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Almost Sure Nonobservability and Unstabilizability of Unstable Noisy Linear Plants via Communication Channels with Packet Losses

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***Abstract*— We show that unstable linear plants affected by uniformly bounded and arbitrarily small disturbances cannot be stabilized or observed with a nonzero probability over communication channels with data dropout. Specifically, both estimation and stabilization errors are almost surely unbounded. This is true for all non-anticipating algorithms of estimation and stabilization with infinite memories, provided that the channel meets some natural requirements. They are satisfied e.g., if packet losses happen with a given nonzero probability and independently of each other and the system noise.**

I. Introduction

In the classic control theory, the standard assumption is that the observations are available constantly and immediately. However in a number of newly arisen engineering applications, observations are transmitted to the decision-maker via communication channels, which provide considerable delays and lose data (see e.g. [1], [4], [5], [8], [11]–[14], [16], [18], [19], [22], [23]). These applications concern e.g., fields such as underwater acoustic or mobile communications, exploration seismology, remote control of a large number of mobile units etc. Other examples are offered by complex networked sensor systems with a large number of low power sensors distributed over a wide area, as well as complex dynamical processes like advanced aircraft, spacecraft, and manufacturing process, where time division multiplexed computer networks are employed for exchange of information between spatially distributed plant components.

Data dropouts and delays degrade the system performance and may cause instability. Recently there was a good deal of research activity in the field of control over delayed communication channels with packet losses e.g., see [1], [4], [5], [8], [9], [11]–[14], [16], [22], [23] and the literature therein. LQG optimization and the minimum variance state estimation over channels with packet losses were addressed in [1], [11]–[14]. In [11]–[14], the packet dropout is modelled as transmission with infinite delay. In [1], dropout in both observation and control loops was treated, and it was assumed that data losses happen independently and are equivalent to receiving the zero signal. The paper [4] discusses a design of a dropout compensator. In [9], observability of linear systems is studied in the situation where the probability of loss of no more than m packets in every lot of $k > m$ ones is kept above a certain level.

In this paper, we consider state estimation and stabilization problems for discrete-time linear unstable plants subjected to external disturbances. The sensor and controller/estimator are physically distant and connected via a communication channel. The objective is to show that by their own rights, packet losses in this channel make it impossible to stabilize the system or observe its state with a nonzero probability. To underscore this, we neglect detrimental effects of the channel that are related to transmission delays and communication noise. For the same reason, we consider the case of the full noiseless measurements, where the controller is directly connected to the plant, and channels with arbitrary (not necessarily finite) alphabets.

The main assumption is that given any previous combination of losses and successful transmissions, there is a nonzero (and bounded from below) probability of the packet loss at the next time instant. This holds e.g., if the losses are independent of each other and occur with a constant positive probability. Another assumption is that the system noises at various times are mutually independent and independent of the communication channel, identically distributed, and have a probability density. (The major points of this paper concern the case where the noises are uniformly and arbitrarily small: the support of the above density is a subset of a small ball centered at 0.) It is shown that then the stabilization and estimation errors are almost surely unbounded for any non-anticipating algorithms of stabilization and estimation with infinite memories. Some other details relevant to the results of this paper can be found in [10].

The outline of the paper is as follows. Sections II, III, and IV present the problem statement, assumptions, and the main result, respectively. Its proof is given in Section V. Section VI offers brief conclusions.

II. Problem statement

We consider discrete time linear systems of the form:

$$x(t+1) = Ax(t) + B(t)u(t) + \xi(t), \quad x(0) = x_0. \quad (1)$$

Here $t = 0, 1, \dots, x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^{n_u}$ is the control, and $\xi(t) \in \mathbb{R}^n$ is a random disturbance. The initial state x_0 is a random vector. The system is unstable: there is an eigenvalue λ of A with $|\lambda| \geq 1$. The objective is to stabilize the system or produce a reliable state estimate.

We suppose that the entire state is accessible for measurements. The channel connecting the sensor and controller/estimator is capable to carry messages from an *al-*

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phabet $\mathcal{E} = \{e\}$. The set $\mathcal{E}$ may be arbitrary; if $\mathcal{E} = \mathbb{R}^n$, the entire current state $x(t)$ can be dispatched over the channel. Any message e is either transmitted correctly and instantaneously or is lost (erased). There is a perfect *feedback communication link* via which a notification about the erasure arrives at the sensor site by time $t+1$. This information may be used in forming the message at time $t+1$. For example, the erased messages may be repeated. The decision about the contents of the current message is made by a *coder*. It may have an access to controls and a side information $\widehat{\omega}(t)$, and is described by an equation of the form

$$e(t) = \mathfrak{E}[t, x_0^t, e_0^{t-1}, u_0^{t-1}, I_0^{t-1}, \widehat{\omega}_0^t]. \tag{2}$$

Here $f_0^t := \{f(\theta)\}_{\theta=0}^t$ for any function $f(\cdot)$ of $t = 0, 1, \ldots$, and $I(t)$ is the indicator of the erasure: $I(t) = 1$ if the erasure occurs at time t and $I(t) = 0$ otherwise.

The *estimator* and *controller* generate the state estimate and control, respectively, on the basis on the data available at the current time t, including a side information $\breve{\omega}(t)$:

$$\begin{aligned} \widehat{x}(t) &= \mathfrak{D}[t, \{e(\theta)\}_{\theta \le t : I(\theta)=0}, u_0^{t-1}, I_0^t, \breve{\omega}_0^t], \\ u(t) &= \mathfrak{C}[t, \{e(\theta)\}_{\theta \le t : I(\theta)=0}, u_0^{t-1}, I_0^t, \breve{\omega}_0^t]. \end{aligned} \tag{3}$$

Do there exist coder and estimator (controller) that keep the estimation (stabilization) error bounded with a nonzero probability?

When treating the estimation problem, we assume that a controller $\mathfrak{C}$ from (3) is given.

The side information $\omega = [\widehat{\omega}, \breve{\omega}]$ may concern e.g., the communication medium or network hosting the channel at hand, which may be useful for prognosis of future erasures. The variable ω may also give the output of a source of a common randomness $\widehat{\omega} = \breve{\omega}$ used for coding-decoding purposes. If ω takes a single value, there is no side information.

For technical reasons, we suppose that $\omega \in \Omega$, where Ω and the channel alphabet $\mathcal{E}$ are separable metric spaces, and consider only measurable functions in (2) and (3).

III. Assumptions

We suppose that given a coder and controller, the process in the system $x(t), e(t), I(t), \omega(t), t = 0, 1, \ldots$ is uniquely determined as a stochastic process defined on a common probability space and satisfies the following assumptions.

Assumption 1: There is a possibility of an erasure $\boldsymbol{P}\big[I(t) = 1 \big| I_0^{t-1} = \overline{I}\big] > 0$ irrespective of which combination of erasures and successful transmissions $\overline{I} = (i_0, \ldots, i_{t-1}), i_\nu = 0, 1$ has occurred before. Moreover,

$$p := \inf_t \inf_{\overline{I}} \boldsymbol{P}\big[I(t) = 1 \big| I_0^{t-1} = \overline{I}\big] > 0. \tag{4}$$

Here the second $\inf$ is over $\overline{I}$ such that $\boldsymbol{P}\big[I_0^{t-1} = \overline{I}\big] > 0$.

For example, this assumption holds if the erasures are mutually independent and $\inf_t \boldsymbol{P}\big[I(t) = 1\big] > 0$.

Assumption 2: In (1), the disturbances $\xi(t)$ are identically distributed according to a probability density $p(\xi)$ with the zero mean and finite variance, mutually independent, and independent of the initial state and the channel. More precisely, they are independent of $x_0, I(t), \omega(t), t = 0, 1, \ldots$.

Assumption 3: If all unstable eigenvalues of A lie on the unit circle, the disturbance $\xi(t)$ has a finite third moment.

By Assumption 2, the side information ω does not concern the disturbances. It may concern the initial state and the erasures. In particular, the initial state may be a part of both $\widehat{\omega}$ and $\breve{\omega}$ and so be known to the coder and controller/estimator. We note also that the above assumptions does not exclude even the extreme idealized case where the entire sequence of erasures is known in advance $\widehat{\omega}(t) = \breve{\omega}(t) = \{I(\theta)\}_{\theta=0}^\infty \, \forall t$.

IV. Main result

Theorem 1: Suppose that the matrix A has an unstable eigenvalue $|\lambda| \ge 1$ and Assumptions 1—3 are true. Then

$$\overline{\lim_{t\to\infty}} |x(t)| = \infty, \quad \overline{\lim_{t\to\infty}} |x(t) - \widehat{x}(t)| = \infty \tag{5}$$

almost surely for any coder, estimator, and controller.

In other words, arbitrarily large estimation and stabilization errors are unavoidably encountered as time progresses. It should be stressed that the level of the plant disturbances is immaterial for this conclusion to hold. In particular, (5) is true even if the disturbances are almost surely, arbitrarily, and uniformly small:

$$\sup_t |\xi(t)| \le \varepsilon \quad \text{a.s., where} \quad \varepsilon \approx 0, \varepsilon > 0. \tag{6}$$

So the behaviour of the closed loop system (1), (2), (3) drastically differs from e.g., that of an asymptotically stable stochastically disturbed linear system $x_{st}(t+1) = \mathcal{A} x_{st}(t) + \xi(t)$. The trajectory of the latter system also satisfies the first relation from (5) if for example, $\{\xi(t)\}$ is a white noise. However this is due to the fact that the sample sequences of the white noise are a.s. unbounded. At the same time, whenever the disturbance is a.s. bounded (6), so is the trajectory $\{x_{st}(t)\}$ of any stable linear system. Moreover, this trajectory is asymptotically bounded by a constant proportional to the noise bound ε from (6) and so $\overline{\lim}_{t\to\infty} |x_{st}(t)| \approx 0$ a.s. whenever $\varepsilon \approx 0$, on contrary to the property (6) $\Rightarrow$ (5) of the system (1), (2), (3) at hand.

In general, $\overline{\lim}$ cannot be replaced by $\lim$ in (5). It is known that the system can be stabilized and observed via the erasure channel in the mean-square sense under certain assumptions [21]. Mean square stability means that though by Theorem 1 arbitrarily large errors are encountered unavoidably, they do not occur frequently. Indeed with the strong law of large numbers in mind, this stability can be viewed as boundedness of the time averaged squared error.

If the channel alphabet is an euclidean space, only coders satisfying a prescribed power constraint $\boldsymbol{E}|e(t)|^2 \le W$ are often considered [2]. The scope of applicability of Theorem 1 includes but is not limited to these coders.

V. Proof of Theorem 1

We start with two preliminary technical lemmas. The proofs of the lemmas can be found in the full version of the paper (see also [10]).

Lemma 1: Whenever Assumption 1 holds, there exists a sequence $\{\tau_i\}_{i=0}^\infty$ of random times such that a. s., $0 =$

$\tau_0, \tau_i > \tau_{i-1} + i \, \forall i \geq 1$, and $i+1$ successive erasures occur at and just before time τ_i for any $i \geq 1$:

$$I(\tau_i - i) = I(\tau_i - i + 1) = \cdots = I(\tau_i - 1) = I(\tau_i) = 1.$$

These times can be chosen so that they depend only on the sequence of the erasures.

Lemma 1 is a key observation underlying Theorem 1. This lemma states that as time progresses, time intervals of arbitrarily large durations are encountered within which all transmitted packets are dropped. This means that the feedback control loop is in fact broken. At the same time, the plant (1) is unstable in the open loop by assumption. Hence an uniformly small but nonzero external disturbance (6) is able to give rise to an error proportional to the duration of the interval. So far as this duration is not limited from above, arbitrarily large errors are unavoidably encountered.

To make these arguments mathematically rigorous, we need one more lemma. It states nothing but that the state of the uncontrolled disturbed unstable plant grows without limits. Moreover, the state becomes more and more distributed over the space.

Lemma 2: For any $b > 0$, the following relation holds:

$$\mu_t(b) := \sup_{\varphi \in \mathbb{R}^n} \boldsymbol{P}\big\{|x^{-c}(t) + \varphi| < b\big\} \xrightarrow{t \to \infty} 0, \qquad (7)$$

where $x^{-c}(t) := \sum_{\theta=0}^{t-1} A^{t-1-\theta}\xi(\theta)$ is the state at time t, provided that $x(0) = 0$ and no control is applied $u(\theta) \equiv 0$.

Further $\boldsymbol{P}(M|Q = q, R)$ and $\boldsymbol{P}_S(ds|Q = q, R)$ stand for the (regular [6]) conditional probability and distribution of the random event M and variable S, respectively, given that another random variable $Q = q$ and the random event R occurs.

Proof of Theorem 1: We employ the random times $\tau_0, \tau_1, \ldots$ from Lemma 1. Since they are determined by the sequence of erasures $I(t)$, the disturbances $\xi(t), t = 0, 1, \ldots$ are independent of these times by Assumption 2.

Suppose that the first relation from (5) fails to be true, i.e., $B := \overline{\lim}_{t \to \infty} |x(t)| < \infty$ with a positive probability. Then there exists a (non-random) constant $b > 0$ such that $B < b - 1$ also with a positive probability. Since $\tau_i \to \infty$ as $i \to \infty$ almost surely, we have $\overline{\lim}_{i \to \infty} |x(\tau_i)| \leq B$ a.s. With this in mind, we get

$$\begin{aligned} \underline{\lim}_{i \to \infty} \boldsymbol{P}\{|x(\tau_i)| < b\} &\geq \underline{\lim}_{i \to \infty} \boldsymbol{P}\{|x(\tau_j)| < b \, \forall j \geq i\} \\ &= \boldsymbol{P}\left(\bigvee_{i=1}^{\infty} \{|x(\tau_j)| < b \, \forall j \geq i\}\right) \\ \geq \boldsymbol{P}\left\{\overline{\lim}_{i \to \infty} |x(\tau_i)| < b - 1\right\} &\geq \boldsymbol{P}\{B < b - 1\} > 0. \qquad (8) \end{aligned}$$

However the first $\underline{\lim}$ equals 0. To show this, we note that $\tau_i \geq i + 1$ a.s. for $i \geq 1$. So

$$\begin{aligned} &\boldsymbol{P}\{|x(\tau_i)| < b\} \\ &= \sum_{k=i+1}^{\infty} \boldsymbol{P}\big\{|x(\tau_i)| < b \big| \tau_i = k\big\} \boldsymbol{P}\{\tau_i = k\} \\ &= \sum_{k=i+1}^{\infty} \boldsymbol{P}\big\{|x(k)| < b \big| \tau_i = k\big\} \boldsymbol{P}\{\tau_i = k\}; \qquad (9) \end{aligned}$$

$$\begin{aligned} &\boldsymbol{P}\big\{|x(k)| < b \big| \tau_i = k\big\} = \\ &\int \boldsymbol{P}\big\{|x(k)| < b \big| x_0^{k-i} = X, I_0^{k-i-1} = \overline{I}, \omega_0^k = \overline{\omega}, \tau_i = k\big\} \\ &\quad \times \boldsymbol{P}_{x_0^{k-i}, I_0^{k-i-1}, \omega_0^k}\big(dX, d\overline{I}, d\overline{\omega} \big| \tau_i = k\big). \qquad (10) \end{aligned}$$

Since $\tau_i = k \Rightarrow I(k-i) = \cdots = I(k) = 1$, no messages arrive at the controller/estimator at times $t = k-i, \ldots, k$. By invoking equations (2), (3), we see that the data $x_0^{k-i} = X, I_0^{k-i-1} = \overline{I}, \omega_0^k = \overline{\omega}$ and $\tau_i = k$ uniquely determine the sequences of controls $u_0^k = U^k[X, \overline{I}, \overline{\omega}, k, i]$ and estimates $\widehat{x}_0^k = \widehat{X}_k[X, \overline{I}, \overline{\omega}, k, i]$ up to time k. These data also determine $x(k-i) = x_{k-i}[X]$. Now we note that

$$x(k) = \underbrace{A^i x(k-i) + \sum\nolimits_{\theta=k-i}^{k-1} A^{k-1-\theta} B(\theta) u(\theta)}_{\varphi} + \sum_{\theta=k-i}^{k-1} A^{k-1-\theta}\xi(\theta).$$

Here φ is a function $\varphi = \Phi[X, \overline{I}, \overline{\omega}, k, i]$ of the above data. The remainder $x_{k-i,k}^{-c} := x(k) - \varphi$ is independent of $x_0^{k-i}, I_0^{k-i-1}, \omega_0^k$, and τ_i due to Assumption 2. Hence

$$\begin{aligned} &\boldsymbol{P}\big\{|x(k)| < b \big| x_0^{k-i} = X, I_0^{k-i-1} = \overline{I}, \omega_0^k = \overline{\omega}, \tau_i = k\big\} \\ &\quad = \boldsymbol{P}\Big\{|x_{k-i,k}^{-c} + \Phi[X, \overline{I}, \overline{\omega}, k, i]| < b\Big\}. \end{aligned}$$

Thanks to Assumption 2, $x_{k-i,k}^{-c}$ and the vector $x^{-c}(i)$ from Lemma 2 are identically distributed. So by (7),

$$\boldsymbol{P}\big\{|x(k)| < b \big| x_0^{k-i} = X, I_0^{k-i-1} = \overline{I}, \omega_0^k = \overline{\omega}, \tau_i = k\big\} \leq \mu_i(b).$$

We proceed by invoking (10)

$$\begin{aligned} &\boldsymbol{P}\big\{|x(\tau_i)| < b \big| \tau_i = k\big\} \\ &\leq \mu_i(b) \int \boldsymbol{P}_{x_0^{k-i}, I_0^{k-i-1}, \omega_0^k}\big(dX, d\overline{I}, d\overline{\omega} \big| \tau_i = k\big) = \mu_i(b), \\ &\boldsymbol{P}\{|x(\tau_i)| < b\} \overset{(9)}{\leq} \sum_{k=i+1}^{\infty} \mu_i(b) \boldsymbol{P}\{\tau_i = k\} = \mu_i(b), \\ &\underline{\lim}_{i \to \infty} \boldsymbol{P}\{|x(\tau_i)| < b\} \leq \underline{\lim}_{i \to \infty} \mu_i(b) \overset{(7)}{=\!=} 0, \end{aligned}$$

in violation of (8). The contradiction obtained proves that the first relation from (5) does hold almost surely. The second one is established likewise.

VI. Conclusions

We studied observability/stabilizability of linear unstable systems over channels, which may lose messages. We followed the natural approach aimed at making the observation/stabilization error small along any (or almost any) trajectory. The main result of this paper means that the plant cannot be stabilized or observed with a nonzero probability. More precisely, arbitrarily large stabilization/observation errors unavoidably occur sooner or later, even if the plant disturbances are almost surely, arbitrarily, and uniformly small. A possible conclusion from these facts is that in face both plant noises and packet losses, "trajectory-wise" observability/stabilizability appears to be too strong property, and weaker ones seem to be more relevant. It should be also remarked that we considered channels for which the number of successive packet losses is not limited. If this number is limited, the above conclusion on nonobservability and unstabilizability does not hold.

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