Proceedings of the
44th IEEE Conference on Decision and Control, and
the European Control Conference 2005
Seville, Spain, December 12-15, 2005

Transverse Feedback Linearization of Multi-Input Systems

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***Abstract*— In this note the problem of feedback linearizing dynamics transverse to controlled invariant manifolds is considered for multi-input control affine systems. Transverse controllability indices are introduced which adapt the familiar notion of controllability indices to assist solving this particular problem. Sufficient conditions for transverse feedback linearization are presented.**

I. Introduction

Many interesting problems in control can be interpreted as the problem of stabilizing the system state to a set in the state space. This is for instance the case for the maneuver regulation (path following) problem. Set stabilization problems have been solved in a variety of ways, see for instance [1], [2], [3]. When the set in question has the structure of a smooth manifold, one approach to solve a set stabilization problem is to transform the dynamics transverse to the manifold into linear controllable form. This is referred to as transverse feedback linearization. Transverse feedback linearization was introduced by Banaszuk and Hauser in [4], where the authors investigate single-input systems and invariant manifolds given by periodic orbits. When feasible, transverse feedback linearization is attractive due to its simplicity and because it allows one to use a wealth of synthesis techniques for linear controllable systems.

In [5], [6], we presented conditions for a system to be transverse feedback linearizable with respect to an arbitrary controlled invariant manifold. Our conditions generalize the results in [4]. In this paper we consider multiple input systems and present sufficient conditions for global transverse feedback linearization.

II. Notation and mathematical preliminaries

Throughout this paper by a *manifold* is meant a *smooth manifold* and by a *submanifold* is meant an *embedded submanifold*. For details on the material presented in this section the reader may refer to [7], [8]. We denote by $\Phi_t^v(x)$ the flow of a smooth vector field v through the point x. Given a distribution D, let $D^\perp$ be its annihilator while $[D(x)]^\perp$ is the orthogonal complement of the vector space $D(x)$.

A. Tangent bundle, contractible manifolds

If $f : M \to N$ is a diffeomorphism of manifolds, then the tangent bundles TM and TN are said to be *equivalent,* denoted $TM \simeq TN$. An m-dimensional manifold M is said to be *parallelizable* if $TM \simeq M \times \mathbb{R}^m$. One has that M is parallelizable if and only if there exist m vector fields $v_1, \ldots, v_m : M \to TM$ such that

$$(\forall p \in M)\ T_pM = \mathrm{span}\{v_1(p), \ldots, v_m(p)\}.$$

A manifold M is *contractible* if there exists a point $p_0 \in M$ and a smooth function $H : M \times [0, 1] \to M$ such that for all $p \in M$

$$H(p, 1) = p, \quad H(p, 0) = p_0.$$

All contractible manifolds are parallelizable. All manifolds that are homeomorphic to $\mathbb{R}^n$ are contractible. The converse is false. Since by definition an n-dimensional manifold is *locally* diffeomorphic to $\mathbb{R}^n$, it is *locally* contractible. The next property of contractible manifolds is used in the sequel.

Theorem II.1 *([9]) Let M be a contractible submanifold of $\mathbb{R}^n$, $v_1, \ldots, v_r : M \to T\mathbb{R}^n$ a set of smooth vector fields in $\mathbb{R}^n$ and $\Delta = \mathrm{span}\{v_1, \ldots, v_r\}$ a distribution in $\mathbb{R}^n$. If Δ has constant dimension k on M, then there exist k smooth vector fields $w_1, \ldots, w_k : M \to \mathbb{R}^n$ such that*

$$(\forall p \in M)\ \Delta(p) = \mathrm{span}\{w_1, \ldots, w_k\}.$$

B. Tubular neighborhoods, retractions

Let M be an m-dimensional submanifold of $\mathbb{R}^n$. Give M the inner product $\langle\ ,\ \rangle : M \times M \to \mathbb{R}$ induced from $\mathbb{R}^n$. The *normal space* of M at p is defined as

$$T_pM^\perp = \{v \in \mathbb{R}^n : \langle v, w\rangle = 0\ \forall w \in T_pM\}.$$

The *normal bundle* of M, denoted $TM^\perp$, is the disjoint union of all normal spaces of M. It is a manifold in its own right, and has dimension $2n - m$. The projection $\pi : TM^\perp \to M$ defined by $\pi : (p, v) \mapsto p$ is smooth. One has that $TM^\perp \simeq M \times \mathbb{R}^{n-m}$ if and only if there exists a function $s : U \to \mathbb{R}^{n-m}$, where U is a subset of $\mathbb{R}^n$ containing M, such that

$$(\forall p \in M)\ \dim(\mathrm{Im}((ds)_p)) = n - m \text{ and } M = s^{-1}(0).$$

The function s is called a *submersion.*

If $\varepsilon > 0$ and $p \in M$, let $D_p(\varepsilon) = \{v \in T_pM^\perp : \|v\| < \varepsilon\}$. If $\epsilon : M \to \mathbb{R}_{>0}$ is a smooth function, let

$$D(\epsilon) = \bigcup_{p \in M} \{p\} \times D_p(\epsilon(p)).$$

Then $D(\epsilon) \subset TM^\perp$ and $M \times 0 = \{(p, v) \in D(\epsilon) : v = 0\} \subset D(\epsilon)$. $D(\epsilon)$ is referred to as the *disk sub-bundle*.

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Theorem II.2 *(Tubular neighborhood theorem) If M is a closed submanifold of $\mathbb{R}^n$ then there exists a smooth function $\epsilon : M \to \mathbb{R}_{>0}$ and a diffeomorphism $t : D(\epsilon) \to \mathbb{R}^n$ onto an open neighborhood of M in $\mathbb{R}^n$ such that $t|_{M\times 0} : (p, 0) \mapsto p$.*

The map t is called a *tubular map* and its image $t(D(\epsilon))$ is called a *tubular neighborhood* of M in $\mathbb{R}^n$, see Figure 1. It is an open set in $\mathbb{R}^n$. When M is compact there exists a constant $\epsilon > 0$ such that $D(\epsilon)$ is a tubular neighborhood. A tubular neighborhood of a contractible submanifold is a contractible manifold.

A *retraction* of a manifold N onto a submanifold M of N is a smooth function $r : N \to M$ such that $r|_M =$ identity on M. The tubular neighborhood theorem implies that any closed submanifold M of $\mathbb{R}^n$ admits a retraction of a tubular neighborhood of M, $t(D(\epsilon))$, onto M. Such a retraction is defined by this commutative diagram

$$\begin{tikzcd} D(\epsilon) \arrow[rr, "t"] \arrow[dr, "\pi"'] & & t(D(\epsilon)) \subset \mathbb{R}^n \arrow[dl, dashed, "\pi\circ t^{-1}"] \\ & M & \end{tikzcd}$$

where π is the natural projection of $TM^{\perp}$ onto M.

Given a submanifold M of $\mathbb{R}^n$, a tubular neighborhood $\mathcal{N}$ of M with associated retraction $r : \mathcal{N} \to M$, and an open subset $V \subset M$, we say that an open subset $U \subset \mathcal{N}$ is a *tubular neighborhood of V adapted from $\mathcal{N}$* if $U = r^{-1}(V)$.

III. Problem formulation and main result

Consider the control system

$$\dot{x} = f(x) + \sum_{i=1}^{m} u_i g_i(x) = f(x) + g(x)u, \qquad (1)$$

where the $f, g_1, \dots, g_m$ are smooth vector fields in $\mathbb{R}^n$. We assume throughout this paper that $\{g_1, \dots, g_m\}$ are linearly independent. Suppose we are given a pair (Γ^*, u^*), where Γ^* is a n^*-dimensional closed and connected submanifold of $\mathbb{R}^n$ which is controlled invariant (assume that $n - n^* \geq m$), and $u^* : \Gamma^* \to \mathbb{R}^m$ is a *friend* of Γ^*, i.e., a smooth feedback which makes Γ^* invariant:

$$\left(f + \sum_{i=1}^{m} u_i^* g_i \right)\Bigg|_{\Gamma^*} : \Gamma^* \to T\Gamma^*.$$

Denote $f^* := (f + \sum_i u_i^* g_i)|_{\Gamma^*}$. We want to solve the following problem.

Problem 1: *Find, if possible, a coordinate transformation*

$$\begin{aligned} \Xi = (r, s) : x &\mapsto (z, \xi) \\ \mathcal{N} &\to \Xi(\mathcal{N}) =: \mathcal{M} \subset \Gamma^* \times \mathbb{R}^{n-n^*} \end{aligned}$$

where $\mathcal{N}$ is a tubular neighborhood of Γ^*, and a feedback transformation

$$v \mapsto u = a(x) + b_1(x)v_1 + b_2(x)v_2,$$

where $u, v = \mathrm{col}(v_1, v_2) \in \mathbb{R}^m$, $a : \mathcal{N} \to \mathbb{R}^m$ is smooth, and $b = [b_1 \ b_2] : \mathcal{N} \to \mathbb{R}^{m\times m}$ is smooth and nonsingular on $\mathcal{N}$, such that

(i) *The restriction of Ξ to Γ^* is*

$$\begin{aligned} \Xi|_{\Gamma^*} : \Gamma^* &\to \Xi(\Gamma^*) \\ z &\mapsto (z, 0). \end{aligned}$$

(ii) *In new coordinates,* (1) *reads as:*

$$\begin{aligned} \dot{z} &= f^0(z, \xi) + g^1(z, \xi)v_1 + g^2(z, \xi)v_2 \\ \dot{\xi} &= A\xi + Bv_1 \end{aligned} \qquad (2)$$

where $v_1 \in \mathbb{R}^{\rho_0}$, ($\rho_0 \leq m$), B is full rank and the pair (A, B) is controllable.

For $i = 0, 1, \dots$, define the distributions

$$G_i = \mathrm{span}\{ad_f^j g_k : 0 \leq j \leq i, 1 \leq k \leq m\}.$$

Problem 1 involves decomposing the dynamics of (1) near the controlled invariant manifold Γ^* into a tangential component (the z subsystem) and a transversal component (the ξ subsystem) which is linear and controllable. This process also involves transforming the set of control inputs into two subsets: v_1 represents a group of controls that can be used to steer the system's state to Γ^*, v_2 represents controls that only affect the dynamics on the manifold. Our main result is a sufficient condition to solve Problem 1.

Theorem III.1 *Suppose that Γ^* is contractible. Then Problem 1 is solvable if*

(a) *$(\forall i \in \{0, \dots, n - n^* - 2\})$, G_i, is involutive in a neighborhood of Γ^*.*
(b) *$(\forall i \in \{0, \dots, n - n^* - 1\})$, G_i, is non-singular in a neighborhood of Γ^*.*
(c) *$(\forall i \in \{0, \dots, n - n^* - 1\})$, $\dim(T_p\Gamma^* + G_i(p))$ is constant on Γ^*.*
(d) *$(\forall p \in \Gamma^*)$ $\dim(T_p\Gamma^* + G_{n-n^*-1}(p)) = n$.*

It turns out that conditions (b)-(d) are also necessary (see Lemma V.1), while condition (a) is not. The theorem above is proved in Section VI. The proof relies on the notion of transverse controllability indices and the subsequent characterization of the directions transverse to $T_p\Gamma^*$ presented in the next section.

IV. Transverse controllability indices

In this section we adapt Brunovský's definition of controllability indices [10] to the framework investigated in this paper. Let $\mathcal{N}$ be a tubular neighborhood of Γ^* with associated retraction $r : \mathcal{N} \to \Gamma^*$. Let V be a contractible open subset of Γ^* and let U be a tubular neighborhood of V adapted from $\mathcal{N}$. Note that U is a contractible manifold (see Section II-B). If Γ^* is contractible then we replace the pair (V, U) by $(\Gamma^*, \mathcal{N})$. Since V is a contractible manifold, it is also parallelizable and there exist vector fields $v_1', \dots, v_{n^*}' : V \to TV$ such that $(\forall p \in V)$ $T_pV = \mathrm{span}\{v_1'(p), \dots, v_{n^*}'(p)\}$. For each $p \in V$ let

$$\begin{aligned} \rho_0(p) &:= \dim(T_pV + G_0(p)) - n^* \\ \rho_i(p) &:= \dim(T_pV + G_i(p)) - \dim(T_pV + G_{i-1}(p)), \end{aligned}$$

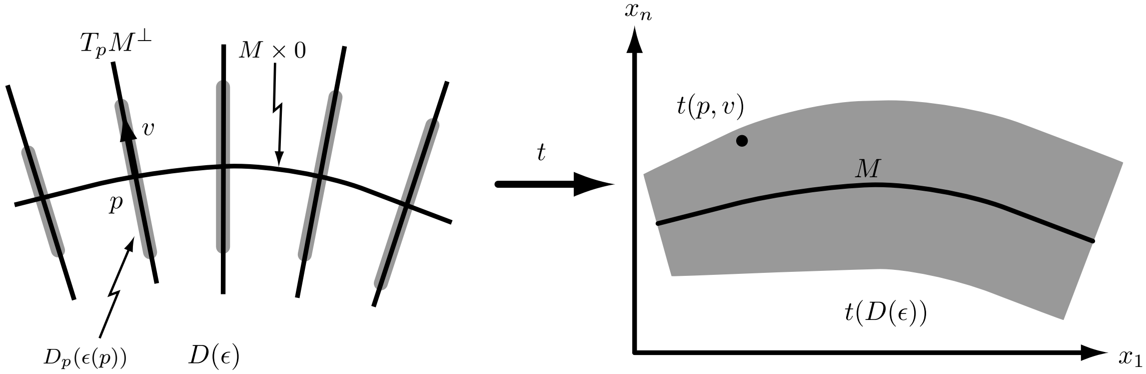

Fig. 1. Illustration of the tubular neighborhood theorem.

$i = 1, 2, \ldots$, so that $\dim(T_pV + G_i(p)) = n^* + \rho_0(p) + \ldots + \rho_i(p)$. When the ρ_i's are constant over V the list $\{\rho_1, \ldots, \rho_{n-n^*-1}\}$ is coordinate and feedback invariant and we have the next result.

Lemma IV.1 *Assume that, for all* $i \in \{0, \ldots, n - n^* - 1\}$,

$$(\forall p \in V) \dim(T_pV + G_i(p)) = \textit{constant}$$
$$(\forall p \in U) \dim(G_i(p)) = \textit{constant}.$$

Then $\rho_0 \geq \rho_1 \geq \cdots \geq \rho_{n-n^*-1}$ *and there exists a smooth feedback transformation on* U, $v \mapsto u = h + Kv$, *such that for all* $p \in V$ *and* $(\forall i \in \{0, \ldots, n - n^* - 1\})$ *the following holds*

$$T_pV + G_i(p) = T_pV \oplus \left(\bigoplus_{j=0}^{i} \mathrm{span}\{ad_f^j g_k : 1 \leq k \leq \rho_j\} \right).$$

Proof: By standing assumption, Γ^* (and hence V) is locally controlled invariant. Use the *friend* u^* to define a preliminary feedback transformation $u = u^* + v'$. Hereafter, without loss of generality, let $f|_{\Gamma^*} = f^*$ and $u = v'$. Let $\Pi(p) = \mathrm{Im}\left([\, g_1 \; \cdots \; g_m \,]\right)(p)$ be the image of the input vector fields prior to any feedback transformations. We will use this matrix function in the final steps of the construction. On V, define the distribution $\bar{G}_0(p) = [G_0(p) \cap T_pV]^\perp \cap G_0(p)$. First we show that $\bar{G}_0$ is a smooth, regular distribution. On V, $G_0(p) \cap T_pV$ is constant dimensional since

$$\dim(T_pV \cap G_0(p)) = \dim(T_pV) + \dim(G_0(p)) - \dim(T_pV + G_0(p)).$$

Since G_0 and T_pV are regular distributions and their intersection is constant dimensional, it follows that $G_0(p) \cap T_pV$ is smooth along with $[G_0(p) \cap T_pV]^\perp$. Thus $\bar{G}_0(p)$ is the intersection of two smooth, regular distributions. In addition, we now show that $\bar{G}_0$ has constant dimension ρ_0 and therefore that it is a smooth distribution. For all $p \in V$,

$$\begin{aligned} \dim(\bar{G}_0(p)) &= n - \dim(G_0(p) \cap T_pV) + \dim(G_0(p)) \\ &\quad - \dim([G_0(p) \cap T_pV]^\perp + G_0(p)) \\ &= \dim(G_0(p)) - \dim(G_0(p) \cap T_pV) \\ &= \rho_0. \end{aligned}$$

This plus the fact that $\bar{G}_0 \subset G_0$ and $\bar{G}_0(p) \cap T_pV = (G_0^\perp(p) + T_pV^\perp) \cap (G_0^\perp(p) + T_pV^\perp)^\perp = 0$, implies that

$$(\forall p \in V)\; G_0(p) = G_0(p) \cap T_pV \oplus \bar{G}_0(p).$$

By Theorem II.1 there exist ρ_0 vector fields $w_1, \ldots, w_{\rho_0}$ such that on V, $\bar{G}_0 = \mathrm{span}\{w_1, \ldots, w_{\rho_0}\}$. Write

$$w_j = \sum_{k=1}^{m} c_k^j g_k, \; j = 1, \ldots \rho_0 \tag{3}$$

where each $c_k^j : V \to \mathbb{R}$ is a $C^\infty(V)$ function. By construction $(\forall j \in \{1, \ldots, \rho_0\})$ $w_j \notin T_pV$. Let

$$[\; \tilde{g}_1 \;\; \cdots \;\; \tilde{g}_{\rho_0} \;] = [\; g_1 \;\; \cdots \;\; g_m \;]\, C_0$$

where C_0 is an $m \times \rho_0$ full rank matrix of real-valued functions obtained from (3). Using a similar procedure, find an additional $m - \rho_0$ vector fields $\tilde{g}_{\rho_0+1}, \ldots, \tilde{g}_m$ such that for all $p \in V$ $\mathrm{span}\{\tilde{g}_1, \ldots, \tilde{g}_m\}(p) = G_0(p)$.

We now have the desired decomposition on V

$$(\forall p \in V)\; T_pV + G_0(p) = T_pV \oplus \mathrm{span}\{\tilde{g}_1, \ldots, \tilde{g}_{\rho_0}\}(p).$$

In the new basis for G_0

$$(\forall p \in V)\; \tilde{g}_{\rho_0+1}(p), \ldots, \tilde{g}_m(p) \in T_pV.$$

Since, on V, $f(p) \in T_pV$, we have[1] $ad_f^j \tilde{g}_k(p) \in T_pV + G_{j-1}(p)$, $\rho_0 + 1 \leq k \leq m$, $j = 0, 1, \ldots$, and so $\rho_1, \ldots, \rho_{n-n^*-1} \leq \rho_0$. Geometrically, on V the vector fields $ad_f^j \tilde{g}_k(p)$, $\rho_0 + 1 \leq k \leq m$, cannot be used to generate directions in $T_pV + G_j(p)$ which are not contained in $T_pV + G_{j-1}(p)$. To simplify notation, relabel these new vector fields $\tilde{g}_1, \ldots, \tilde{g}_m$ as $g_1, \ldots, g_m$ and proceed to perform the induction step. This part of the proof is regrettably omitted due to space restrictions, however the induction step proceeds in a similar manner as above.

Let $\tilde{\Pi}(p) = \mathrm{Im}\left([\, g_1 \; \cdots \; g_m \,]\right)(p)$ be the image of the input vector fields generated during the above process so that so that for all $p \in V$, $\tilde{\Pi}(p) = \Pi(p)$. In conclusion, the feedback transformation, $v \mapsto u = \hat{h} + \hat{K}v := u^* + \left(\Pi^\top \Pi\right)^{-1} \Pi^\top \tilde{\Pi} v$, defined on V, has the required properties.

[1]To unify the notation, it is understood throughout that $G_k(p) = 0$ for $k < 0$.

Finally let $K = \hat{K} \circ r$ and $h = \hat{h} \circ r$ to obtain a feedback transformation defined on U. ■

The next result follows from the proof of Lemma IV.1.

Corollary IV.2 *Assume that Γ^* is contractible and*

$$\begin{aligned} &(\forall p \in \Gamma^*)(\forall i \in \{0,\dots,n-n^*-1\}) \\ &\qquad\qquad \dim(T_p\Gamma^* + G_i(p)) = \textit{constant} \\ &(\forall p \in \mathcal{N})(\forall i \in \{0,\dots,n-n^*-1\}) \\ &\qquad\qquad \dim(G_i(p)) = \textit{constant} \\ &(\forall p \in \Gamma^*) \dim(T_p\Gamma^* + G_{n-n^*-1}(p)) = n. \end{aligned}$$

Then there exists a smooth feedback transformation $v \mapsto u = \alpha + Kv$ defined on $\mathcal{N}$ such that for all $p \in \Gamma^*$, $\mathbb{R}^n$ is isomorphic to

$$T_p\Gamma^* \oplus \mathrm{span}\{ad_f^j g_k(p) : 0 \le j \le n-n^*-1, 1 \le k \le \rho_j\}. \tag{4}$$

In the sequel we will need to identify directions in the intersection $T_pV \cap G_i(p)$. To this end it is useful to define the integers

$$\begin{aligned} \mu_0(p) &:= \dim(T_pV \cap G_0(p)) \\ \mu_i(p) &:= \dim(T_pV \cap G_i(p)) - \dim(T_pV \cap G_{i-1}(p)) \\ \hat{n}_i(p) &:= \sum_{j=0}^{i} \mu_j. \end{aligned}$$

When the ρ_i's and μ_i's are constant over V we have the following result whose proof is omitted for brevity.

Lemma IV.3 *Assume that the conditions of Lemma IV.1 hold. Then, for each $i \in \{0,\dots,n-n^*-1\}$, there exist $\hat{n}_i$ vector fields $\hat{v}_k : U \to T\mathbb{R}^n$ such that, after the feedback transformation in Lemma IV.1, for all $p \in U$*

$$\begin{aligned} &G_i(p) = \\ &\mathrm{span}\{\hat{v}_1,\dots,\hat{v}_{\hat{n}_i}\} \oplus \left(\bigoplus_{j=0}^{i} \mathrm{span}\{ad_f^j g_k : 1 \le k \le \rho_j\} \right) \end{aligned}$$

where $(\forall p \in V)\ \mathrm{span}\{\hat{v}_1,\dots,\hat{v}_{\hat{n}_i}\}(p) \subset T_pV$.

Under the assumptions of Corollary IV.2, we are now ready to define *transverse controllability indices* $k_1,\dots,k_{\rho_0}$:

$$\begin{aligned} k_i := &\text{ number of integers in the list } \{\rho_0,\dots,\rho_{n-n^*-1}\} \\ &\text{ which are } \ge i. \end{aligned}$$

It is easily checked that $k_1 \ge \cdots \ge k_{\rho_0}$ and $\sum_i k_i = n - n^*$. Moreover there is a bijection between the list $\{\rho_0,\dots,\rho_{n-n^*-1}\}$ and the list $\{k_1,\dots,k_{\rho_0}\}$. Using Corollary IV.2 and IV.3, and following Wonham's construction in [11], it is not difficult to see that a reordering of the vector fields in (4) results in the next array of n independent vector fields on $\mathcal{N}$:

$$\begin{array}{l|l} 1 & \hat{v}_1,\dots,\hat{v}_{\hat{n}_0},\ g_1,\dots,g_{\rho_0};\ \cdots\ ;\ \dots, \\ & \quad \hat{v}_{\hat{n}_{k_{\rho_0}-1}},\ ad_f^{k_{\rho_0}-1} g_1,\dots,ad_f^{k_{\rho_0}-1} g_{\rho_0}; \\ 2 & \hat{v}_{\hat{n}_{k_{\rho_0}-1}+1},\dots,\ \hat{v}_{\hat{n}_{k_{\rho_0}}};\ ad_f^{k_{\rho_0}} g_1,\dots,ad_f^{k_{\rho_0}} g_{\rho_0-1}; \\ & \quad \hat{v}_{\hat{n}_{k_{\rho_0-1}-1}},\ ad_f^{k_{\rho_0-1}-1} g_1,\dots,ad_f^{k_{\rho_0-1}-1} g_{\rho_0-1}; \\ & . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\ \rho_0 - 1 & \hat{v}_{\hat{n}_{k_3-1}+1},\dots,\ \hat{v}_{\hat{n}_{k_3}};\ ad_f^{k_3} g_1, ad_f^{k_3} g_2;\ \dots; \\ & \quad ad_f^{k_2-1} g_1, ad_f^{k_2-1} g_2; \\ \rho_0 & \hat{v}_{\hat{n}_{k_2-1}+1},\dots,\ \hat{v}_{\hat{n}_{k_2}};\ ad_f^{k_2} g_1;\ \dots\ ;\ ad_f^{k_1-1} g_1 \\ \rho_0 + 1 & v_{\hat{n}_{k_1-1}+1},,\dots,v_{n^*};. \end{array} \tag{5}$$

Here $\hat{v}_1,\dots,\hat{v}_{\hat{n}_{k_1-1}} : \mathcal{N} \to T\mathbb{R}^n$ are vector fields which restricted to Γ^*, pointwise form a partial basis for $T_p\Gamma^*$. The vector fields $v_{\hat{n}_{k_1-1}+1},\dots,v_{n^*} : \Gamma^* \to T\Gamma^*$ are vector fields defined solely on Γ^*, pointwise completing the basis of $T_p\Gamma^*$. The remaining vector fields of the array point-wise span all directions transverse to $T_p\Gamma^*$. By the construction in the proof of Lemma IV.1, $(\forall i \in \{1,\dots,m\})(\forall j \in \{k_i, k_i+1,\dots\})(\forall k \in \{i,\dots,m\})$ $ad_f^j g_k \in T_p\Gamma^* + G_{j-1}(p)$.

V. Necessary conditions

We present a set of necessary conditions to solve Problem 1.

Lemma V.1 *Suppose that Problem 1 is solvable. Then, for any $x \in \Gamma^*$, there exists a contractible neighborhood V of x in Γ^* such that, letting U be the tubular neighborhood of V adapted from $\mathcal{N}$,*

(a) *For all $p \in V$, $\dim(T_pV + G_i(p)) = \textit{constant}$, $0 \le i \le n-n^*-2$ (i.e., $\rho_0,\dots,\rho_{n-n^*-2} = \textit{constant}$)*
(b) *For all $p \in V$, $\dim(T_p\Gamma^* + G_{n-n^*-1}(p)) = n$ (i.e., $\sum_{i=0}^{n-n^*-1} \rho_i = n - n^*$)*
(c) *The controllability indices of (A,B) in (2) coincide with the transverse controllability indices of (1).*

Proof: Choose V small enough that it is covered by a coordinate chart (V,ϕ) of Γ^*. Since conditions (a)-(c) are coordinate and feedback independent, it is sufficient to show that they hold in (z,ξ) coordinates. Let $\tilde{U} := \Xi(U) \subset \mathcal{M}$ and $\tilde{V} := \Xi(V) = V \times 0$ (the latter equality follows from property (i) in Problem 1). Since U is a tubular neighborhood adapted from $\mathcal{N}$ we have $\tilde{V} \subset \tilde{U} \subset V \times \mathbb{R}^{n-n^*}$.

In (ϕ,ξ) coordinates we have that for any $p \in \tilde{V}$

$$\begin{aligned} &T_{\phi(p)}(\phi(\tilde{V})) + G_i(\phi(p),0) = \\ &\qquad \mathrm{Im}\left(\begin{bmatrix} I_{n^*} & \star & \star & \dots & \star \\ 0_{n-n^* \times n^*} & B & AB & \dots & A^iB \end{bmatrix} \right). \end{aligned} \tag{6}$$

The matrix B is full rank from which it immediately follows that $T_{\phi(p)}(\phi(\tilde{V})) + G_0(\phi(p),0)$ has constant rank which combined with (6) proves (a). By controllability of the pair (A,B), one also has that $T_{\phi(p)}(\phi(\tilde{V})) + G_{n-n^*-1}(\phi(p),0) = \mathbb{R}^n$. From (6) it is also clear that

$$\rho_i = \mathrm{rank}([B\ \cdots\ A^iB]) - \mathrm{rank}([B\ \cdots A^{i-1}B])$$

and property (c) holds. ■

VI. Proof of the main result

The following result is used in the proof of Theorem III.1.

Theorem VI.1 *Problem 1 is solvable if and only if there exist ρ_0 smooth functions $\alpha_1, \ldots, \alpha_{\rho_0} : U \to \mathbb{R}$, where U is a neighborhood of Γ^* in $\mathbb{R}^n$, such that*

(1) $\Gamma^* \subset \{x \in U : \alpha_i(x) = 0,\ i = 1, \ldots, \rho_0\}$

(2) *The system*

$$\begin{aligned} \dot{x} &= f(x) + \sum_{i=1}^{\rho_0} u_i g_i(x) \\ y' &= \alpha(x) \end{aligned} \tag{7}$$

has uniform vector relative degree $\{k_1, \ldots, k_{\rho_0}\}$ over Γ^*.

The proof is conceptually identical to the proof of an analogous result in [12].

Proof: ($\Rightarrow$) Suppose that Problem 1 is solvable. By Lemma V.1, part (c), the pair (A, B) has controllability indices $k_1, \ldots, k_{\rho_0}$. Thus, without loss of generality, we can assume that the pair (A, B) is in Brunovský normal form

$$A = \mathrm{diag}\{A^1, \ldots, A^{\rho_0}\},\ B = \mathrm{diag}\{B^1, \ldots, B^{\rho_0}\},$$

with $A^i \in \mathbb{R}^{k_i \times k_i}$ and $B^i \in \mathbb{R}^{k_i \times 1}$ given by

$$A^i = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \ldots & 1 \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix},\ B^i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

We define α_i's in (z, ξ) coordinates. Let $\alpha = (\alpha_1, \ldots, \alpha_{\rho_0}) : \mathcal{M} \to \mathbb{R}^{\rho_0}$, $(z, \xi) \mapsto C\,\xi$, where

$$C = \mathrm{diag}\{C^1, \ldots, C^{\rho_0}\},\ C^i = [1\ 0\ \ldots\ 0] \text{ (length } k_i).$$

This choice of $\alpha_1, \ldots, \alpha_m$ satisfies conditions (1) and (2).

($\Leftarrow$) The existence of smooth functions $\alpha_1, \ldots, \alpha_{\rho_0} : U \to \mathbb{R}$ yielding a uniform vector relative degree $\{k_1, \ldots, k_m\}$ (with $\sum_i k_i = n - n^*$) over Γ^* implies, by[2] [13, Proposition 11.5.1], that there exists a coordinate transformation $\Xi = (r, s) : \mathcal{N} \to \Xi(\mathcal{N}) \subset \mathcal{Z}^* \times \mathbb{R}^{n-n^*}$, where $\mathcal{N} \subset U$ is a tubular neighborhood of $\mathcal{Z}^*$, yielding the normal form (2), where $\mathcal{Z}^* := \{x : s(x) = 0\}$ is the zero dynamics manifold of the system (7). For any $\bar{x} \in \Gamma^*$, one has $\alpha(\bar{x}) = 0$ and, since $\bar{x}$ belongs to a controlled invariant manifold (Γ^*), it follows that $\bar{x} \in \mathcal{Z}^*$ as well. We have thus shown that $\Gamma^* \subset \mathcal{Z}^*$. Since Γ^* and $\mathcal{Z}^*$ are two connected, closed submanifolds of the same dimension and $\Gamma^* \subset \mathcal{Z}^*$, it follows that $\Gamma^* = \mathcal{Z}^*$. ■

We are now ready to prove the main result of this paper.

Proof of Theorem III.1: Conditions (a) - (c) allow us to apply Lemma IV.1 and Lemma IV.3. Apply the smooth feedback transformation $v \mapsto u = h + Kv$ defined in $\mathcal{N}$ defined therein. We proceed to construct the vector valued function α satisfying the conditions of Theorem VI.1. Consider the n independent vector fields of (5).

Choose any point $p_0 \in \Gamma^*$ as the origin for generating S-coordinates by flowing along the vector fields in (5). Note that all of these vector fields are well defined in $\mathcal{N}$ except for $v_{\hat{n}_{k_1-1}+1}, \ldots, v_{n^*}$ which are defined everywhere on Γ^* and we use these vector fields to generate the mapping $F^0 : (F^0)^{-1}(W) \to W \subset \Gamma^*$

$$\begin{aligned} F^0 : S^0 = (s_1^0, \ldots, &s^0_{n^*-\hat{n}_{k_1-1}}) \\ &\mapsto \Phi^{v_{\hat{n}_{k_1-1}+1}}_{s^0_{n^*-\hat{n}_{k_1-1}}} \circ \cdots \circ \Phi^{v_{n^*}}_{s_1^0}(p_0). \end{aligned}$$

Use the remaining vector fields in (5) to define a sequence of mappings $F_i^{k_j} : (F_i^{k_j})^{-1}(U_i^{k_j}) \to U_i^{k_j} \subset \mathcal{N}$, $j \in \{1, \ldots, \rho_0\}$, $i \in \{k_{j+1}, \ldots, k_j - 1\}$ associated with each layer of bracketing in the array (5). Each map $F_i^{k_j}$ consists of the composition of flows of vector fields which at each point on Γ^* are in $G_i(p)$, not in $G_{i-1}(p)$ (let $k_{\rho_0+1} = 0$ and $\hat{n}_{-1} = 0$)

$$\begin{aligned} F_i^{k_j} : S_i^{k_j} &= \left(s^{k_j}_{(i,1))}, \ldots, s^{k_j}_{(i,j+\mu_i)}\right) \\ &\mapsto \Phi^{\hat{v}_{\hat{n}_{i-1}+1}}_{s^{k_j}_{(i,j+\mu_i)}} \circ \cdots \circ \Phi^{\hat{v}_{\hat{n}_i}}_{s^{k_j}_{(i,j+1)}} \circ \Phi^{ad_f^i g_1}_{s^{k_j}_{(i,j)}} \circ \cdots \circ \Phi^{ad_f^i g_j}_{s^{k_j}_{(i,1)}}(p), \\ &\quad (1 \le j \le \rho_0),\ (k_{j+1} \le i \le k_j - 1). \end{aligned}$$

The notation $F_i^{k_j}$ can be understood as follows: The superscript k_j, $(1 \le j \le \rho_0)$, indicates the row of (5) used in the mapping. The index j in k_j reflects the number of input vector fields, i.e. $g_1(x), \ldots, g_j(x)$ appearing at each order of bracketing in the row. The subscript i, $(k_{j+1} \le i \le k_j - 1)$, gives the order of Lie bracketing. Specifically $F_i^{k_j}$ consists of the vector fields in G_i that are not in G_{i-1}, i.e. j input vector fields and a subset of the tangential vector fields, μ_i of them to be exact.

For $j \in \{1, \ldots, \rho_0\}$, let $F^{k_j} = F^{k_j}_{k_{j+1}} \circ \cdots \circ F^{k_j}_{k_j-1}$. Further compose these mappings to generate S-coordinates via the composite $F : F^{-1}(\mathcal{N}^0) \to \mathcal{N}^0 \subset \mathcal{N}$ defined as

$$F := F^{k_{\rho_0}} \circ \cdots \circ F^{k_1} \circ F^0(p_0). \tag{8}$$

The S-coordinates are given by $S = \mathrm{col}(S^0, \ldots, S^{k_{\rho_0}})$ where $S^{k_j} = \mathrm{col}(S^{k_j}_{k_j-1}, \ldots, S^{k_j}_{k_{j+1}})$. As candidate output functions $(\alpha_1, \ldots, \alpha_{\rho_0})$, choose the time ($S$-coordinate) associated with the highest order Lie bracket of each input vector field. Namely, for $i \in \{1, \ldots, \rho_0\}$ let α_i be the time spent flowing along $ad_f^{k_i-1} g_i$. With this choice for α, we must show that the conditions of Theorem VI.1 are satisfied.

The image of Γ^* in S-coordinates is given by

$$\begin{aligned} F(\Gamma^*) = \{S : s^{k_j}_{(i,1)} = \cdots = s^{k_j}_{(i,j)} = 0, \\ 1 \le j \le \rho_0,\ k_{j+1} \le i \le k_j - 1\}. \end{aligned}$$

The chosen outputs are included in the above set of times and therefore they are identically zero on Γ^*. This shows that $\Gamma^* \subset \{\alpha(x) = 0\}$. Since α_i represents the time flow along vector field $ad_f^{k_i-1} g_i$, we immediately have that for all

[2] While in [13, Proposition 11.5.1] the extra condition that certain vector fields be complete is assumed, here this condition is not needed because the normal form (2) is required to be valid in a neighborhood of Γ^*, rather than the entire $\mathbb{R}^n$.

$p \in \Gamma^*$, $L_{ad_f^{k_i-1} g_i}\alpha_i \neq 0$. In order to show that $L_{ad_f^{\ell} g_j}\alpha_i = 0$ for all $i \in \{1, \ldots, \rho_0\}$, $\ell < k_i - 1$, $j \in \{1, \ldots, m\}$ we appeal to S-coordinates.

Fix a set of times $S^{k_i}_{k_i-1}$, $S^{k_{i-1}}, \ldots, S^0$ to uniquely determine the point $x = F^{k_i}_{k_i-1} \circ F^{k_{i-1}} \circ \cdots \circ F^0(p_0) \in \mathcal{N}^0$. Use this point as the origin for the partial mapping $F^{k_{\rho_0}} \circ \cdots \circ F^{k_i}_{k_i-2}(x)$. The vector fields of this mapping are linearly independent in $\mathcal{N}^0$ so its image is a submanifold. Furthermore, the vector fields span an involutive distribution G_{k_i-2}, so the image of this map is the integral submanifold $\mathcal{G}_{k_i-2}(x)$ of G_{k_i-2}.

The dimension of the fixed times used to obtain the point x is exactly equal to $n - \dim(G_{k_i-2})$. This shows that in S-coordinates

$$F(\mathcal{G}_{k_i-2}(x)) = \{S : S^{k_i}_{k_i-1} = \text{const.},\ S^{k_{i-1}} = \text{const.},\ \ldots,\ S^0 = \text{const.}\}$$

and therefore $T_S F(\mathcal{G}_{k_i-2}(x)) = \mathrm{col}(0,\ I_{\dim(G_{k_i-2})})$. From this it immediately follows that $\left\langle d\alpha_i, ad_f^{\ell} g_j \right\rangle$ is zero for $i \in \{1, \ldots, \rho_0\}$, $\ell < k_i - 1$, $j \in \{1, \ldots, m\}$.

We are left to show that the $\rho_0 \times m$ decoupling matrix is full rank for any $p \in \Gamma^*$. This part of the proof is omitted. We conclude that the vector function $\alpha(x) = \mathrm{col}(\alpha_1(x), \ldots, \alpha_{\rho_0}(x))$ yields a vector relative degree of $\{k_1, \ldots, k_{\rho_0}\}$ thus satisfying condition (2) of Theorem VI.1. ■

Remark VI.1 *Observe that the above proof elucidates the conservativeness of the conditions of Theorem III.1. Theorem III.1 holds if the integer $n - n^*$ in conditions (a) - (d) is replaced with k_1 .*

Example VI.1: Consider the system

$$\dot{x} = \begin{bmatrix} 0 \\ x_4 - x_2x_3 \\ x_1 - x_3 \\ x_5 - x_2x_3 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ x_2 \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u_2. \qquad (9)$$

and the pair

$$(\Gamma^*, u^*) = (\{x : x_1 = x_2 = x_4 = x_5 = 0\}, 0).$$

Here Γ^* is a subspace and hence contractible. Simple calculations reveal $\rho_0 = 2$, $\rho_1 = \rho_2 = 1$ everywhere on Γ^* yielding transverse controllability indices $k_1 = 3$ and $k_2 = 1$. Since the constraints defining Γ^* satisfy property (1) of Theorem VI.1 it makes sense to see if any pair of constraints also satisfy property (2). There is only one choice for y' which yields a well defined relative degree near Γ^*, namely $y' = \mathrm{col}(x_1, x_5)$ with vector relative degree $\{1, 1\} \neq \{k_1, k_2\}$ and so property (2) fails to hold and it is not clear whether input-output linearization can be used to stabilize Γ^*. On the other hand, the sufficient conditions of Theorem III.1 provide an affirmative answer.

The retraction used to generate the feedback transformation of Lemma IV.1 has an especially simple form: $r : \mathrm{col}(x_1, x_2, x_3, x_4, x_5) \mapsto \mathrm{col}(0, 0, x_3, 0, 0)$. The result of the feedback transformation is

$$u = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} v.$$

The distributions G_0 and G_1 are involutive near Γ^* satisfying condition (a) of Theorem III.1. Conditions (b) and (c) are easily checked by writing down the expressions for the vector fields with $T_x\Gamma^* = \frac{\partial}{\partial x_3}$. Finally we have that $(\forall x \in \Gamma^*)\ \dim(T_x\Gamma^* + G_2(x)) = 5$ and condition (d) is satisfied. Applying the preliminary feedback above and following the procedure of Theorem III.1 we obtain the mapping $F(s)$. Taking the inverse, $F^{-1}(x)$ we obtain the function $\alpha(x) = (x_2 e^{-x_3 + x_3(0)}, x_1)$. System (9) with output $y' = \alpha(x)$ satisfies Theorem VI.1 and we can now employ an input-output linearization approach to stabilizing Γ^*. △

VII. Conclusions

This paper presents preliminary results headed toward a characterization of transverse feedback linearization for multi-input systems. The main contributions are a formal problem formulation, the introduction of transverse controllability indices, a methodology for conveniently arranging input vector fields (Lemma IV.1), non-checkable necessary and sufficient conditions (Theorem VI.1) and sufficient checkable conditions (Theorem III.1) for the solvability of Problem 1.

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