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Optimal Control for a Class of Hybrid Systems Under Mixed Costs

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***Abstract*— A finite horizon optimal control problem for a class of discrete time hybrid systems is formulated and solved. The cost to be minimized is a mixture of $\mathcal{H}_2$ and l_∞ criteria. New dynamics are introduced to transform the problem into the minimization problem of a purely l_∞ cost. The transformed problem is solved using dynamic programming methods.**

Keywords: Optimal control, hybrid systems, mixed cost, dynamic programming

I. INTRODUCTION

There is an increasing attention being paid to the study of a class of complex systems named hybrid dynamical systems which involve the interaction of both continuous state and discrete state dynamics. A very general model for hybrid systems is presented in paper [4], and a more mathematically compactly expressed version in [3]. Optimal control problem are also considered in these two papers using dynamic programming which reduces the optimal controller synthesis problem to solving the corresponding dynamic programming equations in the form of System of Quasi-Variational Inequalities (SQVI). However, the SQVIs are coupled to each other by a non-local operator, which makes them difficult to solve. An approach to compute the lower bound of the value function by convex optimization is reported in [9]. In this paper, we propose a framework which leads to uncoupled dynamic programming equations, and the value functions can be computed recursively.

The cost considered in our optimization problem is a mixture of $\mathcal{H}_2$ type and l_∞ type cost. The $\mathcal{H}_2$ type cost is an integral type performance sometimes referred to as soft criteria, while the l_∞ type cost is some kind of absolute bound on some output quantity which is called hard criteria. Techniques for analysis and design using such criteria have begun to emerge, including [5], [7] and [8]. Since a control leading to a small value in either type does not necessarily guarantee a small value for the other cost, in many cases, it produces a bad bound on the other performance, especially for complex system like hybrid system which has far more complex trajectory structure. So it is desirable sometimes to minimize both of them at the same time. One issue with minimizing this kind of mixed cost by dynamic programming is that no dynamic programming principle holds directly for such cost, however, we employ a transformation which takes part of the cost into dynamics to change the cost to an l_∞ type cost to which dynamic programming principle holds.

For simplicity, we consider only a finite horizon optimal control problem for a class of discrete time hybrid system. However, it is not difficult to see that the approach formulated here works for more general problems with hybrid dynamics.

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II. DISCRETE TIME HYBRID DYNAMICS

Hybrid systems are complex systems involve both continuous state dynamics and discrete dynamics. In this paper, our continuous dynamics is a family of discrete time subsystems described by function

$$f : \mathbf{R}^n \times Q \times U \to \mathbf{R}^n, \tag{1}$$

where $Q = \{q_1, q_2, \cdots, q_p\}$ is the finite index set and $U \subset \mathbf{R}^m$ is the compact control set. The discrete state dynamics is represented by the following switching function

$$g : \mathbf{R}^n \times Q \times Q \to \mathbf{R}^n. \tag{2}$$

To understand how these two kinds of dynamics interact, i.e. how a switch happens, we need first describe our hybrid control signal.

Given terminal time $K > 0$, a discrete time hybrid control β_0^q in interval $[0, K]$ is defined as

$$\beta_0^q = (u_{[0,K-1]}, \{\tau_i, q_i\}_{i=0}) \tag{3}$$

where

$$u_{[0,K-1]} = (u_0, u_1, \cdots, u_{K-1}),\ u_i \in U,\ 0 \le i \le K-1$$

is the continuous (state) control sequence and $\{\tau_i, q_i\}_{i=1}$ satisfies

$$\begin{cases} 0 = \tau_0 \le \tau_1 \le \cdots \le \tau_i \le \tau_{i+1} \le \cdots \\ q = q_0, q_i \ne q_{i+1}, q_i \in Q,\ i \ge 0 \end{cases}$$

is the switching sequence.

For a hybrid control β_0^q and any time $0 \le k \le K$, we define the number of switches occurring before time k to be

$$M^\beta(k) = \max\{i : \tau_i < k\} \tag{4}$$

and the number of switches occurring at time k to be

$$N^\beta(k) = |\{i : \tau_i = k\}|. \tag{5}$$

Assume that before some particular switching time τ_i, the system is in state x, then the switch at time τ_i changes the working subsystem from $f(\cdot, q_{i-1}, \cdot)$ to $f(\cdot, q_i, \cdot)$, and resets the initial state for $f(\cdot, q_i, \cdot)$ to $g(x, q_{i-1}, q_i)$, the state evolves forward according to $f(\cdot, q_i, \cdot)$ thereafter until the next switch τ_{i+1}. Note that a switch does not cost any time, while the state of the hybrid system can only go from time k to $k+1$ by some continuous state dynamics with some control input u_k. So it is possible that there are more than one value of state at some time k due to multiple switches at that time, so we need two variables (k, l) to locate the exact position of a state. Let $x_k^l, 0 \le k \le K,\ 0 \le l \le N^\beta(k)$ denote the value of the state at time k after lth switch.

Next, let's define our class of admissible hybrid controls.

Definition 2.1: Given two integers n and N satisfying $0 \le n \le N$, an hybrid control β_0^q is called a class N admissible hybrid control with (freedom) degree n in $[0, K]$ if $N^\beta(k) \le N,\ \forall\ 0 \le k \le K$ and $N^\beta(0) \le n$. Let $\beta_{(0,n,N)}^q$ denote such an admissible control and $\mathcal{B}_{(0,n,N)}^q = \{\beta_{(0,n,N)}^q\}$ the set of all such admissible hybrid controls.

From the above definition, we know that in $\beta_{(0,n,N)}^q$, there can only be at most N switches at any time $0 \le k \le K$, and furthermore, at initial time $k=0$, no more than n switches are allowed. What we are most interested in is the the largest control set $\mathcal{B}_{(0,\infty,\infty)}^q$ which has no constraint on the number of switches.

Given an initial state x_0^0, any admissible hybrid control $\beta_{(0,n,N)}^q \in \mathcal{B}_{(0,n,N)}^q$, using the notations $M^\beta(k)$ and $N^\beta(k)$, we can write down the hybrid dynamics for state $x_k^l, 0 \le k \le K,\ 0 \le l \le N^\beta(k)$ explicitly

$$\left\{ \begin{array}{lcl} x_{k+1}^0 & = & f(x_k^{N^\beta(k)}, q_{(M^\beta(k)+N^\beta(k))}, u_k) \\ x_k^{l+1} & = & g(x_k^l, q_{(M^\beta(k)+l)}, q_{(M^\beta(k)+l+1)}). \end{array} \right. \tag{6}$$

III. MINIMIZATION UNDER MIXED COST

In this section, we define our optimal control problem for the hybrid system in last section, our cost functional to be minimized is a kind of mixture of two different types of costs,

Given initial state $x_0^0 = x$, an admissible hybrid control $\beta_{(0,n,N)}^q \in \mathcal{B}_{(0,n,N)}^q$, define $\mathcal{H}_2$ type cost to be

$$\begin{aligned} &\mathbf{J}_1(x, \beta_{(0,n,N)}^q) = \\ &\sum_{k=0}^{K-1} \ell(x_k^{N^\beta(k)}, q_{(M^\beta(k)+N^\beta(k))}, u_k) + \\ &\sum_{k=0}^{K} \sum_{l=0}^{N^\beta(k)-1} \rho(x_k^l, q_{(M^\beta(k)+l)}, q_{(M^\beta(k)+l+1)}) + \\ &\varphi(x_K^{N^\beta(K)}, q_{M^\beta(K)+N^\beta(K)}), \end{aligned} \tag{7}$$

where the functions $\ell : (\mathbf{R}^n \times Q \times U) \to \mathbf{R}^+$, $\rho : (\mathbf{R}^n \times Q \times Q) \to \mathbf{R}^+$ and $\varphi : (\mathbf{R}^n \times Q) \to \mathbf{R}^+$ are the running cost, switching cost and terminal cost for the $\mathcal{H}_2$ type cost respectively.

Define the l^∞ type cost to be

$$\begin{aligned} &\mathbf{J}_2(x, \beta_{(0,n,N)}^q) = \\ &\max \left\{ \begin{array}{l} \max\limits_{0 \le k \le K-1} \lambda(x_k^{N^\beta(k)}, q_{(M^\beta(k)+N^\beta(k))}, u_k), \\ \max\limits_{0 \le k \le K} \max\limits_{0 \le l \le N^\beta(k)-1} \\ \kappa(x_k^l, q_{(M^\beta(k)+l)}, q_{(M^\beta(k)+l+1)}), \\ \psi(x_K^{N^\beta(K)}, q_{M^\beta(K)+N^\beta(K)}) \end{array} \right\}, \end{aligned} \tag{8}$$

where the functions $\lambda : (\mathbf{R}^n \times Q \times U) \to \mathbf{R}^+$, $\kappa : (\mathbf{R}^n \times Q \times Q) \to \mathbf{R}^+$ and $\psi : (\mathbf{R}^n \times Q) \to \mathbf{R}^+$ are the running cost, switching cost and terminal cost for the l_∞ type cost respectively. We impose the following assumptions on the cost functions

Assumption 3.1: all the cost functions ℓ, λ, ρ, κ, φ, ψ are bounded with upper bounds C_ℓ, C_λ, C_ρ, C_κ, C_φ, C_ψ respectively, and especially, there is a strictly positive lower bound for ρ, i.e.

$$0 < c_\rho \le \rho(x, q, \tilde{q}),\ \forall (x, q, \tilde{q}) \in (\mathbf{R}^n \times Q \times Q).$$

It is known that a control that leads to a small $\mathcal{H}_2$ type cost does not necessary produce a small l_∞ type cost, in many cases, it will produce a bad value for it, and vise versa. In order to minimize these two types of costs at the same time, we introduce the following mixed cost

$$\mathbf{J}(x, \beta_{(0,n,N)}^q) = \max\{\alpha_1 \mathbf{J}_1(x, \beta_{(0,n,N)}^q), \alpha_2 \mathbf{J}_2(x, \beta_{(0,n,N)}^q)\} \tag{9}$$

where $\alpha_1 \ge 0, \alpha_2 \ge 0, \alpha_1 + \alpha_2 = 1$.

The optimal control problem is to find an admissible hybrid control $\breve{\beta}_{(0,n,N)}^q$ such that

$$\mathbf{J}(x, \breve{\beta}_{(0,n,N)}^q) \le \mathbf{J}(x, \beta_{(0,n,N)}^q),\ \forall \beta_{(0,n,N)}^q \in \mathcal{B}_{(0,n,N)}^q$$

for any $x \in \mathbf{R}^n$ and $q \in Q$. Define the value function to be the optimal cost in this control set to be

$$\mathbf{V}^q(x, 0, n, N) = \inf_{\beta_{(0,n,N)}^q \in \mathcal{B}_{(0,n,N)}^q} \mathbf{J}(x, \beta_{(0,n,N)}^q). \tag{10}$$

Similarly, what we really want to compute is the smallest value function $\mathbf{V}^q(x, 0, \infty, \infty)$. We now show that to compute this value function, we don't really need to minimize over the control set $\mathcal{B}_{(0,\infty,\infty)}^q$, it is sufficient to do it in a smaller set $\mathcal{B}_{(0,\bar{N},\bar{N})}^q$ for some finite number $\bar{N}$, under Assumption 3.1.

Theorem 3.2: The value function $\mathbf{V}_{(x,0,n,N)}^q$ satisfies

1) The value function $\mathbf{V}^q(x, 0, 0, 0)$ is nonnegative and bounded, i.e.

$$0 \le \mathbf{V}^q(x, 0, 0, 0) \le C \tag{11}$$

for any $(x, q) \in \mathbf{R}^n \times Q$, where

$$C = \max\{\alpha_1(KC_\ell + C_\varphi), \alpha_2 C_\lambda, \alpha_2 C_\psi\};$$

2) Fix (x, q), the value function $\mathbf{V}^q(x, 0, n, N)$ is non-increasing with respect to n and N;

3) For $\forall\ N \ge \bar{N} = \min\{i : i\ interger,\ i \ge \frac{C}{\alpha_1 c_\rho}\}$

$$\mathbf{V}^q(x, 0, n, N) = \mathbf{V}^q(x, 0, n, \bar{N}) \tag{12}$$

for $\forall\ (x, q, n) \in (\mathbf{R}^n \times Q \times [0, \bar{N}])$ and specifically,

$$\mathbf{V}^q(x, 0, \infty, \infty) = \mathbf{V}^q(x, 0, \bar{N}, \bar{N}), \forall\ (x, q) \in (\mathbf{R}^n \times Q) \tag{13}$$

Proof: Item (1): The nonnegativity of the value function is obvious since all cost functions are nonnegative, to prove the boundedness, notice that $\mathcal{B}_{(0,0,0)}^q$ is a set consisting of non-hybrid controls, hence for any (x, q) and $\beta^q_{(0,0,0)}$, we have

$$\begin{aligned}&\mathbf{V}^q(x,0,0,0)\\ &\leq \mathbf{J}(x,\beta^q_{(0,0,0)})\\ &= \max\{\alpha_1\mathbf{J}_1(x,\beta^q_{(0,0,0)}),\alpha_2\mathbf{J}_2(x,\beta^q_{(0,0,0)}\},\\ &\leq \max\{\alpha_1(KC_\ell+C_\varphi),\alpha_2 C_\lambda,\alpha_2 C_\psi\}.\end{aligned}$$

Item (2): To prove the monotonicity, it is sufficient to notice that for any n, N such that $n \leq N$

$$\mathcal{B}^q_{(0,n,N)} \subset \mathcal{B}^q_{(0,n+1,N)},\ \mathcal{B}^q_{(0,n,N)} \subset \mathcal{B}^q_{(0,n,N+1)}.$$

Item (3): To prove (12), fix any (x, q), $n < \bar{N}$, for $0 < \epsilon < \alpha_1 c_\rho$ and $\hat{\beta}^q_{(0,n,N)}$, $N \geq \bar{N}$ such that

$$\mathbf{V}^q(x,0,n,N)+\epsilon > \mathbf{J}(x,\hat{\beta}^q_{(0,n,N)}). \tag{14}$$

Then we claim the overall switches $M^{\hat{\beta}}(K)+N^{\hat{\beta}}(K) \leq \bar{N}$, otherwise, by the definition of $\bar{N}$

$$\begin{aligned}\mathbf{J}(x,\hat{\beta}^q_{(0,n,N)}) &\geq \alpha_1\mathbf{J}_1(x,\hat{\beta}^q_{(0,n,N)})\\ &\geq \alpha_1(\bar{N}+1)c_\rho\\ &\geq C+\alpha_1 c_\rho\end{aligned}$$

which leads to the contradiction $\mathbf{V}^q(x, 0, n, N) > C$ combined with (14). Hence $N^{\hat{\beta}}(k) \leq M^{\hat{\beta}}(K)+N^{\hat{\beta}}(K) \leq \bar{N}$ for any $0 \leq k \leq K$, which means $\hat{\beta}^q_{(0,n,N)} \in \mathcal{B}^q_{(0,n,\bar{N})}$. Hence

$$\mathbf{J}(x,\hat{\beta}^q_{(0,n,N)}) \geq \mathbf{V}^q(x,0,n,\bar{N}),$$

notice (14), we get

$$\mathbf{V}^q(x,0,n,N) \geq \mathbf{V}^q(x,0,n,\bar{N}).$$

The converse is true due to the monotonicity, so we proved (12).

To prove (13), from (12), let $n = \bar{N}$, we get

$$\mathbf{V}^q(x,0,\bar{N},\infty) = \mathbf{V}^q(x,0,\bar{N},\bar{N}).$$

A similar argument can show for $n \geq \bar{N}$

$$\mathbf{V}^q(x,0,n,\infty) = \mathbf{V}^q(x,0,\bar{N},\bar{N})$$

which directly leads to (13). ■

The above theorem says that in order to get all the value functions $\mathbf{V}^q(x, 0, n, N)$ $0 \leq n \leq N, 0 \leq N \leq \infty$, it is enough to compute $\mathbf{V}^q(x, 0, n, N)$ $0 \leq n \leq N, 0 \leq N \leq \bar{N}$ and all the rest value functions would be the same as $\mathbf{V}^q(x, 0, \bar{N}, \bar{N})$ which is the smallest one.

Dynamic programming does not apply directly to the optimal control problem formulated above; actually examples can be constructed to show that dynamic programming principle does not necessary hold for value function (10). In order to avoid this difficulty, we define a new optimal control problem for an augmented hybrid dynamics under a purely l_∞ type cost. It turns out that the original optimal control problem is embedded in the transformed optimal control problem and the dynamic programming principle holds for the new one.

For an hybrid admissible control $\beta^q_{(0,n,N)}$ and initial value $z^0_0 = z \in \mathbf{R}$, define dynamics for a new quantity $z^l_k \in \mathbf{R}, 0 \leq k \leq K,\ 0 \leq l \leq N^\beta(k)$ to be

$$\left\{\begin{array}{lll} z^0_{k+1} &=& z^{N^\beta(k)}_k + \ell(x^{N^\beta(k)}_k, q_{(M^\beta(k)+N^\beta(k))}, u_k)\\ z^{l+1}_k &=& z^l_k + \rho(x^l_k, q_{(M^\beta(k)+l)}, q_{(M^\beta(k)+l+1)}).\end{array}\right. \tag{15}$$

Now for hybrid dynamics (6) and (15), define new cost corresponding to an admissible hybrid control $\beta^q_{(0,n,N)}$ and any initial state $(x, z) \in \mathbf{R}^{n+1}$,

$$\begin{aligned}&\mathcal{J}(x,z,\beta^q_{(0,n,N)}) =\\ &\max\left\{\begin{array}{l}\alpha_1(z^{N^\beta(K)}_K+\varphi(x^{N^\beta(K)}_K, q_{M^\beta(K)+N^\beta(K)})),\\ \alpha_2\mathbf{J}_2(x,\beta^q_{(0,n,N)})\end{array}\right\}\end{aligned} \tag{16}$$

and new value function

$$\mathcal{V}^q(x,z,0,n,N) = \inf_{\beta^q_{(0,n,N)}\in\mathcal{B}^q_{(0,n,N)}} \mathcal{J}(x,z,\beta^q_{(0,n,N)}). \tag{17}$$

Then it is easy to see that

$$\mathbf{V}^q(x,0,n,N) = \mathcal{V}^q(x,0,0,n,N),\ \forall(x,q)\in(\mathbf{R}^n\times Q). \tag{18}$$

Hence, we can get $\mathbf{V}^q(x, 0, n, N)$ through computing $\mathcal{V}^q(x, z, 0, n, N)$ which we do by dynamic programming.

IV. DYNAMIC PROGRAMMING PRINCIPLE

For any time $0 \leq k \leq K$, let $\beta^q_{(k,n,N)}$ to be an admissible class N hybrid control with (freedom) degree n according to definition 2.1 defined on $[k, K]$, let $\mathcal{J}(x, z, \beta^q_{(k,n,N)})$ and $\mathcal{V}^q(x, z, k, n, N)$ denote the mixed cost and value function for hybrid dynamics (6) and (15) defined the same as (16) and (17) for $\beta^q_{(k,n,N)}$.

From theorem 3.2, we know to compute $\mathbf{V}^q(x, 0, \infty, \infty)$, we only need to focus on $\mathbf{V}^q(x, 0, \bar{N}, \bar{N})$, and this can be done if we can get $\mathcal{V}^q(x, z, 0, \bar{N}, \bar{N})$, the following theorem shows how this value function can be obtained recursively.

Theorem 4.1: The value function $\mathcal{V}^q(x, z, k, n, \bar{N})$ satisfies

1) Terminal condition, $k = K, n = 0$

$$\mathcal{V}^q(x,z,K,0,\bar{N}) = \max\{\alpha_1(z+\varphi(x,q)),\alpha_2\psi(x,q)\}; \tag{19}$$

2) $k = K, 0 < n \leq \bar{N}$

$$\begin{aligned}&\mathcal{V}^q(x,z,K,n,\bar{N}) =\\ &\min\left\{\begin{array}{l}(\mathcal{MV})^q(x,z,K,n-1,\bar{N}),\\ \mathcal{V}^q(x,z,K,0,\bar{N})\end{array}\right\};\end{aligned} \tag{20}$$

3) $0 \leq k < K,\ n = 0$

$$\mathcal{V}^q(x,z,k,0,\bar{N}) = (\mathcal{HV})^q(x,z,k+1,\bar{N},\bar{N}); \tag{21}$$

4) $0 \leq k < K,\ 0 < n \leq \bar{N}$

$$\begin{aligned}&\mathcal{V}^q(x,z,k,n,\bar{N}) =\\ &\min\left\{\begin{array}{l}(\mathcal{MV})^q(x,z,k,n-1,\bar{N})\\ (\mathcal{HV})^q(x,z,k+1,\bar{N},\bar{N})\end{array}\right\}.\end{aligned} \tag{22}$$

where the operators $\mathcal{M}$ and $\mathcal{H}$ are defined as

$$(\mathcal{M}\mathcal{V})^q(x,z,k,n,\bar{N}) = \min_{\tilde{q}\neq q}\max\left\{\begin{array}{l}\alpha_2\kappa(x,q,\tilde{q}),\\ \mathcal{V}^{\tilde{q}}(g(x,q,\tilde{q}),z+\rho(x,q,\tilde{q}),k,n,\bar{N})\end{array}\right\} \tag{23}$$

and

$$(\mathcal{H}\mathcal{V})^q(x,z,k,n,\bar{N}) = \min_{u\in U}\max\left\{\begin{array}{l}\alpha_2\lambda(x,q,u),\\ \mathcal{V}^{q}(f(x,q,u),z+\ell(x,q,u),k,n,\bar{N})\end{array}\right\}. \tag{24}$$

Proof: (Due to space limitation, only the proofs of item 4 are attached, proofs of item 2 and 3 are similar.)

Item 4: For $0\le k<K$, and $0<n\le\bar{N}$

Fix any $\tilde{q}\neq q$ and any $\bar{\beta}^{\tilde{q}}_{(k,n-1,\bar{N})} = (\bar{u}_{[k,K-1]},\{\bar{\tau}_i,\bar{q}_i\}_{i=0}) \in \mathcal{B}^{\tilde{q}}_{(k,n-1,\bar{N})}$, construct new hybrid control $\beta^q_k=(u_{[k,K-1]},\{\tau_i,q_i\}_{i=0})$ such that

$$\left\{\begin{array}{l}u_j=\bar{u}_j, k\le j\le K-1\\ \tau_i=\bar{\tau}_{i-1}, q_i=\bar{q}_{i-1}, \forall i\ge 1,\end{array}\right.$$

then $\beta^q_k\in\mathcal{B}^q_{(k,n,\bar{N})}$, we write $\beta^q_k=\beta^q_{(k,n,\bar{N})}$, then

$$\begin{array}{l}\mathcal{V}^q(x,z,k,n,\bar{N})\\ \le \mathcal{J}(x,z,\beta^q_{(k,n,\bar{N})})\\ = \max\left\{\begin{array}{l}\alpha_2\kappa(x,q,\tilde{q}),\\ \mathcal{J}(g(x,q,\tilde{q}),z+\rho(x,q,\tilde{q}),\bar{\beta}^{\tilde{q}}_{(k,n-1,\bar{N})})\end{array}\right\},\end{array}$$

by minimizing over $\mathcal{B}^{\tilde{q}}_{(k,n-1,\bar{N})}$ and $Q\backslash\{q\}$ on the righthand, we proved

$$\mathcal{V}^q(x,z,k,n,\bar{N})\le(\mathcal{M}\mathcal{V})^q(x,z,k,n-1,\bar{N}).$$

Next fix any $u\in U$ and any $\bar{\beta}^q_{(k+1,\bar{N},\bar{N})} = (\bar{u}_{[k+1,K-1]},\{\bar{\tau}_i,\bar{q}_i\}_{i=0}) \in \mathcal{B}^q_{(k+1,\bar{N},\bar{N})}$, construct new hybrid control $\beta^q_k=(u_{[k,K-1]},\{\tau_i,q_i\}_{i=0})$ such that

$$\left\{\begin{array}{l}u_k=u, u_j=\bar{u}_j, k+1\le j\le K-1\\ \tau_i=\bar{\tau}_i, q_i=\bar{q}_i, i\ge 1\end{array}\right.$$

then $\beta^q_k\in\mathcal{B}^q_{(k,0,\bar{N})}\subset\mathcal{B}^q_{(k,n,\bar{N})}$, denote it as $\beta^q_k=\beta^q_{(k,n,\bar{N})}$, then

$$\begin{array}{l}\mathcal{V}^q(x,z,k,n,\bar{N})\\ \le \mathcal{J}(x,z,\beta^q_{(k,n,\bar{N})})\\ = \max\left\{\begin{array}{l}\alpha_2\lambda(x,q,u),\\ \mathcal{J}(f(x,q,u),z+\ell(x,q,u),\bar{\beta}^q_{(k+1,\bar{N},\bar{N})})\end{array}\right\}.\end{array}$$

A minimization over $\mathcal{B}^q_{(k+1,\bar{N},\bar{N})}$ then minimize over control set U leads to

$$\mathcal{V}^q(x,z,k,n,\bar{N})\le(\mathcal{H}\mathcal{V})^q(x,z,k+1,\bar{N},\bar{N}).$$

Hence we finished proving the " $\le$"part of equation (22).

What is the left to be shown is the converse inequality. For any $\epsilon>0$, let the hybrid control $\beta^q_{(k,n,\bar{N})} = (u_{[k,K-1]},\{\tau_i,q_i\}_{i=0})$ be such that

$$\mathcal{V}^q(x,z,k,n,\bar{N})+\epsilon>\mathcal{J}(x,z,\beta^q_{(k,n,\bar{N})}).$$

Two cases arise

Case 1: $\tau_1>k$, i.e. $\beta^q_{(k,n,\bar{N})}\in\mathcal{B}^q_{(k,0,\bar{N})}$, then construct a new control $\beta^q_{k+1}=(\bar{u}_{[k+1,K-1]},\{\bar{\tau}_i,\bar{q}_i\}_{i=0})$ such that

$$\left\{\begin{array}{l}\bar{u}_j=u_j, k+1\le j\le K-1\\ \bar{\tau}_i=\tau_i, \bar{q}_i=q_i, i\ge 1,\end{array}\right.$$

then $\beta^q_{k+1}\in\mathcal{B}^q_{(k+1,\bar{N},\bar{N})}$, denote it as $\beta^q_{k+1}=\bar{\beta}^q_{(k,0,\bar{N})}$, then

$$\begin{array}{l}\mathcal{V}^q(x,z,k,0,\bar{N})+\epsilon\\ > \mathcal{J}(x,z,\beta^q_{(k,0,\bar{N})})\\ = \max\left\{\begin{array}{l}\alpha_2\lambda(x,q,u),\\ \mathcal{J}(f(x,q,u),z+\ell(x,q,u),\bar{\beta}^q_{(k+1,\bar{N},\bar{N})}\end{array}\right\}\\ \ge \max\left\{\begin{array}{l}\alpha_2\lambda(x,q,u),\\ \mathcal{V}^q(f(x,q,u),z+\ell(x,q,u),k+1,\bar{N},\bar{N})\end{array}\right\}.\\ \ge(\mathcal{H}\mathcal{V})^q(x,z,k+1,\bar{N},\bar{N}),\end{array}$$

Case 2: $\tau_1=k$, we construct a new control $\beta^{q_1}_k = (\bar{u}_{[k,K-1]},\{\bar{\tau}_i,\bar{q}_i\}_{i=0})$ such that

$$\left\{\begin{array}{l}\bar{u}_j=u_j, k\le j\le K-1\\ \bar{\tau}_i=\tau_{i+1}, \bar{q}_i=q_{i+1}, i\ge 0\end{array}\right.$$

Then $\beta^{q_1}_k\in\mathcal{B}^{q_1}_{(k,n-1,\bar{N})}$, we write it to be $\bar{\beta}^{q_1}_k=\beta^{q_1}_{(k,n-1,\bar{N})}$, then

$$\begin{array}{l}\mathcal{V}^q(x,z,k,n,\bar{N})+\epsilon\\ > \mathcal{J}(x,z,\beta^q_{(k,n,\bar{N})})\\ = \mathcal{V}^q(x,z,k,n,\bar{N})\\ = \max\left\{\begin{array}{l}\alpha_2\kappa(x,q,q_1),\\ \mathcal{J}(g(x,q,q_1),z+\rho(x,q,q_1),\bar{\beta}^{q_1}_{(k,n-1,\bar{N})})\end{array}\right\}\\ \ge \max\left\{\begin{array}{l}\alpha_2\kappa(x,q,q_1),\\ \mathcal{V}^{q_1}(g(x,q,q_1),z+\rho(x,q,q_1),k,n-1,\bar{N})\end{array}\right\}\\ \ge(\mathcal{M}\mathcal{V})^q(x,z,k,n-1,\bar{N}).\end{array}$$

Combining the two cases, we proved the "$\ge$"part of (22). ■

The significance of the above equations is that it allows to compute $\mathcal{V}^q(x,z,0,\bar{N},\bar{N})$ starting from the terminal condition $\mathcal{V}^q(x,z,T,0,\bar{N})$ recursively backwards. The procedure is as follows: Start from $\mathcal{V}^q(x,z,K,0,\bar{N})$, according to (20), we could get $\mathcal{V}^q(x,z,K,\bar{N},\bar{N})$, then by (21), we can obtain $\mathcal{V}^q(x,z,K-1,0,\bar{N})$, and by (22), we can compute $\mathcal{V}^q(x,z,K-1,\bar{N},\bar{N})$, repeat the process we finally get $\mathcal{V}^q(x,z,0,\bar{N},\bar{N})$.

V. OPTIMAL HYBRID CONTROLLERS

It is known that dynamic programming allows to compute the optimal feedback control law at the same time when computing the value function. In our case, the optimal controller is a hybrid controller which can be obtained from the equations (19)-(22). At time (k,n), assume we computed the value function $\mathcal{V}^q(x,z,k,n,\bar{N})$, the optimal hybrid controller at (k,n) is a tripe $(\mathcal{S}(k,n),q^*_{(k,n)}(x,z,q),u^*_{(k,n)}(x,z,q))$ where $S(k,n)\subset(\mathbf{R}^{n+1}\times Q)$ is the optimal switching set, $q^*_{(k,n)}:S\to Q$ is the optimal switcher, and $u^*_{(k,n)}:(\mathbf{R}^{n+1}\times Q)\backslash S\to U$ is the optimal continuous state controller respectively, they can be constructed according to

1) Optimal switching set

$$\mathcal{S}(k,n) = \left\{ (x,z,q) : \begin{array}{l} \mathcal{V}^q(x,z,k,n,\bar{N}) = \\ (\mathcal{M}\mathcal{V})^q(x,z,k,n-1,\bar{N}) \end{array} \right\}$$

for $0 \le k \le K$ and $1 \le n \le \bar{N}$;

2) Optimal switcher

$$q^*_{(k,n)}(x,z,q) = \arg((\mathcal{M}\mathcal{V})^q(x,z,k,n-1,\bar{N}))$$

for $0 \le k \le K$, $1 \le n \le \bar{N}$ and $(x,z,q) \in \mathcal{S}(k,n)$;

3) Optimal continuous state controller

$$u^*_{(k,n)}(x,z,q) = \arg((\mathcal{H}\mathcal{V})^q(x,z,k+1,\bar{N},\bar{N}))$$

for $0 \le k \le K-1$.

Note $\mathcal{S}(k,0), 0 \le k \le K$ are not defined above, however, since no switches happen when degree $n=0$, so we can just let $\mathcal{S}(k,0) = \varnothing,\ \forall\ 0 \le k \le K$, this will simplify the following algorithm.

Provided with all the hybrid optimal controllers, the optimal hybrid control $\breve{\beta}^q_{(0,\bar{N},\bar{N})} = (\breve{u}_{[0,K-1]}, \{\breve{\tau}_i, \breve{q}_i\}_{i=0})$ as well as the optimal trajectory $(\breve{x}^l_k, \breve{z}^l_k)$ corresponding to initial condition $(x,z) \in \mathbf{R}^{n+1}$ and initial index $q \in Q$ are produced as follows

Step 1: **Initialization**: $k=0$, $l=0$, $n=\bar{N}$, $i=0$, and $\breve{x}^0_0 = x$, $\breve{z}^0_0 = z$, $\breve{\tau}_0 = 0$, $\breve{q}_0 = q$;

Step 2: **While** $k \le K-1$, **do**

If $(\breve{x}^l_k, \breve{z}^l_k, \breve{q}_i) \in \mathcal{S}(k,n)$,

Then "**SWITCH**"and choose the optimal switcher

$$\begin{cases} \breve{\tau}_{i+1} = k \\ \breve{q}_{i+1} = q^*_{(k,n)}(\breve{x}^l_k, \breve{z}^l_k, \breve{q}_i) \end{cases}$$

and set optimal trajectory

$$\begin{cases} \breve{x}^{l+1}_k = g(\breve{x}^l_k, \breve{q}_i, \breve{q}_{i+1}) \\ \breve{z}^{l+1}_k = \breve{z}^l_k + \rho(\breve{x}^l_k, \breve{q}_i, \breve{q}_{i+1}) \end{cases}$$

and n decrease by 1;

Else "**GO ONE STEP FORWARD**"and choose the optimal continuous state control

$$\breve{u}_k = u^*_{(k,n)}(\breve{x}^l_k, \breve{z}^l_k, \breve{q}_i)$$

and set optimal trajectory

$$\begin{cases} \breve{x}^0_{k+1} = f(\breve{x}^l_k, \breve{q}_i, \breve{u}_k) \\ \breve{z}^0_{k+1} = \breve{z}^l_k + \ell(\breve{x}^l_k, \breve{q}_i, \breve{u}_k) \end{cases}$$

and let k increase by 1 and set $n = \bar{N}$.

End(If)

End(While)

Step 3: **While** $n \ge 0$, **Do**

If $(\breve{x}^l_K, \breve{z}^l_K, \breve{q}_i) \in \mathcal{S}(K,n)$,

Then "**SWITCH**"and choose the optimal switcher

$$\begin{cases} \breve{\tau}_{i+1} = K \\ \breve{q}_{i+1} = q^*_{(k,n)}(\breve{x}^l_K, \breve{z}^l_K, \breve{q}_i) \end{cases}$$

and set optimal trajectory

$$\begin{cases} \breve{x}^{l+1}_K = g(\breve{x}^l_K, \breve{q}_i, \breve{q}_{i+1}) \\ \breve{z}^{l+1}_K = \breve{z}^l_K + \rho(\breve{x}^l_K, \breve{q}_i, \breve{q}_{i+1}) \end{cases}$$

and n decrease by 1;

Else "**STOP**"

End(If)

End(While)

VI. NUMERICAL EXAMPLE

Consider a hybrid system consists of two subsystems, i.e. $Q = \{0,1\}$, the subsystems are simple one dimensional linear systems

$$f(x,q,u) = \begin{cases} x + 0.3u, & for\ q = 0 \\ -x + 0.3u, & for\ q = 1, \end{cases}$$

the control space $U = [-1,1]$.

The switching functions are

$$g(x,q,\tilde{q}) = \begin{cases} x + 0.3, & for\ q = 0, \tilde{q} = 1 \\ x - 0.3, & for\ q = 1, \tilde{q} = 0. \end{cases}$$

Let the $\mathcal{H}_2$ type running cost functions and terminal cost functions be bounded quadratic functions

$$\ell(x,0,u) = \begin{cases} 1.2x^2 + 0.8u^2, & |x| < 1 \\ 1.2 + 0.8u^2, & else \end{cases}$$

$$\ell(x,1,u) = \begin{cases} 0.8x^2 + 0.5u^2, & |x| < 1 \\ 0.8 + 0.5u^2, & else \end{cases}$$

and

$$\varphi(x,0) = \begin{cases} 0.6x^2, & |x| < 1 \\ 0.6, & else \end{cases}$$

$$\varphi(x,1) = \begin{cases} x^2, & |x| < 1 \\ 1, & else. \end{cases}$$

We assume a constant switching cost for $\mathcal{H}_2$ type

$$\rho(x,0,1) = \rho(x,1,0) = 0.2.$$

The l_∞ type cost functions are as follows

$$\lambda(x,0,u) = \begin{cases} 0.2|x| + 0.5|u|, & |x| < 1 \\ 0.2 + 0.5|u|, & else \end{cases}$$

$$\lambda(x,1,u) = \begin{cases} 0.8|x| + 1.2|u|, & |x| < 1 \\ 0.8 + 1.2|u|, & else \end{cases}$$

$$\psi(x,0) = \begin{cases} 1.2|x|, & |x| < 1 \\ 1.2, & else \end{cases}$$

$$\psi(x,1) = \begin{cases} |x|, & |x| < 1 \\ 1, & else \end{cases}$$

and

$$\kappa(x,0,1) = \kappa(x,1,0) = \begin{cases} 0.3|x|, & |x| < 1 \\ 0.3, & else. \end{cases}$$

In the definition of our mixed cost, let $K = 2$, $\alpha_1 = 0.3$, $\alpha_2 = 0.7$. In this example, the upper bound is $C = 1.4$, so $\bar{N} = 24$.

The following three pictures are the simulation results for the example. Figure 1 and Figure 2 are the value functions computed according to the dynamic programming equations (19)-(22). Figure 1 is the value function $\mathcal{V}^1(x,z,0,\bar{N},\bar{N})$, Figure 2 is the projection to x variable when $z = 0$, we know it is $\mathbf{V}^1(x,0,\bar{N},\bar{N})$, and from theorem 3.2, it actually is $\mathbf{V}^1(x, 0, \infty, \infty)$. Figure 3 is the optimal trajectories of x (dotted line) and z (solid line) starting from initial states $(x, z) = (0.6, 0)$ and $q = 1$, from the figure we can see that there is a switch at time $k = 0$. The corresponding optimal hybrid controls $(\breve{u}_{[0,1]}, \{\breve{\tau}_i, \breve{q}_i\}_{i=0}^{i=1})$ are $[\breve{u}_1, \breve{u}_2] = [-0.2, -0.2]$ and $\breve{\tau}_0 = \breve{\tau}_1 = 0$, $\breve{q}_0 = 1$, $\breve{q}_1 = 0$.

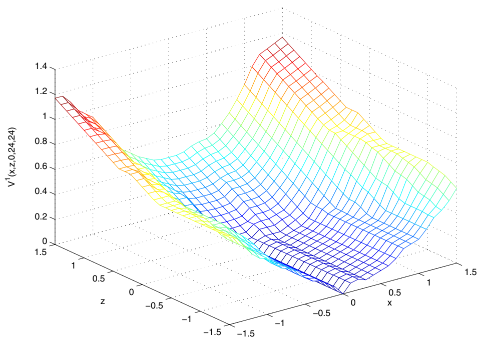

Fig. 1. Value function $\mathcal{V}^1(x, z, 0, 24, 24)$

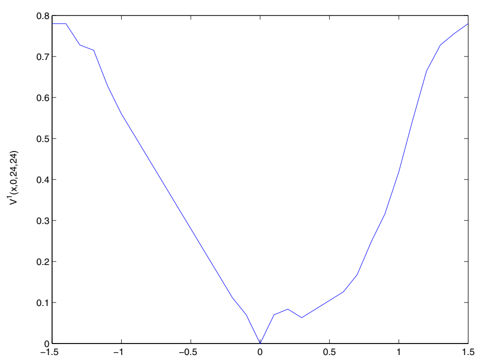

Fig. 2. Value function $\mathbf{V}^1(x, 0, 24, 24)$

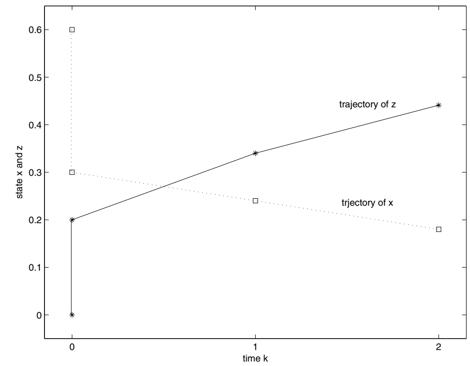

Fig. 3. Optimal state trajectories starting from $(x, z) = (0.6, 0)$

VII. CONCLUDING REMARKS

In this paper, we formulated a problem of minimizing a mixed cost for a class of discrete time hybrid systems. We have shown that the minimization problem can be solved by methods of dynamic programming if we transform the mixed cost into unmixed cost by taking part of the cost into the dynamics. By performing a recursion in both time and number of switches, we can compute the value function from the dynamic programming equations. We conclude with two remarks.

Remark 7.1: The cost defined in (9) is one kind of mixture of $\mathcal{H}_2$ type and l_∞ type costs, other kinds of mixture are possible like

$$\mathbf{J}(x, \beta^q_{(0,n,N)}) = \alpha_1 \mathbf{J}_1(x, \beta^q_{(0,n,N)}) + \alpha_2 \mathbf{J}_2(x, \beta^q_{(0,n,N)}).$$

This cost can be treated similarly.

Remark 7.2: To get the value function $\mathcal{V}^q(x, z, 0, \bar{N}, \bar{N})$ from the terminal condition $\mathcal{V}^q(x, z, K, 0, \bar{N})$, we need to compute $(K + 1)(\bar{N} + 1)$ value functions including the terminal condition. So the efficiency of the algorithm relies on the magnitude of $\bar{N}$, which is the upper bound on the number of possible switches could happen at a single time in the optimal hybrid control. In this paper, we use the upper bound of the number of overall switches could happen for an optimal hybrid control. The reason is that it is possible in the worst case that all the switches happen in one single time. in many cases, the maximum number of switches happened at a single time for an optimal hybrid control is far less than that, in our example, although, $\bar{N} = 24$, actually, we found that only 1 switch happened at one single time for the optimal control, So for some particular problems, find a smaller $\bar{N}$ will reduce the computation dramatically.

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