ \right\}. \quad (3)$$

The set of candidate controllers K is denoted as $\mathbb{K}$. A scalar valued function, $V : \mathbb{K} \times \mathbb{D} \times \mathfrak{R} \rightarrow \mathfrak{R}$ is called a *cost function*. It is used to evaluate candidate controllers K based on past data d_τ in [12] and is also closely related to the cost functions employed in unfalsified control methods [17]. The cost $V(K, d, \tau)$ is assumed to be causally dependent on d; that is, for all $\tau > 0$ and all $d \in \mathfrak{L}_{2\mathfrak{e}}$,

$$V(K, d, \tau) = V(K, d_\tau, \tau)$$

Definition 3: (Unfalsification at a cost level) Given V, $\mathbb{K}$ and a scalar $\gamma \in \mathfrak{R}$, we say that a controller $K \in \mathbb{K}$ is *falsified* at time τ with respect to cost level γ by past measurement information d_τ if $V(K, d, \tau) > \gamma$. Otherwise the control law K is said to be *unfalsified* by d_τ. The least value of γ for which K is unfalsified by data d_τ is the *unfalsified cost level* of K at time τ. The set of unfalsified controllers having an unfalsified cost level of γ or less at time τ is called *unfalsified controller set* $\mathbb{K}_{unf}(\gamma, \tau)$. □

The foregoing definition of unfalsification at a cost level is a minor extension of the 'unfalsification' definition in [11], where only falsification with respect to cost level $\gamma = 0$ was considered. Clearly, for all time

$$\mathbb{K}_{unf}(\gamma_1, \tau) \subset \mathbb{K}_{unf}(\gamma_2, \tau)$$

if $\gamma_1 < \gamma_2$.

Definition 4: [11] Given plant data d and a candidate controller K, a *fictitious reference signal* for K, when it exists, is a hypothetical signal $\tilde{r}$ that would have produced exactly the same d had the controller K with noise $s = 0$ been in the feedback loop with the plant during the entire time period over which d were collected. □

For brevity, we denote $\tilde{r}(K, d)$ as $\tilde{r}_K$. When it exists, the induced map $d \mapsto \tilde{r}(K, d)$ is called the *fictitious reference signal generator* of the controller K, denoted in the operator form as $\mathfrak{K}_{CLI}(d)$ or $\mathfrak{K}_{CLI}$ (the subscript "CLI" stands for "causally left invertible", the reason for which will be explained shortly) .

Fictitious reference signal is in general not the true reference signal([11], [18]), hence the name fictitious. Given data $d = (u_0, y_0)$ and a controller K having graph K, the fictitious reference signals are the $\tilde{r}(K, d)$ that satisfy $(\tilde{r}(K, d), y_0, u_0) \in D_\tau \cap \mathsf{K}$. As noted in [17], fictitious reference signals allow unfalsified control performance goals of the form $J(r, y, u, \tau) \leq \gamma$ to be expressed in a form suitable for use in conjunction with the convergence lemma of [12]:

$$V(K, d, \tau) = J(\tilde{r}(K, d), d, \tau). \quad (4)$$

Example 1: For example, a controller, K, with the structure in Fig. 2 is such a controller. Its fictitious reference signal would be

$$\begin{aligned} \tilde{r}(K, d) &= \mathfrak{K}_{CLI}(d) \\ &= \frac{1}{k}[u - \theta_1 W_1(s) u - \theta_2 W_2 y - \theta_0 y], \quad (5) \end{aligned}$$

which is shown in Fig.3. Of particular interest is the case when $\mathfrak{K}_{CLI}$ is incrementally stable, in which case K has the following property. □

Definition 5: (CLI and SCLI Controllers). A controller K with input (r, y) and output u is said to be *Causally Left Invertible (CLI)*, if $\mathfrak{K}_{CLI}$ exists and is causal. If additionally $\mathfrak{K}_{CLI}$ is incrementally stable, then we say it is *Stably Causally Left Invertible (SCLI)*. □

Since the $\mathfrak{K}_{CLI}$ in (5) is the fictitious reference signal generator for the controllers of the standard form shown in Fig. 2, it is clear that such controllers are SCLI if and only if $W_1(s)$ and $W_2(s)$ are stable and $k_I \neq 0$.

Definition 6: (Unfalsified stability). Given an input output pair (v, w) of a system, we say that *stability of the system is unfalsified* by (v, w) if there exist $\beta, \alpha \geq 0$ such that (1) holds; otherwise, we say *the stability of the system is falsified* by (v, w). □

Unfalsified stability is determined from (1) based on the data from one infinite-duration experiment for one input, while 'stability' requires additionally that (1) hold for the data from every possible input.

Definition 7: (Cost-detectability). Let r denote the input and let $d \triangleq \begin{bmatrix} u \\ y \end{bmatrix}$ denote the resulting plant data collected while $\widehat{K}(t, d)$ is in the loop. Consider the adaptive control system $\Sigma(P, \widehat{K})$ of Figure 1 with input r and output

$d \triangleq \begin{bmatrix} u \\ y \end{bmatrix}$. The pair $(V, \mathbb{K})$ is said to be *cost-detectable* if, without any assumptions on the plant P and for every $\widehat{K}(t,d) \in \mathbb{K}$ with finitely many switching times, the following statements are equivalent:
1). $V(K^f, d, \tau)$ is bounded as τ increases to infinity;
2). Stability of the system $\Sigma(P, \widehat{K}(\tau, d))$ is unfalsified by (r, d). □

Cost-detectability is controller-dependent, but plant-independent. Cost-detectability is in a sense the dual of the plant-dependent, but controller-independent property of tunability introduced by Morse *et al.* [12]. Cost-detectability can substituted for tunability in adaptive stability proofs, allowing one to remove the need for most plant assumptions. The key ideas is that when we have cost-detectability, then we can use the cost $V(K, d, \tau)$ to reliably detect any instability exhibited by the adaptive system $\Sigma(P, \widehat{K}(\tau, d))$, *even when initially the plant is completely unknown*. The implication is that one can completely circumvent "model-mismatch" instability problems that would otherwise arise in cases where the plant turns out not to conform to assumptions.

Definition 8: ($\mathfrak{L}_{2e}$-gain-related cost). Given a cost/candidate controller-set pair $(V, \mathbb{K})$, we say that the cost V is $\mathfrak{L}_{2e}$*-gain-related* if for each $d \in \mathfrak{L}_{2e}$ and $K \in \mathbb{K}$,

1) $V(K, d, \tau)$ is monotone in τ,
2) the fictitious reference signal $\tilde{r}_\tau(K, d) \in \mathfrak{L}_{2e}$ exists and,
3) For every $K \in \mathbb{K}$ and $d \in \mathfrak{L}_{2e}$, $V(K, d, \tau)$ is bounded as τ increases to infinity if, and only if, stability is unfalsified by the input-output pair $(\tilde{r}_\tau(K, d)), d)$. □

The third condition in Definition 8 requires that the cost $V(K, d, \tau)$ be bounded uniformly with respect to τ if and only if $\mathfrak{L}_{2e}$-stability is unfalsified by $(\tilde{r}(K, d), d)$; this is the motivation for the choice of terminology '$\mathfrak{L}_{2e}$-gain-related'. Clearly, cost detectability implies $\mathfrak{L}_{2e}$-gain-relatedness. In fact, $\mathfrak{L}_{2e}$-gain-relatedness is simply cost detectability of V for the special case where $\widehat{K}(t, d) = K \in \mathbb{K}$ is a constant, unswitched non-adaptive controller .

$\mathfrak{L}_{2e}$-gain-related cost functions are easily identified. For example, consider weighted 'mixed sensitivity' cost functions of the form (cf. Tsao and Safonov [11])

$$V(K, d, \tau) \triangleq \max_{t \le \tau} \frac{\left\| \begin{bmatrix} [W_1\ (\tilde{r}(K, d) - y)]_t \\ [W_2 * u]_t \end{bmatrix} \right\|}{\|\tilde{r}_t(K, d)\|}. \qquad (6)$$

provided that the 'weights' W_1 and W_2 are stable operators with stable inverses. If the controllers K in $\mathbb{K}$ is SCLI, then clearly the cost (6) is $\mathfrak{L}_{2e}$-gain-related.

Like cost-detectability, the $\mathfrak{L}_{2e}$-gain-relatedness is a plant-independent concept. It turns out, as we shall show, that for some broad classes $\mathbb{K}$ of candidate controllers, the $\mathfrak{L}_{2e}$-gain-relatedness of $V(K, d, \tau)$ implies cost-detectability of $V(K, d, \tau)$. As we shall show, this together with the Morse-Mayne-Goodwin convergence lemma [12] will lead to a solution to the safe adaptive control problem.

IV. Properties of Adaptive Control

Assumption 1: The cost function $V(d, K, \tau)$ is $\mathfrak{L}_{2e}$-gain-related.

Assumption 2: Each candidate controller $K \in \mathbb{K}$ is SCLI;

Lemma 1: (*Stability of* $r \mapsto \tilde{r}_{K^f}$) Consider the adaptive system $\Sigma(P, \widehat{K}(t, d))$ of Figure 1. Suppose that controller switching eventually stops; i.e., for each r, there exists $t^f \geq 0$ such that,

$$\widehat{K}(t, d) = K^f \in \mathbb{K},\ \forall t > t^f.$$

If the final controller K^f is SCLI, then the mapping $r \mapsto \tilde{r}_{K^f}$ is stable with gain $\beta = 1$, i.e.,

$$\|(\tilde{r}_{K^f})_\tau\| \leq \|r_\tau\| + \alpha < \infty,\ \forall \tau \geq 0. \qquad (7)$$

Proof. Please refer to Appendix VII-A. □

Theorem 1: (*Cost-detectability*) Suppose Assumption 1 holds. Then, a sufficient condition for the pair $(V, \mathbb{K})$ to be cost-detectable is that the candidate controllers be SCLI. If, additionally, the candidate controllers are LTI, then it is also necessary.

Proof. For sufficiency, please refer to Appendix VII-B. It remains to establish necessity. Since by hypothesis Assumption 1 holds, it follows that $\tilde{r}(K, d)$ is defined and hence the fictitious reference signal generator $\mathfrak{K}_{CLI}$ exists and is causal. Suppose $\mathfrak{K}_{CLI}$ is not stable. Then, the dominant pole of $\mathfrak{K}_{CLI}$ has a non-negative real part, say $\sigma_0 \geq 0$. Since by definition cost-detectability is a plant independent property, it must hold for every plant mapping $\mathfrak{L}_{2e}$ into $\mathfrak{L}_{2e}$. Choose P so that $\Sigma(P, K)$ has its dominant closed loop pole at σ_0. Choose bounded duration inputs $r, s \in \mathfrak{L}_2$ so that the modes of $\mathfrak{K}_{CLI}$ and $\Sigma(P, K)$ associated with the unstable dominant poles with real part σ_0 are both excited. Then, since the fictitious reference signal $\tilde{r}(K, d)$ is unstable with the same growth rate $e^{\sigma_0 t}$ as the unstable closed loop response $d(t)$, there exists a constant β such that $\|d_\tau\| \leq \beta \|\tilde{r}_\tau(K, d)\| + \alpha\ \forall \tau \geq 0$ holds. Hence, by Assumption 1, the cost $\lim_{t \to \infty} V(K, d, t)$ is finite. On the other hand, stability of $\Sigma(P, K)$ is falsified by (r, d), which contradicts cost-detectability. Therefore the LTI controller K must be $SCLI$. □

V. Multiple Controller Adaptive Control (MCAC)

A. Algorithm 1

Consider a deterministic switching adaptive control system in Fig. 4, with reference signal r, and measurable control output and system output (u, y). For simplicity,

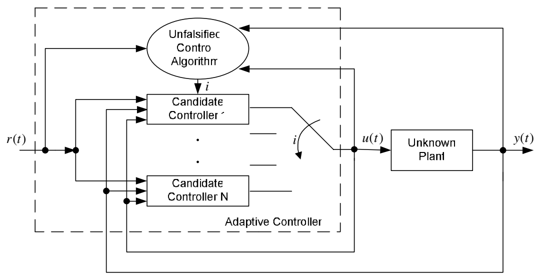

Fig. 4. Unfalsified adaptive control system using multiple controllers

noise, disturbance and initial conditions x_0 in Fig. 1 are assumed zero. The plant is considered to be unknown. We are given a finite set of candidate controllers $\mathbb{K} = \{K_i\}, i = 1, 2, \ldots, N$. At each time instant, say τ, if the current controller's cost exceeds the minimal cost by more than a pre-specific small number ϵ, the task is to identify and switch to the optimal controller $K^\star(\tau)$, *i.e.* .,

$$K^\star(\tau) = arg \min_{K_i \in \mathbb{K}} V(K_i, d, \tau) \text{ and } \widehat{K}(\tau) = K^\star(\tau), \quad (8)$$

where $V(K_i, d, \tau)$ is a given cost function. The steps of the algorithm are:

Algorithm 1: (ϵ-Hysteresis Algorithm [12])

1) Initialize: Let $t = 0$, $\tau = 0$; choose $\epsilon > 0$.
Let $\hat{K}(0) = K_0, K_0 \in \mathbb{K}$, be the first controller in the loop.
2) $\tau \leftarrow \tau + 1$.
If $V(\hat{K}(\tau - 1), d, \tau) > \min_{K \in \mathbb{K}} V(K, d, \tau) + \epsilon$ then $\hat{K}(\tau) \leftarrow \arg\min_{K \in \mathbb{K}} V(K, d, \tau)$,
else $\hat{K}(\tau) \leftarrow \hat{K}(\tau - 1)$
3) go to 2. □

Suppose the unfalsified controller set at each time τ is denoted by $\mathbb{K}_{unf}(\gamma, \tau)$. If the cost function is chosen so that $V(K, d, \tau)$ is monotone non-decreasing in τ for all $K \in \mathbb{K}$ and all $d \in \mathfrak{L}_{2e}$, then for each $\gamma \in \Re$ the unfalsified set $\mathbb{K}_{unf}(\gamma, \tau)$ shrinks monotonically as τ increases; that is, if $\tau_1 < \tau_2$, $\mathbb{K}_{unf}(\gamma, \tau_1) \subset \mathbb{K}_{unf}(\gamma, \tau_2)$. Note that candidate controllers can be reused and no controller is ever discarded here. Candidate controllers are grouped into different set according to their different cost levels.

Algorithm 1 is essentially the same as the 'hysteresis algorithm' of [12]. To solve the safe adaptive control problem, we will use Algorithm 1 with an $\mathfrak{L}_{2e}$-gain-related cost function—cf. Assumption 1 in section IV. An important property of Algorithm 1 is the *Hysteresis Switching Lemma* [12], which says essentially.

If $V(K, d, \tau)$ is monotone in τ and $\min_K \sup_{\tau,d} V(K, d, \tau) < M < \infty$, then there is a time $t^f < \infty$ beyond which the controller switching in Algorithm 1 stops and, moreover, $V(\hat{K}(\tau), d, \tau) < M + \epsilon \; \forall \tau$. □

This suggests that if $(V, \mathbb{K})$ is cost-detectable, then Algorithm 1 can be used to solve our safe adaptive control problem.

B. Stability of adaptive control system using Algorithm 1

Now consider MCAC using Algorithm 1. To deduce the following Theorem 2, it requires Assumptions 1 and 2 to be satisfied, and additionally, the following Assumption 3 should be satisfied too:

Assumption 3: The safe adaptive control problem is feasible.

Lemma 2: (*Convergence*) If (1) the cost function $V(K, \tau, d)$ is monotone increasing in τ and (2) the safe adaptive control problem is feasible, then using Algorithm 1 for any input r, there are finitely many switches among candidate controllers before switching stops, and the cost $V(K^f, d, \tau)$ remains bounded as τ increases to infinity.

Proof. Available in [1] for the finite controller set (also in [17] for the infinite controller set). □

Theorem 2: (*Stability Theorem*) If Assumptions 1, 2 and 3 hold, the unfalsified MCAC system is stable.

Proof. By Theorem 1, $(V, \mathbb{K})$ is cost-detectable. Also by Lemma 2, for every $r \in \mathfrak{L}_{2e}$ there are finitely many switches among candidate controllers before switching stops, and the cost $V(K^f, d, \tau)$ remains bounded as τ increases to infinity where $K^f \in \mathbb{K}$ denotes the final controller. Therefore, since $(V, \mathbb{K})$ is cost-detectable, it follows that stability of the system $\Sigma(P, \widehat{K}(\tau, d))$ is unfalsified by any possible (r, d), *i.e.*, the system is stable. □

VI. CONCLUSION

A mismatch between plant model assumptions and reality poses a risk for adaptive control systems. With a view towards reducing this risk, we have re-examined the Morse-Mayne-Goodwin hysteresis algorithm for adaptive control from the perspective of unfalsified control theory. We have proved that using an $\mathfrak{L}_{2e}$-gain-related cost function together with SCLI candidate controllers is sufficient to ensure that the hysteresis algorithm correctly detects destabilizing candidate controllers without assumptions on the plant, thereby eliminating plant-model mismatch instability problems. The conditions ensure that the cost function correctly orders controllers, so that the hysteresis algorithm yields a *safe adaptive controller* that is guaranteed to be stable without plant model assumptions, subject only to the feasibility requirement that there exists at least one stabilizing controller amongst the candidate controllers. For

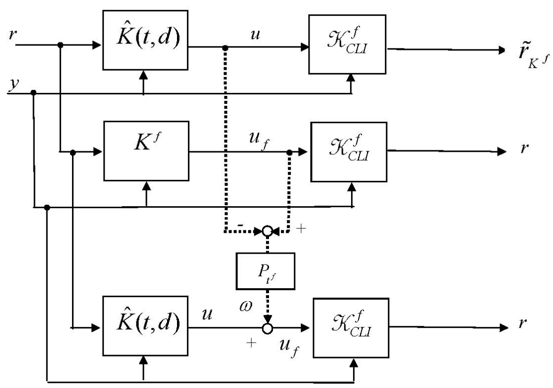

Fig. 5. Generators of the true and fictitious reference signals.

LTI candidate controllers, we proved that these conditions are also necessary.

VII. APPENDIX

A. *Proof of Lemma 1*

Stability of the mapping $r \mapsto \tilde{r}_{K^f}$ will be proven without constraints on linearity or time-invariance of the candidate controllers.

By assumption, controller switching eventually stops; *i.e.* for each r, there exists $t^f \geq 0$ such that,

$$\widehat{K}(t,d) = K^f \in \mathbb{K},\ \forall t > t^f.$$

Consider the control configuration in Fig. 5. The top branch generates the fictitious reference signal of the controller K^f. Its inputs are the measured data (y,u), and its output is $\tilde{r}_{K^f}$. The output is generated by the fictitious reference signal generator for the controller K^f, denoted $\mathfrak{K}^f_{CLI}$, which exists by assumption that K^f is SCLI, and is causal and incrementally stable. In the middle interconnection, the signal u_f, generated as the output of the final controller K^f excited by the actually applied signals r and y, is simply inverted by passing through the causal left inverse $\mathfrak{K}^f_{CLI}$. Finally, the bottom interconnection has the identical structure as the top interconnection (series connection of $\hat{K}(t,d)$ and $\mathfrak{K}^f_{CLI}$), except that it should generate the actual reference signal r. To this end, another input to the bottom interconnection is added (denoted ω), as shown in Fig. 5. This additional input ω can be thought of as a compensating (bias) signal, that accounts for the difference between the subsystems generating r and $\tilde{r}_{K^f}$ before the time of the last switch. In particular, it can be shown (as seen in Fig. 5) that $\omega \doteq P_{t^f}(u_f - u)$ (due to the fact that $u_f \equiv u,\ \forall t \geq t^f$).

As stated above, $\mathfrak{K}^f_{CLI}$ is incrementally stable. Thus, there exist constants $\tilde{\beta}, \tilde{\alpha} \geq 0$ such that

$$\begin{aligned} \|(\tilde{r}_{K^f} - r)_t\| &\leq \tilde{\beta}\cdot\|(u-u_f)_t\| + \tilde{\alpha} && (9)\\ &\leq \tilde{\beta}\cdot\|\omega_{t_f}\| + \tilde{\alpha} && (10)\\ &< \infty\ \forall t \geq 0. && (11) \end{aligned}$$

Whence by the triangle inequality for norms, inequality (7) holds with

$$\alpha = \tilde{\beta}\cdot\|\omega\|_{t_f} + \tilde{\alpha}. \qquad (12)$$

□

B. *Proof of Theorem 1*

The proof is immediately obtained from the definition of unfalsified stability by the data $(\tilde{r}_{K^f}, d)$ and Lemma 1.

□

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