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# A New Adaptive Parameter Estimation Algorithm for Robust Constrained Predictive Control

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***Abstract*— Model predictive control (MPC) combined with adaptation mechanism has been one of the new research topics for the control of constrained uncertain systems. This paper proposes a new adaptive parameter estimation algorithm using input and output data which can explicitly evaluate that the upper bound of parameter estimation error will become smaller in the future. That is, this method enables us to predict the monotonously decreasing future worst-case estimation error bound of uncertain system parameters. This distinctive feature contributes to the development of less conservative robust adaptive-type MPC schemes. Moreover, it basically can be incorporated into not only state-feedback but also output-feedback MPC. As one of the applications, we first present how the discrete-time adaptive state-feedback MPC scheme can be developed based on the proposed method. Then the numerical example shows its key features.**

## I. Introduction

Model predictive control (MPC) is one of the most promising and significant control approaches because of its ability to handle control problems for the systems having input, state and output constraints (see [9] for an overview). It determines the input sequence by solving on-line, at each time step, an open-loop constrained finite horizon optimization problem and subsequently only the first control in this sequence is implemented. In this approach, a model is used to predict the future behavior of the real system and yield an optimal open-loop control. Therefore, a model quality plays a vital role in MPC, but in reality there always exist model uncertainties and these may cause a significantly large effect on the system performance.

One of the attractive ways to cope with such modeling uncertainties is to use an adaptive-type MPC scheme which updates the prediction model on-line, at each time step, based on the various measurement data [4][5][9]. However, as far as the authors know, there are few reports on the development of MPC method combined with adaptation mechanism for the full constrained systems [9] other than our own [4][5]. One of the main reasons is that it is difficult under both a receding horizon control strategy and a presence of adaptation mechanism to guarantee the fulfillment of system constraints. Moreover, it seems to be extremely difficult to guarantee both the feasibility and the stability theoretically.

One of the approaches to overcome these problems is to predict the estimation error bound of uncertain parameters over the prediction horizon while updating parameters on-line at each time step. Then, in order to guarantee the robust stability, the given constraints are tightened based on the information for the future estimation error bound. Finally, the constrained finite horizon optimal control problem based on these modified constraints is solved to determine the control input. As one of possible approaches, we proposed an adaptive parameter estimation method in recent research [4][5]. However, in practice, it was derived under somewhat restrictive assumption that the system states are exactly known. Thus, there is a drawback such that it may be difficult to extend it to output-feedback MPC schemes.

In this paper, we first present a new adaptive parameter estimation algorithm using only input and output data. Therefore, this approach could be applied into not only state-feedback but also output-feedback MPC schemes. It is derived based on the moving horizon estimation and enables us to predict the monotonously decreasing future worst-case estimation error bound of uncertain system parameters. Then, in order to show its effectiveness, we present how the discrete-time adaptive state-feedback MPC scheme can be developed based on the proposed method. This is one of the extensions of adaptive MPC method for continuous systems that we have proposed previously [4] to constrained discrete systems. Finally, the advantages of using the proposed approach are illustrated by numerical examples.

**Notation and Preliminaries:** $x_i$ denotes the $i$th element of a vector $x$. The notation $\|x\|$ and $\|x\|_p$ denote the Euclidean and $p$-norms of a vector $x$; when $x$ is a scalar $|x|$ denotes its absolute value. We denote the largest singular value of a matrix $M$ as $\overline{\sigma}(M)$. The maximum and minimum eigenvalues of a matrix $M$ are denoted by $\overline{\lambda}(M)$ and $\underline{\lambda}(M)$, respectively. The notation $M \succ 0$ means that $M$ is a symmetric positive definite; $M^{\frac{1}{2}}$ denotes the unique positive definite square root of $M \succ 0$.

## II. New Adaptive Parameter Estimation

### A. System Description

Consider a $n$th order single-input single-output (SISO) discrete systems represented as follows:

$$A(z)y(t) = B(z)u(t) \qquad (1)$$

where only $y(t) \in \mathbb{R}$ and $u(t) \in \mathbb{R}$ are available for measurement and $A(z)$ and $B(z)$ are in the form

$$\begin{aligned} A(z) &:= 1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_n z^{-n} \\ B(z) &:= b_1 z^{-1} + b_2 z^{-2} + \cdots + b_m z^{-m} \end{aligned} \qquad (2)$$

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for $n \geq m$. In (2), $a_i$ and $b_i$ denote the uncertain time-invariant system parameters. A general parameterization of (1) is given to separate uncertain parameters from the known signals as follows:

$$\begin{aligned} y(t) &= -\sum_{i=1}^{n} a_i y(t-i) + \sum_{i=1}^{m} b_i u(t-i) =: \phi^T(t)\theta^*, \\ \theta^* &:= [-a_1, -a_2, \cdots, -a_n, b_1, b_2, \cdots, b_m]^T, \\ \phi(t) &:= [y(t-1), \cdots, y(t-n), u(t-1), \cdots, u(t-m)]^T, \end{aligned}$$

where $\phi(t) \in \mathbb{R}^{n+m}$ is the regression vector at time step $t$ and $\theta^* \in \mathbb{R}^{n+m}$ is a vector of uncertain system parameters to be estimated. We assume that the initial estimate $\theta_0$ of $\theta^*$ and its estimation error bound $\nu_0$ satisfying

$$\|\theta_0 - \theta^*\| \leq \nu_0 \tag{3}$$

are given. The aim of the parameter estimation procedure is to estimate uncertain $\theta^*$ on-line recursively using all available data.

### B. New Adaptive Parameter Estimation Algorithm

In this subsection, we propose the following recursive parameter estimation algorithm. It is assumed that $\phi(\tau) := 0$ for $\tau < 0$ and $\alpha \geq 0$. The estimation horizon $N_e$ and $\kappa$ are design parameters. We define

$$\begin{aligned} \Gamma(t) &= \sqrt{\overline{\lambda}(\Xi^T(t)\Xi(t))}, \\ \Xi(t) &:= I_n - \frac{\kappa \sum_{s=t}^{t-N_e+1} \phi(s)\phi^T(s)}{\alpha + \sum_{s=t}^{t-N_e+1} \phi^T(s)\phi(s)}, \end{aligned} \tag{4}$$

and

$$\begin{aligned} F_1(t) &:= \frac{\sum_{s=t}^{t-N_e+1} \phi(s)y(s)}{\alpha + \sum_{s=t}^{t-N_e+1} \phi^T(s)\phi(s)}, \\ F_2(t) &:= \frac{\sum_{s=t}^{t-N_e+1} \phi(s)\phi^T(s)}{\alpha + \sum_{s=t}^{t-N_e+1} \phi^T(s)\phi(s)} \end{aligned} \tag{5}$$

Then, the parameter estimation algorithm at the current time instant $t$ is described as follows:

**Step 0**: At the update starting time $t = 0$, initialize $\overline{\gamma}(0)$ and $f_i(0)$ as follows and go to Step 2.

$$\overline{\gamma}(0) = \Gamma(0),\ f_i(0) = F_i(0),\ i = 1, 2.$$

**Step 1**: Let $t = t + 1$. Then if $\overline{\gamma}(t-1) \leq \Gamma(t)$, update $\overline{\gamma}(t)$ and $f_i(t)$ as follows

$$\overline{\gamma}(t) = \overline{\gamma}(t-1),\ f_i(t) = f_i(t-1),\ i = 1, 2,$$

otherwise

$$\overline{\gamma}(t) = \Gamma(t),\ f_i(t) = F_i(t),\ i = 1, 2.$$

Then, go to Step 2.
**Step 2**: Apply the following parameter update law.

$$\theta(t) = \theta(t-1) + \kappa(f_1(t) - f_2(t)\theta(t-1)) \tag{6}$$

Then, at the next time instant, go to Step 1.

One of the key features of this algorithm is to use the summation $F_1(t)$ and $F_2(t)$ as in (5) over the estimation horizon $N_e$. The value of $\overline{\gamma}$ determined in the above algorithm at time $t$ satisfies

$$\overline{\gamma}(t) := \min_{0 \leq t_i \leq t} \sqrt{\overline{\lambda}(\Xi^T(t_i)\Xi(t_i))} \tag{7}$$

where $t_i$ denotes the time instant. Although the future value of $\|\tilde{\theta}(t+k)\|$ where $\tilde{\theta}(t) := \theta(t) - \theta^*$ is unknown, its upper bound $\nu_{t+k|t}$ can be predicted as follows:

$$\textstyle\sum_{S1} : \begin{array}{l} \nu_{t+k+1|t} = \overline{\gamma}(t)\nu_{t+k|t}, \\ \nu_{t|t} = \nu(t),\ k = 0, 1, \cdots, N-1, \end{array} \tag{8}$$

where $\nu_{t+k|t}$ denotes the predicted upper bound of the parameter estimation error at the current time $t$ and $N$ is any positive constant. $\nu_{t+k|t}$ in (8) satisfies the following lemma.

*Lemma 1: The parameter estimation error bound satisfies*

$$\|\tilde{\theta}(t+k)\| \leq \nu_{t+k|t}, \quad k = 0, 1, \cdots, N. \tag{9}$$

The scalar system (8) shows that the proposed algorithm tries, at each time instant, to choose the "best" data set in the sense that the predicted estimation error bound $\nu_{t+k|t}$ is minimized more rapidly than that of the previous time instant. As mentioned in Section I, this approach can predict the future worst-case estimation error bound explicitly as in Lemma 1 and, since only input and output data are used, it may be applicable to both state-feedback and output-feedback MPC schemes. In the next section, we present how this approach can be incorporated into a state-feedback MPC to develop the adaptive-type MPC scheme. It is one of the extensions of the adaptive MPC method for the constrained continuous-time systems that we have previously proposed [4] to the constrained uncertain discrete-time linear systems.

*Remark 1:* The conventional parameter estimation method [1][7] which is described as

$$\theta(t) = \theta(t-1) + \frac{\kappa\phi(t)}{\alpha + \phi^T(t)\phi(t)}(y(t) - \phi^T(t)\theta(t-1))$$

may can be incorporated into a MPC method to develop an adaptive-type MPC algorithm. However, this method makes the MPC too conservative in the sense that the future value of error bound $\nu_{t+k|t}$ is fixed. That is, since $\overline{\gamma}(t) = 1$ in (8) at all time instants, $\nu_{t+k|t} = \nu(t)$ for $k = 1, \cdots, N$. Therefore, although the estimation error bound $\|\tilde{\theta}(t+k)\|$ could be decreased, it cannot be considered and evaluated explicitly over the prediction horizon by the conventional methods.

## III. Adaptive State-feedback MPC Algorithm

### A. Problem Formulation

A discrete-time linear dynamical system $\sum_{G^*}$ described by the difference equation can be derived from an equivalent input-output representation in (1) as follows:

$$\textstyle\sum_{G^*} : \begin{array}{l} x(t+1) = A(\theta^*)x(t) + Bu(t), \\ y(t) = Cx(t),\ x(0) = x_0, \end{array} \tag{10}$$

where

$$A(\theta^*) = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_n \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ & I_{n-1} & \vdots & \mathbf{0} \end{bmatrix} \in \mathbb{R}^{n\times n},$$
$$B = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}^T \in \mathbb{R}^n,$$
$$C = \begin{bmatrix} b_1 & \cdots & b_m & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^n, \tag{11}$$

and $C \in \mathbb{R}^n$ is assumed to be known. Then, the uncertain parameter vector to be estimated is denoted as $\theta^* = -(a_1, \cdots, a_n)^T \in \mathbb{R}^n$. It is also assumed that the system (10) has state and input constraints, which should be fulfilled at all time instants $t \geq 0$, such as

$$\begin{aligned} &x(t) \in X, \quad X := \{x(t) \in \mathbb{R}^n : \ |x_i(t)| \leq \psi_i, \ \forall i\}, \\ &u(t) \in U, \quad U := \{u(t) \in \mathbb{R} : \ |u(t)| \leq \eta\}. \end{aligned} \tag{12}$$

We also assume that the following terminal set $X_f$

$$X_f := \{x(t) \in \mathbb{R}^n : V(x) \leq 1\}, \quad V(x) := \sqrt{x^T P x} \tag{13}$$

and $K_0 \in \mathbb{R}^n$, $P \succ 0$ are given and satisfy the similar assumption to those used in Kim, *et al.* [4][5] as follows:

*Assumption 1:*

$$1 \leq \sqrt{\underline{\lambda}(P)} \min_i \psi_i, \quad \| K_0 \| \leq \sqrt{\underline{\lambda}(P)} \ \eta$$

*Assumption 2:*

$$0 < \sqrt{1 - \frac{\underline{\lambda}(Q)}{\overline{\lambda}(P)}} + \frac{\overline{\sigma}(P^{\frac{1}{2}}B)\nu_0}{\sqrt{\underline{\lambda}(P)}} < 1,$$

*where*

$$Q := P - F^T P F, \quad F := A(\theta_0) + BK_0.$$

Note that Assumption 1 implies that the given feedback control $u(t) = K_0 x(t)$ always satisfies the given constraints in $X_f$. Assumption 2, on the other hand, implies that $X_f$ is a robustly invariant set by $u(t) = K_0 x(t)$ [2][3][4][5].

### B. Extended Adaptive MPC Algorithm

In this research, we adopt the following feedback control which consists of feedback gain $K(\theta(t))$, depending on $\theta(t)$ which is updated at each time instant, and open-loop trajectory $\tilde{u}(t)$ determined by solving the constrained finite horizon optimization problem.

$$\begin{aligned} &u(t) = K(\theta(t))x(t) + \tilde{u}(t), \\ &K(\theta(t)) := -\theta^T(t) + \theta_0^T + K_0. \end{aligned} \tag{14}$$

From (14), the real system $\sum_{G^*}$ can be described as follows:

$$\textstyle\sum_G : \begin{aligned} &x(t+1) = Fx(t) + B\tilde{u}(t) - Bd(t), \\ &d(t) := \tilde{\theta}^T(t)x(t). \end{aligned} \tag{15}$$

In MPC scheme, we predict the state $x(t+k)$ of $\sum_G$ based on the prediction model $\sum_{M^*}$ such as

$$\textstyle\sum_{M^*} : \begin{aligned} &\hat{x}_{t+k+1|t} = A(\theta_t(t+k))\hat{x}_{t+k|t} + B\hat{u}_{t+k|t}, \\ &\hat{x}_{t|t} = x(t), \ k = 0, 1, \cdots, N-1, \end{aligned} \tag{16}$$

where $\hat{x}_{t+k|t}$ and $\theta_t(t+k)$ denote the predicted value of $x(t+k)$ and the estimated parameters based on the proposed approach in (6) at the current time step $t$, respectively. The predictive control $\hat{u}_{t+k|t}$ in $\sum_{M^*}$ has a similar form in (14) as

$$\begin{aligned} \hat{u}_{t+k|t} &= K(\theta_t(t+k))\hat{x}_{t+k|t} + \tilde{u}_{t+k|t}, \\ &= (-\theta_t^T(t+k) + \theta_0^T + K_0)\hat{x}_{t+k|t} + \tilde{u}_{t+k|t}. \end{aligned} \tag{17}$$

Therefore, the model $\sum_{M^*}$ can be described as follows

$$\textstyle\sum_M : \ \hat{x}_{t+k+1|t} = F\hat{x}_{t+k|t} + B\tilde{u}_{t+k|t}. \tag{18}$$

As mentioned in Section II, we can predict the future worst-case upper bound $\nu_{t+k|t}$ of $\|\tilde{\theta}(t+k)\|$ as shown in Lemma 1 and thus $d_{t+k|t}$ in (15) is bounded as follows:

$$\begin{aligned} &d(t+k) \in D, \quad k = 0, 1, \cdots, N-1, \\ &D := \{d \in \mathbb{R} : \ \|d(t+k)\| \leq \nu_{t+k|t}\|x(t+k)\|\}. \end{aligned} \tag{19}$$

Thus, in order to guarantee the robust stability despite of the uncertain system parameters, the prediction error of $\hat{x}_{t+k|t}$ in $\sum_M$ due to the parameter estimation error $d(t+k)$ should be evaluated exactly. That is, the given constraints (12) should be tightened based on (9) and (19). However, since the estimation error $d(t+k) \in D$ depends also on the state $x(t+k)$ of the real system $\sum_G$ as shown in (19), it is not so easy to evaluate the prediction error from only the prediction model $\sum_M$. Therefore, we propose the following additional scalar system, comparison model, constructed based on *a priori* information on the worst-case estimation error bound in (8) and (9) as a similar method described in [3][4][5].

$$\textstyle\sum_{S2} : \begin{aligned} &w_{t+k+1|t} = a_{t+k|t} w_{t+k|t} + b|\tilde{u}_{t+k|t}|, \\ &a_{t+k|t} := \sqrt{1 - \frac{\underline{\lambda}(Q)}{\overline{\lambda}(P)}} + \frac{\overline{\sigma}(P^{\frac{1}{2}}B)\nu_{t+k|t}}{\sqrt{\underline{\lambda}(P)}}, \\ &b := \|P^{\frac{1}{2}}B\|, \ w_{t|t} = V(x(t)). \end{aligned} \tag{20}$$

This comparison model enables us to obtain an upper bound of the future value of $V(x)$ as shown in the following lemma.

*Lemma 2: For any $\tilde{u}_{t+k|t}$, the states of comparison model in (20) and real system $\sum_G$ in (15) satisfy*

$$V(x(t+k)) \leq w_{t+k|t}, \ k = 0, 1, \cdots, N. \tag{21}$$

Since we can evaluate the prediction error caused by the uncertain parameters $\theta^*$ using two scalar systems $\sum_{S1}$ and $\sum_{S2}$, the given constraint sets $X$ and $U$ can be modified based on these informations. The modified constraint sets $\hat{X}$ and $\hat{U}$ for $\hat{x}_{\cdot|t}$ and $\tilde{u}_{\cdot|t}$ depend on the states $\nu_{\cdot|t}$ of $\sum_{S1}$ and $w_{\cdot|t}$ of $\sum_{S2}$ as follows:

$$\begin{aligned} &\hat{X}(t+k+1, \nu_{t+k+1|t}, w_{t+k+1|t}) \\ &:= \{\hat{x} \in \mathbb{R}^n : |\hat{x}_i| \leq \psi_i - \hat{\psi}_i(t+k+1, \nu_{\cdot|t}, w_{\cdot|t}), \forall i\}, \\ &\hat{U}(t+k, \nu_{t+k|t}, w_{t+k|t}) \\ &:= \{K_0\hat{x} + \tilde{u} \in \mathbb{R} : |K_0\hat{x} + \tilde{u}| \leq \eta - \hat{\eta}(t+k, \nu_{\cdot|t}, w_{\cdot|t})\}, \end{aligned}$$

where

$$\begin{aligned}
&\hat{\psi}_i(t+k+1,\nu_{\cdot|t},w_{\cdot|t}) := \sum_{s=0}^{k} |\zeta_i(s)| \frac{\nu_{t+k-s|t}w_{t+k-s|t}}{\sqrt{\underline{\lambda}(P)}},\\
&\hat{\eta}(t+k,\nu_{\cdot|t},w_{\cdot|t}) := (\nu_{t+k|t}+\nu_0)(2\hat{\eta}_1+\hat{\eta}_3)+\hat{\eta}_2,\\
&\hat{\eta}_1(t+k,\nu_{\cdot|t},w_{\cdot|t}) := \sum_{s=0}^{k} \|\xi(s)\| \frac{\nu_{t+k-s|t}w_{t+k-s|t}}{\sqrt{\underline{\lambda}(P)}},\\
&\hat{\eta}_2(t+k,\nu_{\cdot|t},w_{\cdot|t}) := \sum_{s=0}^{k} |K_0\xi(s)| \frac{\nu_{t+k-s|t}w_{t+k-s|t}}{\sqrt{\underline{\lambda}(P)}},\\
&\hat{\eta}_3(t+k,\nu_{\cdot|t},w_{\cdot|t}) := \nu_{t+k|t}/\sqrt{\underline{\lambda}(P)},\\
&\zeta(s) := F^sB, \quad \xi(s) := \begin{cases} F^{s-1}B, & for\ s \neq 0 \\ 0 \quad , & for\ s = 0 \end{cases}
\end{aligned}$$

and $\zeta_i(s)$ denotes the $i$th row of $\zeta(s)$. The modified constraint sets $\hat{X}$ and $\hat{U}$ above and two scalar systems $\sum_{S1}$ and $\sum_{S2}$ satisfy the following key properties.

*Theorem 1: For a given $x(t) \in \mathbb{R}^n$, any open-loop trajectory $\tilde{u}_{t+k|t}$, $k = 0, \cdots, N-1$, which satisfies*

$$\begin{aligned}
&\hat{x}_{t+k+1|t} = F\hat{x}_{t+k|t} + B\tilde{u}_{t+k|t},\ \hat{x}_{t|t} = x(t),\\
&\nu_{t+k+1|t} = \overline{\gamma}(t)\nu_{t+k|t},\ \nu_{t|t} = \nu(t),\\
&w_{t+k+1|t} = a_{t+k|t}w_{t+k|t} + b|\tilde{u}_{t+k|t}|,\ w_{t|t} = V(x(t)),\\
&\hat{x}_{t+k+1|t} \in \hat{X}(t+k+1,\nu_{t+k+1|t},w_{t+k+1|t}),\\
&K_0\hat{x}_{t+k|t} + \tilde{u}_{t+k|t} \in \hat{U}(t+k,\nu_{t+k|t},w_{t+k|t}),
\end{aligned} \tag{22}$$

*also satisfies the constraints for the real system $\sum_{G^*}$ in (10)*

$$\begin{aligned}
&x(t+k+1) \in X,\\
&K(\theta_t(t+k))x(t+k) + \tilde{u}_{t+k|t} \in U,
\end{aligned} \tag{23}$$

*for all possible $d(t+k) \in D$.*

The constrained finite horizon optimization problem for the proposed adaptive MPC method is described for a given diagonal matrix $R \succ 0$ as follows:

**Finite horizon optimization problem for MPC:**

$$\begin{aligned}
&\min_{\tilde{u}} J(x(t),\tilde{u}_{t+k|t}) := \sum_{k=0}^{N-1} \tilde{u}_{t+k|t}^T R\tilde{u}_{t+k|t}\\
&subject\ to\ (22)\ and\\
&w_{t+k|t} \leq \omega,\ w_{t+N|t} \leq 1,\ \text{for}\ k = 0,1,\cdots,N-1.
\end{aligned} \tag{24}$$

At each time instant, the above finite horizon optimal control problem without the disturbance term $d(t+k)$ is solved online to determine the optimal control sequence based on the measured state $x(t)$. Then subsequently only the first control in this sequence is implemented. It is easily verified from Theorem 1 that, if the problem in (24) is feasible at each update time, the given constraints (12) are always satisfied. In (24), the additional constraints for $w_{t+k|t}$ were introduced to guarantee the feasibility at each time instant. The constant $\omega$ is a number satisfying $\omega \geq \max\{V(x_0), 1\}$ and the terminal condition $\omega_{t+N|t} \leq 1$ guarantees $x(t+N) \in X_f$ for the real system. Although $\omega$ is desired to be as large as possible for the feasibility at the current time instant, the value of $\omega$ should be bounded to guarantee the feasibility at the next time instant, which is verified in Section IV.

*Remark 2:* The third constraint in (22) is a nonlinear equation of $\tilde{u}_{t+k|t}$. By introducing a new variable $\chi_{t+k|t} \in \mathbb{R}$, we can modify this constraint to

$$\begin{aligned}
&w_{t+k+1|t} = a_{t+k|t}w_{t+k|t} + b\chi_{t+k|t},\\
&|\tilde{u}_{t+k|t}| \leq \chi_{t+k|t},\ w_{t|t} = V(x(t)),
\end{aligned} \tag{25}$$

and the cost function $J(x(t),\tilde{u}_{t+k|t})$ to $J(x(t),\chi_{t+k|t})$. Therefore, the modified optimization problem has only linear constraints and can be solved by the standard quadratic programming (QP) method with free variables $\tilde{u}_{t+k|t}$ and $\chi_{t+k|t}$.

## IV. Feasibility and Stability

In order to ensure that the optimization problem in (24) is feasible at each time step, we need the following assumption defining the upper bound of $\omega$ as mentioned in Section III.

*Assumption 3: The given $\omega$ ($\geq \max\{V(x_0), 1\}$) in (24) satisfies*

$$\begin{aligned}
&(i)\ \omega\nu(t)c_\zeta \leq \sqrt{\underline{\lambda}(P)} \min_i \psi_i - 1,\\
&(ii)\ \omega\big((\nu(t)+\nu_0)(2\nu(t)c_{\xi1}+1)+\nu(t)c_{\xi2}\big)\\
&\qquad \leq \eta\sqrt{\underline{\lambda}(P)} - \|K_0\|,
\end{aligned}$$

*where*

$$\begin{aligned}
&c_\zeta := \sum_{s=0}^{N-1} \|\zeta(s)\|_\infty,\ c_{\xi1} := \sum_{s=0}^{N-1} \|\xi(s)\|,\\
&c_{\xi2} := \sum_{s=0}^{N-1} |K_0\xi(s)|.
\end{aligned}$$

Note that Assumption 3 is a sufficient condition for Assumption 1. If Assumption 3 cannot be satisfied for any $\omega \geq \max\{V(x_0), 1\}$, we need to consider a smaller terminal set $X_f$ or modify the term $K_0$ of a feedback gain in (17). It is also important to notice that once the state is steered into the robust invariant constraint set $X_f$, the control law in (17) is completely switched to the feedback law $\hat{u} = K(\theta_t)\hat{x}$, since it is the optimal control in $X_f$ in terms of the cost function in (24). That is, the control law of the proposed method converges to the given feedback law $\hat{u} = K(\theta_t)\hat{x}$.

The following Theorem 2 and Lemma 3 describe the properties of the feasibility and stability of the proposed adaptive MPC method.

*Theorem 2: (Stability) Assume the optimization in (24) is feasible at $t = 0$ for $\omega$ which satisfies Assumption 3. Then the proposed MPC method has the following properties.*

*(i) The optimization in (24) is feasible at time $t > 0$,*
*(ii) $x(t)$ converges to the origin as $t \to \infty$.*

The following Lemma 3 is a key result to prove Theorem 2 and means that if the proposed MPC problem is feasible at $t = 0$ then the optimization problem is feasible at all time steps.

*Lemma 3: (Feasibility) Assume the optimization problem in (24) is feasible at the current time $t$ for $\omega$ which satisfies Assumption 3. Then, at the next time step $t + 1$,*

$$\tilde{u}_{t+k|t+1} = \begin{cases} \tilde{u}^*_{t+k|t}, & k = 1, 2, \cdots, N-1 \\ 0 \quad, & k = N \end{cases} \tag{26}$$

*could be a feasible solution of (24), where $\tilde{u}^*_{t+k|t}$ denotes the optimal solution at $t$.*

*Proof:* In order to prove Lemma 3, we use the following three facts, Lemmas 4, 5 and 6. However, the detailed proofs of these and Lemma 3 is omitted on account of limited space.

*Lemma 4: The scalar system $\sum_{S1}$ satisfies*

$$\nu_{t+k|t+1} \leq \nu_{t+k|t}, \quad k = 1, 2, \cdots, N.$$

*Lemma 5: For the given trajectory $\tilde{u}^*_{t+k|t}$, $k = 0, 1, \cdots, N-1$, and*

$$\tilde{u}_{t+k|t+1} = \tilde{u}^*_{t+k|t}, \quad k = 1, 2, \cdots, N-1,$$

*the scalar system $\sum_{S2}$ satisfies*

$$w_{t+k|t+1} \leq w_{t+k|t}, \quad k = 1, 2, \cdots, N.$$

*Lemma 6: Assume a predicted trajectory $w_{\cdot|t}$ in $\sum_{S2}$ satisfies*

$$w_{t+k|t} \leq \omega, \quad k = 0, 1, \cdots, N,$$

*for $\tilde{u}_{t+k|t}$ at the current time $t$. Then, for the input $\tilde{u}_{t+k|t+1}$, $k = 1, 2, \cdots, N-1$, in Lemma 3, we have*

$$\begin{aligned} &\underline{\psi} \leq \psi_i - \hat{\psi}_i(t+k+1, \nu_{t+k+1|t+1}, w_{t+k+1|t+1}) \\ &\underline{\eta} \leq \eta - \hat{\eta}(t+k+1, \nu_{t+k+1|t+1}, w_{t+k+1|t+1}) \end{aligned}$$

*at the next time step $t + 1$, where*

$$\begin{aligned} &\underline{\psi} := \min_i \psi_i - \omega\nu(t)/\sqrt{\underline{\lambda}(P)} \\ &\underline{\eta} := \eta - \frac{\omega}{\sqrt{\underline{\lambda}(P)}}((\nu(t) + \nu_0)(2\nu(t)c_{\xi 1} + 1) + \nu(t)c_{\xi 2}) \end{aligned}$$

*Proof of Theorem 2:* We show (i) by induction. The optimization problem is feasible at $t = 0$ by the assumption. Assume now it is feasible at each $t = i$ $(i = 1, \cdots, k)$. Then, from Lemma 3, the optimization problem is feasible at $t = i+1$. Therefore (i) is proved. In order to prove (ii), we next show that the optimal cost $J(x(t), \tilde{u}^*)$ is nonincreasing. At the time step $t + 1$, the feasible solution in Lemma 3 satisfies

$$J(x(t+1), \tilde{u}_{t+k|t+1}) \leq J(x(t), \tilde{u}^*_{t+k|t}) \tag{26}$$

since

$$\begin{aligned} &J(x(t+1), \tilde{u}_{t+k|t+1}) - J(x(t), \tilde{u}^*_{t+k|t}) \\ &= \sum_{k=1}^{N} \tilde{u}^T_{t+k|t+1} R \tilde{u}_{t+k|t+1} - \sum_{k=0}^{N-1} \tilde{u}^{*T}_{t+k|t} R \tilde{u}^*_{t+k|t} \\ &= -\tilde{u}^{*T}_{t|t} R \tilde{u}^*_{t|t} \ \leq \ 0. \end{aligned} \tag{27}$$

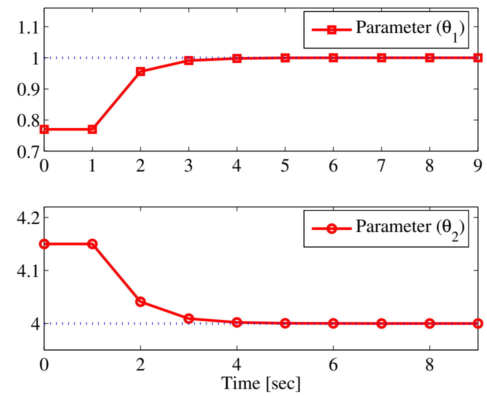


Fig. 1. Estimated parameter values

It is also satisfied that

$$J(x(t+1), \tilde{u}^*_{t+k|t+1}) \leq J(x(t), \tilde{u}_{t+k|t+1}) \tag{28}$$

from the optimality of $J(x(t+1), \tilde{u}^*_{t+k|t+1})$. From (26)and (28), the optimal cost is nonincreasing, that is,

$$J(x(t+1), \tilde{u}^*_{t+k|t+1}) \leq J(x(t), \tilde{u}^*_{t+k|t}). \tag{29}$$

Since the optimal cost is nonincreasing and bounded by 0 from below, it satisfies $J(x(t+1), \tilde{u}^*_{t+k|t+1}) \to c$ as $t \to \infty$ for a constant $c \geq 0$. This implies that, for each $\epsilon > 0$, there exists $t_1 \geq 0$ such that

$$0 \leq J(x(t), \tilde{u}^*_{t+k|t}) - J(x(t+1), \tilde{u}^*_{t+k|t+1}) < \epsilon, \quad \forall t \geq t_1. \tag{30}$$

But, from (26),(27) and (28), we have

$$\begin{aligned} \tilde{u}^{*T}_{t|t} R \tilde{u}^*_{t|t} &= J(x(t), \tilde{u}^*_{t+k|t}) - J(x(t+1), \tilde{u}_{t+k|t+1}) \\ &\leq J(x(t), \tilde{u}^*_{t+k|t}) - J(x(t+1), \tilde{u}^*_{t+k|t+1}). \end{aligned} \tag{31}$$

Thus, (30) and (31) imply

$$\tilde{u}^*_{t|t} \to 0 \text{ as } t \to \infty. \tag{32}$$

From (32), we can choose $t_1$ which satisfies

$$|\tilde{u}^*_{t|t}| \leq \epsilon_1, \ \forall t \geq t_1 \tag{33}$$

for any $\epsilon_1 > 0$. From $\sum_{S2}$ and (33), it follows that

$$V(x(t+1)) \leq a_{t_1|t_1} V(x(t)) + b|\tilde{u}^*_{t|t}| \leq \breve{a}V(x(t)) + \breve{b}, \tag{34}$$

where $\forall t \geq t_1$, $\breve{b} := b\epsilon_1$, $\breve{a} := a_{t_1|t_1}$. Therefore

$$\begin{aligned} V(x(t_1+k)) &\leq \breve{a}^k V(x(t_1)) + \sum_{s=0}^{k-1} \breve{a}^s \breve{b} \\ &= \breve{a}^k V(x(t_1)) + \frac{\breve{b}(1-\breve{a}^k)}{1-\breve{a}} \end{aligned} \tag{35}$$

and the right-hand side of (35) converges to $\breve{b}/(1-\breve{a})$ as $k \to \infty$. Therefore, $x(t) \to 0$ as $t \to \infty$, since $\breve{b} := b\epsilon_1$ can be chosen arbitrarily. $\square$

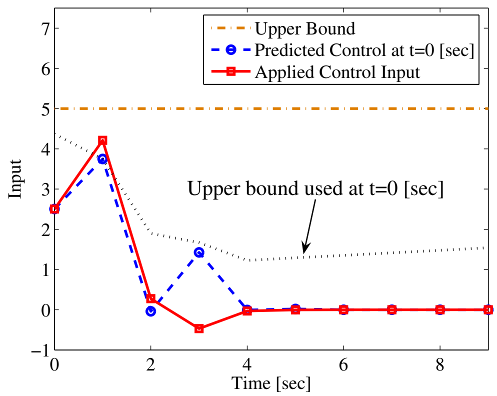


Fig. 2. Time plot of control input

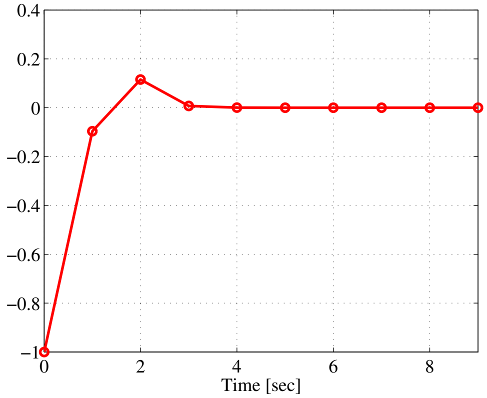


Fig. 3. Time plot of output

## V. Numerical Example

Consider the following linear time-invariant discrete-time system in a controllable canonical form:

$$x(t+1) = \begin{bmatrix} 1 & 4 \\ 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t), \;\; x_0 = \begin{bmatrix} -1 \\ -0.4 \end{bmatrix}$$
$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t). \tag{36}$$

We assume that the uncertain parameter vector as $\theta^* = (1,4)^T$ and the initial estimate $\theta_0$ and the estimation error bound $\nu_0$ such as

$$\theta_0 = [0.77,\ 4.15]^T, \quad \nu_0 = 0.275 \tag{37}$$

are given and satisfy (4). The state and input constraints are given as

$$|x_i(t)| \le 1.0, \quad |u(t)| \le 5.0, \quad i = 1, 2. \tag{38}$$

A feedback gain $K_0$ and matrix $P$ are chosen as

$$K_0 = [-0.7745 \;\; -4.1370], \;\; P = \begin{bmatrix} 15.0054 & -0.0689 \\ -0.0689 & 30.0054 \end{bmatrix}.$$

We choose the length of prediction horizon $N = 130$ and the terminal set as follows

$$X_f = \{x(t) \in \mathbb{R}^n : \; V(x(t)) \le 0.7\}. \tag{39}$$

For an adaptive parameter estimation algorithm, we choose the estimation horizon $N_e = 10$ and other design parameters $\alpha = 0.1$ and $\kappa = 1.8$. The estimated parameters, at each time step of MPC, by the proposed method in Section II are shown in Fig. 1. In Fig. 2, the solid line shows the applied control $u(t)$ and the dashed line shows the predicted control sequence $K_0\hat{x}_{t+k|t} + \tilde{u}_{t+k|t}$ at $t = 0$, which is obtained by the proposed adaptive MPC method in Section III. This shows that the control input $u(t)$ obtained by the proposed adaptive MPC method satisfies the given constraints (38). The dotted line shows the upper bound calculated based on the modified constraint sets $\hat{X}$ and $\hat{U}$ in (22) at time step $t = 0$[sec]. At each time instant, we iterate the same procedure and determine the input sequence. The trajectory of output which converge to zero as $t \to \infty$ is shown in Fig. 3.

## VI. Conclusion

In this paper, we have proposed a new adaptive parameter estimation algorithm. The recursive parameter estimation is performed, at each time step, based on only the known input and output data, which is different from the previous method that we have proposed. The key feature of this approach is that it can predict the monotonically decreasing future worst-case estimation error bound of the uncertain system parameters. It will ultimately contribute to the development of less conservative adaptive-type robust MPC schemes that guarantee the feasibility and the stability. As a first attempt to show its effectiveness, we have presented that the state-feedback adaptive MPC scheme based on this approach and robust MPC method can be developed. Moreover its performance has been verified in a numerical example. As a future work, we will develop the output-feedback MPC algorithm based on the parameter estimation method proposed in this paper.

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