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Convergence of Pseudospectral Methods for a Class of Discontinuous Optimal Control

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Abstract— **We consider the optimal control of feedback linearizable dynamic systems subject to mixed state and control constraints. The optimal controller is allowed to be discontinuous including bang-bang control. Although the nonlinear system is assumed to be feedback linearizable, in general, the optimal control does not linearize the dynamics. The continuous optimal control problem is discretized using pseudospectral (PS) methods. We prove that the discretized problem is always feasible and that the optimal solution to the discretized, constrained problem converges to the possibly discontinuous optimal control of the continuous-time problem.**

I. INTRODUCTION

Over the last decade, pseudospectral (PS) methods have been used to solve a broad class of industrial-strength optimal control problems arising in low-thrust orbit transfers [3], spacecraft attitude control [17], ascent guidance [9], [11] and many others [12]. One of the main reasons for the popularity of PS methods is that they offer an exponential convergence rate for the approximation of analytic functions while providing Eulerian-like simplicity. This property is particularly attractive for control applications as it places real-time computation within easy reach of modern computational power [14]. PS methods also offer a ready approach to exploiting differential-geometric properties of a control system such as convexity and differential flatness [14]. For a recent result on flatness-based PS method, see [12].

The essential idea of pseudospectral methods is to approximate the continuous optimal control problems by PS discretization and solve the resulting optimization problem. The simplicity of this approach masks a wide range of deeply theoretical issues that lie at the intersection of approximation theory and control theory. Significant progress has been made in answering some fundamental questions. For instance, in [3], [13] a detailed relation between the necessary conditions of the continuous optimal control problem and the Karush-Kuhn-Tucker (KKT) condition of the discrete optimization problem is revealed. In [7], the feasibility of PS discretization is proved with relaxed inequality constraints. In [6], the existence and convergence results are proved for feedback linearizable nonlinear systems in an approach similar to the theory of consistent approximations [10]. All these results rely on a key assumption that the optimal controller is at least continuous. Unfortunately, for many optimal control problems, this assumption is not valid, especially when the controller is constrained. In this situation, the optimal controller is likely to be discontinuous.

The consideration of discontinuous controller in computational optimal control is a very challenging problem. In [15], a PS knotting method is proposed to handle the discontinuities. A careful analysis of this method requires tools from nonsmooth analysis. Before such an analysis can be carried out, it is necessary to analyze the convergence of smooth approximations to nonsmooth functions. In the present paper, we address this problem. We provide existence and convergence results for a Legendre PS method for optimal control problems with discontinuous controller and feedback linearizable dynamics. We assume the dynamic system can be written in normal form. It permits a modification of the standard pseudospectral method [2], [13] in a manner that is similar to dynamic inversion. That is, we seek polynomial approximations of the state trajectories while the controls are determined by an exact satisfaction of the dynamics. This modification of a pseudospectral method permits us to prove sufficient conditions for the existence and convergence of PS discretizations with discontinuous controller. Furthermore, our method allows one to easily incorporate state and control constraints including mixed state and control constraints. Note that we do not linearize the dynamics by feedback control; rather, we find the optimal control for a generic cost function. Such problems are particularly common in astronautical applications where stringent performance requirements demand that the control be optimal rather than feasible as implied by the linearizing control. We show that, under mild conditions, the PS discretized optimization problem always has a feasible solution even when the controller is discontinuous. Further, we prove that the numerical solution converges to the solution of the original continuous-time constrained optimal control problem.

II. THE PROBLEM AND ITS DISCRETIZATION

We consider the following constrained nonlinear Bolza problem (Problem B) with feedback linearizable dynamics.

Problem B: Minimize

$$J[x(\cdot), u(\cdot)] = \int_{-1}^{1} F(x(t), u(t))\ dt + E(x(-1), x(1)) \qquad (1)$$

subject to the dynamics

$$\begin{aligned} \dot{x}_i(t) &= x_{i+1}(t), \qquad i = 1, \ldots, r-1 \\ \dot{x}_r(t) &= f(x(t)) + g(x(t))u(t) \end{aligned} \qquad (2)$$

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almost everywhere on $t \in [-1,1]$, and the constraints

$$h(x(t),u(t)) \leq 0 \quad (3)$$
$$e(x(-1),x(1)) = 0 \quad (4)$$

where $x \in \mathbb{R}^{\mathrm{r}}$, $u \in \mathbb{R}$, and $F : \mathbb{R}^{\mathrm{r}} \times \mathbb{R} \to \mathbb{R}$, $E : \mathbb{R}^{\mathrm{r}} \times \mathbb{R}^{\mathrm{r}} \to \mathbb{R}$, $f : \mathbb{R}^{\mathrm{r}} \to \mathbb{R}$, $g : \mathbb{R}^{\mathrm{r}} \to \mathbb{R}$, $e : \mathbb{R}^{\mathrm{r}} \times \mathbb{R}^{\mathrm{r}} \to \mathbb{R}^{\mathrm{N_e}}$ and $h : \mathbb{R}^{\mathrm{r}} \times \mathbb{R} \to \mathrm{R}$ are Lipschitz continuous (over the domain) with respect to their arguments. For controllability reasons, we assume $g(x) \neq 0$ for all x. We also assume that an optimal solution, $(x^*(t), u^*(t))$, of Problem B exists.

Definition 1: A function $\psi(t) : [-1,1] \to \mathbb{R}^{\mathrm{k}}$ is called piecewise C^1, if there exist finite many points $\tau_0 = -1 < \tau_1 < \cdots < \tau_{s+1} = 1$ such that, on every subinterval (τ_i, τ_{i+1}), $i = 0, \ldots, s$, $\psi(t)$ is continuously differentiable and both $\psi(t)$ and its derivative, $\dot{\psi}(t)$, are bounded. Furthermore, $\psi(t)$ is continuous at τ_j from at least one side, i.e., $\lim_{t\to\tau_j^-} \psi(t) = \psi(\tau_j)$ or $\lim_{t\to\tau_j^+} \psi(t) = \psi(\tau_j)$.

Assumption 1: The optimal state, $x_r^*(t)$, is assumed to be continuous and piecewise C^1. The optimal control input, $u^*(t)$, is assumed to be piecewise C^1.

Note that, according to Definition 1 and Assumption 1, $u^*(t)$ could have finite many discontinuous points.

Assumption 2: The set $\{(x,u)|h(x,u) \leq 0\}$ is convex.

In the next, we apply a pseudospectral (PS) computational method to discretize the continuous optimal control Problem B. We focus on the Legendre PS method for the purpose of brevity; the extension to other PS methods is trivial. The basic idea of Legendre PS method is to approximate $(x_1(t), \ldots, x_r(t))$ by N-th order polynomials $(x_1^N(t), \ldots, x_r^N(t))$ based on Lagrange interpolation of their values at the Legendre-Gauss-Lobatto (LGL) node points. Let $t_0 = -1 < t_1 < \cdots < t_N = 1$ be the LGL nodes defined as, $t_0 = -1$, $t_N = 1$, and for $k = 1, 2, \ldots, N-1$, t_k are the roots of $\dot{L}_N(t)$, where $\dot{L}_N(t)$ is the derivative of the N-th order Legendre polynomial $L_N(t)$. Let $\bar{x}_k^N$ and $\bar{u}_k^N$ be an approximation of a feasible solution $(x(t), u(t))$ evaluated at the node t_k. Then, $x^N(t)$ is used to approximate $x(t)$ by,

$$x(t) \approx x^N(t) = \sum_{k=0}^N \bar{x}_k^N \phi_k(t), \quad (5)$$

where $\phi_k(t)$ is defined by

$$\phi_k(t) = \frac{1}{N(N+1)L_N(t_k)} \frac{(t^2-1)\dot{L}_N(t)}{t - t_k}. \quad (6)$$

It is readily verifiable that $\phi_k(t_j) = 1$, if $k = j$ and $\phi_k(t_j) = 0$, if $k \neq j$. The precise nature of the approximation indicated in (5) is the main focus of this paper. From (2), the control that generates the approximate state is given by,

$$u^N(t) = (\dot{x}_r^N(t) - f(x^N(t)))/g(x^N(t)) \quad (7)$$

Note that $u^N(t)$ is not necessarily a polynomial and hence differs from a standard pseudospectral approximation. The derivative of $x_i^N(t)$ at the LGL node t_k is given by

$$\dot{x}_i^N(t_k) = \sum_{j=0}^N D_{kj}\, x_i^N(t_j), \quad i = 1, 2, \ldots, r$$

where the differentiation matrix D is defined by

$$D_{ik} = \begin{cases} \frac{L_N(t_i)}{L_N(t_k)} \frac{1}{t_i - t_k}, & \text{if } i \neq k; \\ -0.25N(N+1), & \text{if } i = k = 0; \\ 0.25N(N+1), & \text{if } i = k = N; \\ 0, & \text{otherwise} \end{cases}$$

Throughout the paper, we use the "bar" notation to denote corresponding variables in the discrete space, and the superscript N to denote the number of nodes used in discretization. Thus, let

$$\bar{x}_0^N = [\bar{x}_{10}^N \cdots \bar{x}_{r0}^N]^T, \ \cdots, \ \bar{x}_N^N = [\bar{x}_{1N}^N \cdots \bar{x}_{rN}^N]^T$$

Note that the subscript in $\bar{x}_k^N \in \mathbb{R}^{\mathrm{r}}$ denotes an evaluation of the approximate state, $x^N(t) \in \mathbb{R}^{\mathrm{r}}$, at the node t_k whereas $x_k(t)$ denotes the k-th component of the exact state.

With these preliminaries, it is apparent that the approximate solutions must satisfy the following algebraic equations

$$D \begin{pmatrix} \bar{x}_{i0}^N \\ \vdots \\ \bar{x}_{iN}^N \end{pmatrix} = \begin{pmatrix} \bar{x}_{i+1,0}^N \\ \vdots \\ \bar{x}_{i+1,N}^N \end{pmatrix}, \quad i = 1, \ldots, n-1 \quad (8)$$
$$D \begin{pmatrix} \bar{x}_{r0}^N \\ \vdots \\ \bar{x}_{rN}^N \end{pmatrix} = \begin{pmatrix} f(\bar{x}_0^N) + g(\bar{x}_0^N)\bar{u}_0^N \\ \vdots \\ f(\bar{x}_N^N) + g(\bar{x}_N^N)\bar{u}_N^N \end{pmatrix}$$

for feasibility with respect to the dynamics. In a standard pseudospectral method, it is quite common [14], [3], [2], [12] to discretize the mixed state- and control constraints as,

$$h(\bar{x}_k^N, \bar{u}_k^N) \leq 0, \quad k = 0, 1, \ldots, N \quad (9)$$

Here, we propose the following relaxation,

$$h(\bar{x}_k^N, \bar{u}_k^N) \leq (N-r)^{-\frac{1}{4}} \cdot \mathbf{1}, \quad k = 0, 1, \ldots, N \quad (10)$$

where $\mathbf{1}$ denotes $[1, \ldots, 1]^T$. When N tends to infinity, the difference between (9) and (10) vanishes. The purpose of this relaxation will be evident later. Similarly, we relax the endpoint condition $e(x(-1), x(1)) = 0$, to an inequality, i.e.,

$$\|e(\bar{x}_0^N, \bar{x}_N^N)\|_\infty \leq (N-r)^{-\frac{1}{4}}. \quad (11)$$

Finally, the cost functional $J[x(\cdot), u(\cdot)]$ is approximated by the Gauss-Lobatto integration rule,

$$J[\cdot] \approx \bar{J}^N(\bar{X}, \bar{U}) = \sum_{k=0}^N F(\bar{x}_k^N, \bar{u}_k^N) w_k + E(\bar{x}_0^N, \bar{x}_N^N)$$

where w_k are the LGL weights given by

$$w_k = \frac{2}{N(N+1)} \frac{1}{[L_N(t_k)]^2},$$

and $\bar{X} = [\bar{x}_0^N, \ldots, \bar{x}_N^N]$, $\bar{U} = [\bar{u}_0^N, \ldots, \bar{u}_N^N]$. Hence, the optimal control Problem B can be approximated by a nonlinear programming with $\bar{J}^N$ as the objective function and (8)-(10)- (11) as constraints; this is summarized as:

Problem $\mathbf{B^N}$: Find $\bar{x}_k^N \in \mathbb{R}^{\mathrm{r}}$ and $\bar{u}_k^N \in \mathbb{R}$ that minimize

$$\bar{J}^N(\bar{X}, \bar{U}) = \sum_{k=0}^N F(\bar{x}_k^N, \bar{u}_k^N) w_k + E(\bar{x}_0^N, \bar{x}_N^N) \quad (12)$$

subject to

$$D \begin{pmatrix} \bar{x}_{i-1,0}^N \\ \vdots \\ \bar{x}_{i-1,N}^N \end{pmatrix} = \begin{pmatrix} \bar{x}_{i0}^N \\ \vdots \\ \bar{x}_{iN}^N \end{pmatrix}, \ i = 1, \ldots, r-1 \quad (13)$$
$$D \begin{pmatrix} \bar{x}_{r0}^N \\ \vdots \\ \bar{x}_{rN}^N \end{pmatrix} = \begin{pmatrix} f(\bar{x}_0^N) + g(\bar{x}_0^N)\bar{u}_0^N \\ \vdots \\ f(\bar{x}_N^N) + g(\bar{x}_N^N)\bar{u}_N^N \end{pmatrix}$$
$$h(\bar{x}_k^N, \bar{u}_k^N) \leq (N-r)^{-\frac{1}{4}} \cdot \mathbf{1}, \quad (14)$$
$$\|e(\bar{x}_0^N, \bar{x}_N^N)\|_\infty \leq (N-r)^{-\frac{1}{4}} \quad (15)$$

for all $0 \le k \le N$.

Problem $\mathrm{B^N}$ can be solved by an appropriate globally-convergent algorithm [5], for example, a sequential-quadratic programming method. This approach has been successfully used in solving an impressive array of problems (see for example, [9], [2], [11]).

III. Existence Results

For Problem $\mathrm{B^N}$, a fundamental question that needs to be answered is the following: does a feasible solution satisfying the discretized constraints exist around a feasible solution of the continuous problem? In [6], the feasibility of Problem $\mathrm{B^N}$ is guaranteed under a critical assumption: the controller $u(t)$ is continuous. However, in many problems, the optimal controller is discontinuous, for instance, bang-bang controller. In this section, we extend the result in [6], and show that Problem $\mathrm{B^N}$ is always feasible even the optimal control of Problem B is discontinuous.

Let $(x(t), u(t))$ be any feasible solution of Problem B, i.e., $(x(t), u(t))$ satisfying differential equation (2), constraint (3) and endpoint condition (4). Suppose Assumption 1 holds for $(x(t), u(t))$. Let $-1 < \tau_1 < \cdots < \tau_s < 1$ represent the discontinuity points of $u(t)$, and define

$$I_\delta = [-1,1] \setminus \bigcup_{j=1}^{s} (\tau_j - \delta, \tau_j + \delta) \quad (16)$$

where $\delta = (N-r)^{-\frac{1}{2}}$. In other words, I_δ represents the closed set in $[-1,1]$ by removing a δ neighborhood around the discontinuous points of $u(t)$.

Lemma 1: For any feasible solution, $(x(t), u(t))$, of Problem B, satisfying Assumption 1-2, there exist continuous and piecewise C^1 functions $(z_1(t), \ldots, z_r(t), v(t))$, such that $(z_1(t), \ldots, z_r(t), v(t))$ satisfy differential equation (2) and the following conditions

$$\begin{aligned} h(z(t), v(t)) &\le C_1 (N-r)^{-\frac{1}{2}} \cdot \mathbf{1} && (17)\\ \|e(z(-1), z(1))\|_\infty &\le C_2 (N-r)^{-\frac{1}{2}} && (18)\\ \|z(t) - x(t)\|_\infty &\le C_3 (N-r)^{-\frac{1}{2}} && (19)\\ |u(t) - v(t)| &\le C_4 (N-r)^{-\frac{1}{2}}, \ \forall t \in I_\delta && (20)\\ \sum_{i=1}^{2} \|z_r^{(i)}(t)\|_\infty &\le C_5 + C_6 (N-r)^{\frac{1}{2}} && (21) \end{aligned}$$

where C_i, $1 \le i \le 6$, are positive constants independent of N and $z_r^{(i)}$ denotes the i-th order distribution derivative of $z_r(t)$ (see [1] for the definition of distribution derivative).

Proof: Define a continuous function $\hat{u}(t)$ as follows,

$$\hat{u}(t) = \begin{cases} (1-\alpha) u(\tau_i - \delta) + \alpha u(\tau_i + \delta), \\ \qquad \text{if } t \in [\tau_i - \delta, \tau_i + \delta], \quad 1 \le i \le s; \\ u(t), \ \text{otherwise} \end{cases} \quad (22)$$

where $\alpha = \frac{1}{2\delta}(t - \tau_i + \delta)$ and $\delta = (N-r)^{-\frac{1}{2}}$. So, $\hat{u}(t)$ agrees with $u(t)$ if t is not close to any point of discontinuity. If t is in a δ neighborhood of discontinuity, $\hat{u}(t)$ interpolates the points $(\tau_i - \delta, u(\tau_i - \delta))$ and $(\tau_i + \delta, u(\tau_i + \delta))$ by a straight line. A similar concept has been used in [15] to justify the PS knotting method. Let $q(t) = f(x(t)) + g(x(t))\hat{u}(t)$, $t \in [-1,1]$. Both $\hat{u}(t)$ and $q(t)$ are bounded, continuous, and piecewise C^1. Next, define

$$\begin{aligned} z_r(t) &= \int_{-1}^{t} q(\tau) d\tau + x_r(0) \\ z_{i-1}(t) &= \int_{-1}^{t} z_i(\tau) d\tau + x_{i-1}(0), \ i = r, \ldots, 2 \\ v(t) &= [q(t) - f(z(t))]/g(z(t)) \end{aligned}$$

Apparently, $(z(t), v(t))$ satisfy differential equation (2). In the next we will show that they also satisfy (17)-(20).

Denote M_1 as the upper bound of $|u(t)|$ for $t \in [-1,1]$. From the definition of $\hat{u}(t)$, we have

$$\begin{aligned} ||u(t) - \hat{u}(t)||_{L^1} &= \sum_{i=1}^{s} \int_{\tau_i - \delta}^{\tau_i + \delta} |(1-\alpha)(u(\tau_i - \delta) - u(t)) \\ &\quad + \alpha(u(\tau_i + \delta) - u(t))| dt \ \le \ 4 s M_1 (N-r)^{-\frac{1}{2}} \end{aligned}$$

Therefore,

$$\begin{aligned} ||\dot{x}_r(t) - q(t)||_{L^1} &= ||g(x(t))(u(t) - \hat{u}(t))||_{L^1} \\ &\le 4 s M_1 M_2 (N-r)^{-\frac{1}{2}} \quad (23) \end{aligned}$$

where M_2 is an upper bound of $|g(x(t))|$ for $t \in [-1,1]$. From (23), it is not difficult to show, $\forall t \in [-1,1]$,

$$|x_i(t) - z_i(t)| \le 2^{r-i+2} s M_1 M_2 (N-r)^{-\frac{1}{2}} \quad (24)$$

where $i = 1, 2, \ldots, r$. Hence, (19) holds with $C_3 = 2^{r+1} s M_1 M_2$. Next, $\forall t \in [-1,1]$,

$$\begin{aligned} |v(t) - \hat{u}(t)| &= \left| \frac{q(t) - f(z(t))}{g(z(t))} - \frac{q(t) - f(x(t))}{g(x(t))} \right| \\ &\le r K_1 \|z(t) - x(t)\|_\infty \le 2^{r+1} s r M_1 M_2 K_1 (N-r)^{-\frac{1}{2}} \quad (25) \end{aligned}$$

where K_1 is determined by the upper bound of $q(t)$ and the Lipschitz constants of $1/g(x)$ and $f(x)/g(x)$. By definition, $u(t) = \hat{u}(t)$ for all $t \in I_\delta$; therefore, (20) is true with $C_4 = 2^{r+1} s r M_1 M_2 K_1$.

For constraint (17), if $|t - \tau_i| > \delta$,

$$h(x(t), \hat{u}(t)) = h(x(t), u(t)) \le 0$$

If $|t - \tau_i| \le \delta$, the convexity Assumption 2 implies

$$\begin{aligned} & h(x(t), \hat{u}(t)) = h((1-\alpha)x(\tau_i - \delta) + \alpha x(\tau_i + \delta), \hat{u}(t)) \\ & + h(x(t), \hat{u}(t)) - h((1-\alpha)x(\tau_i - \delta) + \alpha x(\tau_i + \delta), \hat{u}(t)) \\ & \le 0 + r K_2 \|x(t) - ((1-\alpha)x(\tau_i - \delta) + \alpha x(\tau_i + \delta))\|_\infty \\ & \le 2 r K_2 M_3 (N-r)^{-\frac{1}{2}} \quad (26) \end{aligned}$$

In the above derivation, K_2 represents a Lipschitz constant of $h(\cdot)$; M_3 is an upper bound of $|\dot{x}_i(t)|$, for $i = 1, \cdots, r$ and $t \in [-1,1]$. From (24)-(25)-(26),

$$\begin{aligned} & h(z(t), v(t)) = h(x(t), \hat{u}(t)) + h(z(t), v(t)) - h(x(t), \hat{u}(t)) \\ & \le 2 r K_2 M_3 (N-r)^{-\frac{1}{2}} \cdot \mathbf{1} + K_2 (r \|z(t) - x(t)\|_\infty + \\ & \qquad \|v(t) - \hat{u}(t)\|_\infty) \cdot \mathbf{1} \\ & \le (2M_3 + (K_1 + 1) 2^{r+1} s M_1 M_2) r K_2 (N-r)^{-\frac{1}{2}} \cdot \mathbf{1} \end{aligned}$$

Hence, constraint (17) holds with $C_1 = (2M_3 + (K_1 + 1) 2^{r+1} s M_1 M_2) r K_2$. Similarly, (18) can be verified with K_3 be the Lipschitz constant of $e(\cdot)$. Finally, from the fact that $\dot{z}_r(t) = q(t)$, and using the definition of $\hat{u}(t)$, one can easily show (21). ∎

With Lemma 1 in hand, we can further prove the following existence result.

Theorem 1: Given any feasible solution $(x(t), u(t))$ of (2)-(3)-(4) in Problem B. Suppose Assumption 1-2 hold. Then there exists a positive integer N_1 such that, for any $N > N_1$, the constraints (13)-(14)-(15) of Problem $\mathrm{B^N}$ has a feasible solution $(\bar{x}_k^N, \bar{u}_k^N)$. Furthermore, the feasible solution satisfies

$$\begin{aligned} \|x(t_k) - \bar{x}_k^N\|_\infty &\leq (N-r)^{-\frac{1}{4}}, \quad 0 \leq k \leq N \quad (27)\\ |u(t_k) - \bar{u}_k^N| &\leq (N-r)^{-\frac{1}{4}}, \quad \forall\ t_k \in I_\delta \quad (28)\end{aligned}$$

where I_δ is defined in (16).

Proof: From Lemma 1, there exist continuous and piecewise C^1 function pair $(z(t), v(t))$ satisfying the differential equations (2) and inequalities (17)-(20). Let $p(t)$ be the $(N-r)$-th order best approximation polynomial of $\dot{z}_r(t)$ in the norm of $L^\infty(-1,1)$. The following estimation has been proved in the literature of spectral methods [1]

$$|\dot{z}_r(t) - p(t)| \leq C_0(N-r)^{-1}\textstyle\sum_{i=1}^2 \|z_r^{(i)}\|_{L^\infty(-1,1)} \quad (29)$$

$\forall t \in [-1,1]$. Substituting (21) to (29) leads to

$$|\dot{z}_r(t) - p(t)| \leq C_0C_5(N-r)^{-1} + C_0C_6(N-r)^{-\frac{1}{2}} \quad (30)$$

Let us define

$$\begin{aligned} \hat{x}_r(t) &= \int_{-1}^t p(\tau)d\tau + x_r(-1)\\ \hat{x}_{i-1}(t) &= \int_{-1}^t \hat{x}_i(\tau)d\tau + x_{i-1}(-1),\ i = r\ldots,2\\ \hat{v}(t) &= \frac{p(t) - f(\hat{x}_1(t),\ldots,\hat{x}_r(t))}{g(\hat{x}_1(t),\ldots,\hat{x}_r(t))}\end{aligned}$$

From (30), it is easy to show, $\forall t \in [-1,1]$

$$|z_i(t) - \hat{x}_i(t)| \leq 2^{r-i+1}C_0[C_5(N-r)^{-1} + C_6(N-r)^{-\frac{1}{2}}] \quad (31)$$

and

$$\begin{aligned} |v(t) - \hat{v}(t)| &= \left|\frac{\dot{z}_r(t) - f(z(t))}{g(z(t))} - \frac{p(t) - f(\hat{x}(t))}{g(\hat{x}(t))}\right|\\ &\leq K_1(|\dot{z}_r(t) - p(t)| + r\|z(t) - \hat{x}(t)\|_\infty)\\ &\leq C_0K_1(1 + r2^r)(C_5(N-r)^{-1} + C_6(N-r)^{-\frac{1}{2}}) \quad (32)\end{aligned}$$

Define $\bar{x}_k^N = \hat{x}(t_k)$, $\bar{u}_k^N = \hat{v}(t_k)$. In the following, we prove that $(\bar{x}_k^N, \bar{u}_k^N)$ is a feasible solution of (13)-(14)-(15) Apparently, $\hat{x}_1(t), \ldots, \hat{x}_r(t)$ are polynomials of degree less than or equal to N. Moreover, $(\hat{x}(t), \hat{v}(t))$ satisfies the differential equation (2) and has the same initial condition as $x(-1)$. Given any polynomial of degree less than or equal to N, it is known (see [1]) that its derivative at the nodes $t_0, \ldots, t_N$ is exactly equal to the value of the polynomial at the nodes multiplied by the differential matrix D. Thus

$$D\begin{pmatrix}\bar{x}_{i0}^N\\ \vdots\\ \bar{x}_{iN}^N\end{pmatrix} = \begin{pmatrix}\dot{\hat{x}}_i(t_0)\\ \vdots\\ \dot{\hat{x}}_i(t_N)\end{pmatrix} = \begin{pmatrix}\bar{x}_{i+1,0}^N\\ \vdots\\ \bar{x}_{i+1,N}^N\end{pmatrix}$$

where $i = 1, 2, \ldots, r-1$ and $\bar{x}_{ik}^N$ is the i-th component of $\bar{x}_k^N$. At $i = r$, we have

$$D\begin{pmatrix}\bar{x}_{r0}^N\\ \vdots\\ \bar{x}_{rN}^N\end{pmatrix} = \begin{pmatrix}f(\hat{x}(t_0)) + g(\hat{x}(t_0))\hat{v}(t_0)\\ \vdots\\ f(\hat{x}(t_N)) + g(\hat{x}(t_N))\hat{v}(t_N)\end{pmatrix}$$

Therefore $(\bar{x}_k^N, \bar{u}_k^N)$, $k = 0, 1, \ldots, N$, satisfy the constraint equations in (13). In the next, we prove that the mixed state-control constraint (14) is also satisfied. Because $h(\cdot)$ is Lipschitz continuous, the following estimation holds.

$$\begin{aligned} &\|h(z(t), v(t)) - h(\hat{x}(t), \hat{v}(t))\|_\infty\\ \leq\ &K_2(r\|z(t) - \hat{x}(t)\|_\infty + |v(t) - \hat{v}(t)|)\\ \leq\ &K_2C_0(r2^r + K_1 + r2^rK_1)[C_5(N-r)^{-1} + C_6(N-r)^{-\frac{1}{2}}]\end{aligned}$$

Hence, by (17),

$$h(\hat{x}(t), \hat{v}(t)) \leq (L_1(N-r)^{-1} + L_2(N-r)^{-\frac{1}{2}}) \cdot \mathbf{1}$$

where

$$\begin{aligned} L_1 &= K_2C_0C_5(r2^r + K_1 + r2^rK_1)\\ L_2 &= K_2C_0C_6(r2^r + K_1 + r2^rK_1) + C_1\end{aligned}$$

Since constants L_1 and L_2 are independent of N, there exists a positive integer N_1 such that, for all $N > N_1$,

$$L_1(N-r)^{-1} + L_2(N-r)^{-\frac{1}{2}} \leq (N-r)^{-\frac{1}{4}}$$

Therefore $\hat{x}_1(t_k), \ldots, \hat{x}_r(t_k)$, $\hat{u}(t_k)$, $k = 0, 1, \ldots, N$, satisfy mixed state and control constraint (14) for all $N > N_1$. The end-point condition (15) can be proved in the same way. Thus, $(\bar{x}_k^N, \bar{u}_k^N)$ is a feasible solution to Problem $\mathrm{B^N}$.

As for (27)-(28), they can be easily deduced from (31)-(32) and (19)-(20) in Lemma 1. ■

Remark 1: Theorem 1 guarantees that Problem $\mathrm{B^N}$ is well-posed with a nonempty feasible set as long as a sufficient number of nodes are chosen. More importantly, (27)-(28) show the existence of a feasible discrete solution inside any neighborhood around the continuous trajectory.

Remark 2: In the proof of Theorem 1 and Lemma 1, we actually established a stronger result than (27)-(28). That is

$$\begin{aligned} \|x(t) - \hat{x}(t)\|_\infty &\leq (N-r)^{-\frac{1}{4}}, \forall\ t \in [-1,1]\\ |u(t) - \hat{v}(t)| &\leq (N-r)^{-\frac{1}{4}}, \quad \forall\ t \in I_\delta\end{aligned}$$

These properties will be used later in the proof of the convergence of Legendre PS method.

IV. Convergence Results

Let $(\bar{x}_k^N, \bar{u}_k^N)$, $k = 0, 1, \ldots, N$, be a feasible solution to Problem $\mathrm{B^N}$, and $x^N(t) \in \mathbb{R}^r$ be the Nth order interpolating polynomials of $(\bar{x}_0^N, \ldots, \bar{x}_N^N)$, i.e.

$$x^N(t) = \textstyle\sum_{k=0}^N \bar{x}_k^N \phi_k(t), \quad (33)$$

where $\phi_k(t)$ is defined by (6). Also denote

$$u^N(t) = [\dot{x}_r^N(t) - f(x^N(t))]/g(x^N(t))$$

By the definition of $u^N(t)$, it is easy to show $u^N(t_k) = \bar{u}_k^N$.

Now consider a sequence of discrete feasible solution $\{(\bar{x}_k^N, \bar{u}_k^N), k = 0, \ldots, N\}_{N=N_1}^\infty$ and the corresponding interpolating polynomial sequence $\{x^N(t)\}_{N=N_1}^\infty$ and the non-polynomial sequence $\{u^N(t)\}_{N=N_1}^\infty$.

Assumption 3: (a) For all $1 \leq i \leq r$, the sequences $\{\bar{x}_{i0}^N\}_{N=N_1}^\infty$ converges as $N \to \infty$; (b) $\dot{x}_r^N(t)$ is uniformly bounded for $N \geq N_1$ and $t \in [-1,1]$; (c) there exists a piecewise C^1 function $q(t)$ such that, for any fixed $\epsilon > 0$, $\dot{x}_r^N(t)$ converges to $q(t)$ uniformly on interval I_ϵ, where

$$I_\epsilon = [-1,1] \setminus \textstyle\bigcup_{j=1}^s (\tau_j - \epsilon, \tau_j + \epsilon) \quad (34)$$

and $-1 < \tau_1 < \cdots < \tau_s < 1$ are the discontinuity points of $q(t)$.

Theorem 2: Consider a sequence of feasible solutions $(\bar{x}_k^N, \bar{u}_k^N)$, $k = 0, 1, \ldots, N$, of (13)-(14)-(15) in Problem B^N. Suppose Assumption 3 holds. Then there exists a feasible solution, $(x^\infty(t), u^\infty(t))$, of (2)-(3)-(4) in the continuous optimal control Problem B such that the limit

$$\lim_{N\to\infty}(x^N(t) - x^\infty(t)) = 0 \tag{35}$$

converges uniformly on $[-1, 1]$, and the limit

$$\lim_{N\to\infty}(u^N(t) - u^\infty(t)) = 0 \tag{36}$$

converges uniformly on any closed set I_ϵ.

Proof: Let x_{i0} be the limit of $\{\bar{x}_{i0}^N\}_{N=N_1}^\infty$. Then, define the following functions

$$\begin{aligned} x_r^\infty(t) &= \int_{-1}^t q(\tau)d\tau + x_{r0} \\ x_{i-1}^\infty(t) &= \int_{-1}^t x_i^\infty(\tau)d\tau + x_{i-1,0}, \quad i = r, \ldots, 2 \\ u^\infty(t) &= \frac{q(t) - f(x_1^\infty(t), \ldots, x_r^\infty(t))}{g(x_1^\infty(t), \ldots, x_r^\infty(t))} \end{aligned}$$

Obviously, $(x^\infty(t), u^\infty(t))$ satisfies the differential equation (2). Next, we prove (35)-(36) and the fact that $(x^\infty(t), u^\infty(t))$ satisfies both the mixed constraints in (3) and end-point condition (4).

Let $x_i^N(t)$ be the interpolating polynomial of $\bar{x}_{i0}^N, \cdots, \bar{x}_{iN}^N$. Because $(\bar{x}_k^N, \bar{u}_k^N)$ satisfies discrete state equation (13), it is easy to see

$$\begin{pmatrix} \dot{x}_i^N(t_0) \\ \vdots \\ \dot{x}_i^N(t_N) \end{pmatrix} = D \begin{pmatrix} \bar{x}_{i0}^N \\ \vdots \\ \bar{x}_{iN}^N \end{pmatrix} = \begin{pmatrix} x_{i+1}^N(t_0) \\ \vdots \\ x_{i+1}^N(t_N) \end{pmatrix}$$

for $i = 1, 2, \ldots, r-1$. Hence the N-th order polynomial:

$$\dot{x}_i^N(t) - x_{i+1}^N(t) \tag{37}$$

has $N+1$ different roots: $t_0, \ldots, t_N$. Therefore, $\dot{x}_i^N(t) = x_{i+1}^N(t)$, $i = 1, \ldots, r-1$. Under Assumption 3, $\dot{x}_r^N(t)$ is a bounded sequence that converges to $q(t)$ almost everywhere, then $\dot{x}_r^N(t)$ converges to $q(t)$ in L^1. Therefore,

$$\begin{aligned} \lim_{N\to\infty} |x_r^N(t) - x_r^\infty(t)| &= \lim_{N\to\infty} \left| \int_{-1}^t (\dot{x}_r^N(\tau) - q(t))d\tau \right| \\ &\le \lim_{N\to\infty} \int_{-1}^1 \left| \dot{x}_r^N(\tau) - q(t)) \right| d\tau = 0 \end{aligned}$$

Moreover, the limit converge uniformly in t. Hence, the following limit is also uniformly convergent

$$\lim_{N\to\infty} x_{r-1}^N(t) = \lim_{N\to\infty} \int_{-1}^t x_r^N(\tau)d\tau + x_{r-1,0} = x_{r-1}^\infty(t)$$

Following the same procedure, one can show

$$\lim_{N\to\infty} x_i^N(t) = x_i^\infty(t), \qquad i = 1, 2, \ldots, r$$

uniformly in t. Thus, (35) is proved. As for (36), it follows by the following inequality

$$\begin{aligned} \left| u^N(t) - u^\infty(t) \right| &= \left| \frac{\dot{x}_r^N(t) - f(x^N(t))}{g(x^N(t))} - \frac{q(t)}{g(x^N(t))} \right. \\ &\left. + \frac{q(t)}{g(x^N(t))} - \frac{q(t) - f(x^\infty(t))}{g(x^\infty(t))} \right| \\ &\le K_1|\dot{x}_r^N(t) - q(t)| + rK_1\|x^N(t) - x^\infty(t)\|_\infty \end{aligned}$$

and the fact that both $\dot{x}_r^N(t) - q(t)$ and $x^N(t) - x^\infty(t)$ converges to zero uniformly on any closed set I_ϵ.

The endpoint condition $e(x^\infty(-1), x^\infty(1)) = 0$ follows directly from the convergence property. Now, to show $(x^\infty(t), u^\infty(t))$ is a feasible solution of Problem B, it is enough to prove the mixed state-control constraint $h(x^\infty(t), u^\infty(t)) \le 0$. Using contradiction argument, suppose at a time instance $\tau' \in (-1, 1)$, there is a constraint $h_i(\cdot)$, $i \in \{1, 2, \ldots, l\}$, so that

$$h_i(x^\infty(\tau'), u^\infty(\tau')) > 0. \tag{38}$$

Since $x^\infty(t)$ is continuous and $u^\infty(t)$ is piecewise C^1, we can choose τ' that is not in the set $\{\tau_1, \cdots, \tau_s\}$ without loss of generality. By the fact that the nodes t_k are getting dense as N tends to infinity [4], there exist a sequence j^N and a sufficiently small ϵ such that, $0 \le j^N \le N$, the LGL nodes $t_{j^N} \in I_\epsilon$ and $\lim_{N\to\infty} t_{j^N} = \tau'$. Then (35) and (36) imply

$$\lim_{N\to\infty} h_i(\bar{x}_{j^N}^N, \bar{u}_{j^N}^N) = h_i(x^\infty(\tau'), u^\infty(\tau')) > 0$$

It contradicts the mixed state-control constraint (14), in which the right side of the inequality approaches zero as N approaching infinity. ∎

In Theorem 2, we proved a sufficient condition under which a sequence of discrete feasible solutions of Problem $\mathrm{B^N}$ converges to a feasible solution of the original continuous optimal control problem. Next, we study the optimal solution sequence of discrete Problem $\mathrm{B^N}$. Before introducing our finial convergence result, we need the following lemmas. Lemma 2-3 are known results (see [4]). The proof of Lemma 4 is omitted to save the space.

Lemma 2: Let t_k, $k = 0, 1, \ldots, N$, be the LGL nodes, and w_k be the LGL weights. Suppose $\xi(t)$ is Riemann integrable; then, $\int_{-1}^1 \xi(t)dt = \lim_{N\to\infty} \sum_{k=0}^N \xi(t_k)w_k$.

Lemma 3: Given any interval $[a, b] \subseteq [-1, 1]$. Then

$$\lim_{N\to\infty} \sum_{t_k \in [a,b]} \omega_k = b - a \tag{39}$$

where t_k are LGL nodes.

Lemma 4: Suppose $\{x^N(t)\}_{N\ge1}$, $x(t)$ are continuous and $\{u^N(t)\}_{N\ge1}$, $u(t)$ are piecewise C^1. Suppose $u^N(t)$ is uniformly bounded for all $N \ge 1$ and $t \in [-1, 1]$. Moreover, assume the limit, $\lim_{N\to\infty} x^N(t) = x(t)$, converges uniformly on $[-1, 1]$ and the limit, $\lim_{N\to\infty} u^N(t) = u(t)$, converges uniformly on any I_ϵ, the closed set defined by ϵ and the discontinuous points of $u(t)$. Then we have

$$\begin{aligned} &\lim_{N\to\infty} \left\{ \sum_{k=0}^N F(x^N(t_k), u^N(t_k))\omega_k + E(x^N(-1), x^N(1)) \right\} \\ &\quad = \int_{-1}^1 F(x(t), u(t))dt + E(x(-1), x(1)) \end{aligned}$$

Theorem 3: Suppose Problem B satisfies Assumption 1-2. Let $(\bar{x}_k^{*N}, \bar{u}_k^{*N})$, $k = 0, 1, \ldots, N$, be a sequence of discrete optimal solutions of Problem B^N. Assume the sequence satisfies Assumption 3. Then, there exists an optimal solution $(x^*(t), u^*(t))$ of the continuous optimal control Problem B such that the following limits converge uniformly

$$\begin{aligned} \lim_{N\to\infty} (\bar{x}_k^{*N} - x^*(t_k)) &= 0 \\ \lim_{N\to\infty} (\bar{u}_k^{*N} - u^*(t_k)) &= 0, \quad t_k \in I_\epsilon \\ \lim_{N\to\infty} \bar{J}^N(\bar{X}^*, \bar{U}^*) &= J(x^*(\cdot), u^*(\cdot)) \end{aligned}$$

for all $0 \leq k \leq N$ and any fixed $\epsilon > 0$.

Proof: According to Theorem 2, we know that the discrete optimal solutions uniformly converge to a feasible trajectory of the continuous problem. More specifically, there exists a continuous feasible solution, $(x^\infty(t), u^\infty(t))$, of (2)-(3)-(4) in Problem B such that

$$\begin{aligned} \lim_{N\to\infty} (\bar{x}_k^* - x^\infty(t_k)) &= 0 \\ \lim_{N\to\infty} (\bar{u}_k^* - u^\infty(t_k)) &= 0, \quad t_k \in I_\epsilon \end{aligned}$$

uniformly for $0 \leq k \leq N$ and any fixed $\epsilon > 0$. In the next, we prove that $(x^\infty(t), u^\infty(t))$ is indeed an optimal solution of the continuous optimal control problem. To this end, denote $\bar{J}^N(\bar{X}^*, \bar{U}^*)$ and $J(x^*(\cdot), u^*(\cdot))$ the optimal cost of Problem B^N and Problem B respectively, i.e.,

$$\begin{aligned} \bar{J}^N(\bar{X}^*, \bar{U}^*) &= E(\bar{x}_0^*, \bar{x}_N^*) + \sum_{k=0}^{N} F(\bar{x}_k^*, \bar{u}_k^*) w_k \\ J(x^*(\cdot), u^*(\cdot)) &= E(x^*(-1), x^*(1)) + \int_{-1}^{1} F(x^*(t), u^*(t)) dt \end{aligned}$$

where $(x^*(t), u^*(t))$ denotes any optimal solution of Problem B (the optimal solution may not be unique). According to Theorem 1, there exists a sequence of feasible solutions, $(\tilde{x}_k^N, \tilde{u}_k^N)$, of (13)-(14)-(15) that uniformly converges to $(x^*(t), u^*(t))$. Now, from Lemma 4 and the optimality of $(x^*(t), u^*(t))$ and $(\bar{x}_k^{*N}, \bar{u}_k^{*N})$, we have

$$\begin{aligned} & J(x^*(\cdot), u^*(\cdot)) \leq J(x^\infty(\cdot), u^\infty(\cdot)) = \\ & \lim_{N\to\infty} \bar{J}^N(\bar{X}^*, \bar{U}^*) \leq \lim_{N\to\infty} \bar{J}^N(\tilde{X}, \tilde{U}) = J(x^*(\cdot), u^*(\cdot)) \end{aligned}$$

The last equation is deduced from Lemma 4 and Remark 2. Therefore, we proved $J(x^*(\cdot), u^*(\cdot)) = J(x^\infty(\cdot), u^\infty(\cdot))$. It is equivalent to say that $(x^\infty(t), u^\infty(t))$ is a feasible solution that achieves optimal cost. Hence, $(x^\infty(t), u^\infty(t))$ is an optimal solution to optimal control Problem B. ■

To illustrate the practical convergence of the PS method, consider the following problem from [8].

$$\begin{aligned} \text{Min.} \quad J &= \int_0^3 x(t) dt \\ \text{s.t.} \quad \dot{x} &= u \\ x(0) &= 1, \quad x(3) = 1 \\ |u| &\leq 1, \quad x \geq 0 \end{aligned}$$

In Figure 1, we show the time histories of the discrete-time optimal controller and the optimal state calculated by DIDO [16], a software package that implements many of the ideas articulated in this paper. A plot of the analytic optimal solution,

$$x^* = \begin{cases} 1-t \\ 0 \\ t-2 \end{cases}, \quad u^* = \begin{cases} -1 \\ 0 \\ 1 \end{cases}, \text{ for } t \in \begin{cases} [0,1) \\ [1,2] \\ (2,3] \end{cases}$$

is also shown in Figure 1. Clearly, the discrete controller for $N = 64$ is a very accurate approximation of the analytic optimal controller except in a small neighborhood around the discontinuous point. A plot of the maximum errors in the state with respect to nodes reveals the convergence property of the PS method. A more complex problem including feedback principles for discontinuous control is presented in [17].

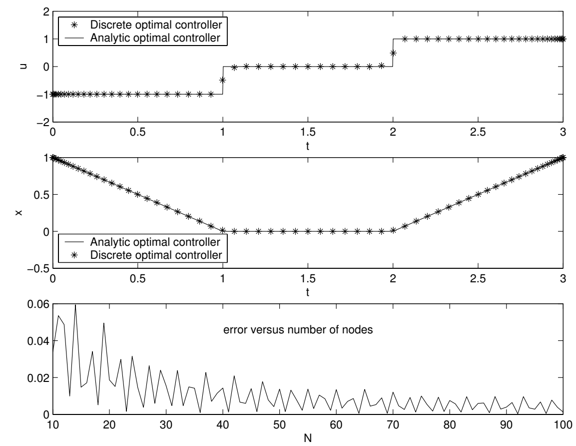

Fig. 1. Optimal solutions

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