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On the Use of the Averaging Method for the Characteristic Multiplier Placement Problem

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Abstract— **This paper proposes a novel approach for designing a periodic state feedback control law for a class of continuous-time linear periodic systems. The control law generates the sufficiently high frequency sinusoidal in the coefficient of the closed loop systems, then the closed loop solution is shown to be exponentially stable based on the averaging method. It is also shown that the closed loop solution is approximated by the solution of the average system.**

I. Introduction

The specification of the characteristic multipliers is one of the fundamental problems for linear periodic systems and has been addressed by a number of authors from the following perspectives:

i) When can we arbitrarily assign the characteristic multipliers by continuous periodic state feedback ?
ii) How can we practically compute the continuous feedback gain ?
iii) How can we improve the transient response during a period ?

In his fundamental paper, Brunovsky [2] gave the complete solution for the first question above: if the system is controllable, there exists a continuous periodic feedback that allows all the characteristic multipliers to be freely assigned. However this approach does not give any solution for the other problems. Since the implicit function theorem is used in his constructive proof, it is difficult to compute the periodic feedback gain. Since the periodic gain becomes impulsive, the transient response is stimulated by the impulsive input.

Kabamba [3] gave the alternative solution for the first two questions in the framework of sampled control. He gave the explicit formula for designing the control law by reducing this problem into the pole placement problem for discrete-time linear-time-invariant systems. In addition, the use of piecewise continuous gains are proposed to answer the last question [1], [5]. The transient response can be improved at finite points during a period, but it is still vibrating except for those points. So this modification is not sufficient for iii). In addition, the sampled control law can not be always implemented by continuous time feedback. So this approach is not satisfactory for i). Since the state transition matrix and the controllability Gramian are used, it is necessary to numerically integrate them in general. So this approach is also unsatisfactory for ii).

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Tornambe [7] also gave the alternative solution for the first question. He gave the explicit procedure to shift the characteristic multiplier to the desired place (under the strong assumption that all the characteristic multiplies are positive and real). It is necessary to compute the left eigenvector and indefinite integral, so this approach is not acceptable for ii) in addition to iii).

Therefore we restrict our attention to the following system

$$\dot{x} = Ax + b\,g(t)\,u \tag{1}$$

$$g(t) := \prod_{l=1}^{L} \sin\left(\frac{2\pi}{\omega}(t - \phi_l \omega)\right) \tag{2}$$

where $A \in \mathbf{R}^{n\times n}, b \in \mathbf{R}^n$ are constant matrices, L is a positive integer and $\phi_l\,(0 \le \phi_l < 1)$ are rational numbers which are assumed to be different to each other for simplicity.

We note that this class is restrictive but significant, since a class of attitude stabilization problems for small satellites can be described in this framework as shown in Section IV. We also note that it is not possible to apply the input transformation such as $u = \frac{1}{g(t)}v$ for a new input v, since $\frac{1}{g(t)}$ becomes unbounded.

The aim of this paper is to give answers for the three questions above for the system (1). The proposed method consists of two steps. In the first step, a constant matrix is selected to specify the desired trajectory. In the second step, a scalar periodic function is selected to generate a sufficiently high frequency trigonometric function in the coefficient of the closed loop system. A linear periodic gain is constructed as the multiplication of the selected constant matrix and the selected scalar periodic function. Based on the averaging method [4], it is shown that the closed loop system is exponentially stable. Furthermore the closed loop solution is approximated by the solution of the average system.

In summary, we give the explicit formula for designing a linear periodic state feedback control law such that (see Section III)

i') the characteristic multipliers are arbitrarily asymptotically assigned
ii') the periodic feedback gain is computed by symbolic computations
iii') the transient response is arbitrarily approximated by that of the linear time invariant system

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II. Basic Idea

In this section we introduce our basic idea with an unstable scalar AR model with the sinusoidal coefficient

$$\dot{x} = ax + \sin\left(\frac{2\pi}{\omega}t\right)u, \quad a > 0 \tag{3}$$

where $x \in \mathbf{R}$ is the state variable, $u \in \mathbf{R}$ is the input. The control objective is to find a stabilizing state feedback control as well as to improve the transient response during a period.

A. Proportional Control

Let us start with the proportional control

$$u = kx.$$

The initial response of the closed loop system is given by

$$x(t) = \exp\left(at + \frac{\omega k}{2\pi}\left(1 - \cos\left(\frac{2\pi}{\omega}t\right)\right)\right)x(0),$$

whose absolute value is bounded from below

$$|x(t)| \geq \exp\left(at - \frac{\omega|k|}{\pi}\right)|x(0)|.$$

It then follows that

$$\lim_{t\to\infty}|x(t)| = \infty$$

for any k and $x(0) \neq 0$, therefore there exists no stabilizing proportional feedback for (3).

B. Periodic Control

The main difficulty arises from the sign indefiniteness of the input coefficient, which prevents negative feedback control. In order to avoid this difficulty, let us continue with the periodic control

$$u = 2k\sin\left(\frac{2\pi}{\omega}t\right)x.$$

The initial response of the closed loop system is given by

$$x(t) = \exp\left((a+k)t - \frac{\omega k}{4\pi}\sin\left(\frac{4\pi}{\omega}t\right)\right)x(0), \tag{4}$$

whose absolute value is bounded from above

$$|x(t)| \leq \exp\left((a+k)t + \frac{\omega|k|}{4\pi}\right)|x(0)|.$$

It then follows that

$$\lim_{t\to\infty}|x(t)| = 0$$

for any $k < -a$ and $x(0)$. We note that the pole of the average system

$$\dot{x} = (a+k)x \tag{5}$$

is shifted by the use of the identity

$$1 - \cos(2p) = 2\sin^2(p),$$

and this is the key idea to stabilize the closed loop system.

Now we have the freedom of the choice of k. Let us compare two extreme cases to choose an appropriate k.

Case 1: For sufficiently small k, *i.e.* $k \to -\infty$, the closed loop solution (4) converges to 0 sufficiently fast. However, since $|\frac{\omega k}{4\pi}| \to \infty$, the closed loop solution (4) is extremely corrupted by $\frac{\omega k}{4\pi}\sin(\frac{4\pi}{\omega}t)$.

Case 2: If we could choose k satisfying $|\omega k| \simeq 0$, the closed loop solution (4) converges to the solution of the average system (5). However, for fixed a and ω, the closed loop system becomes unstable for $k \simeq 0$. Hence it is not possible to choose such k.

Therefore the choice of k involves a certain trade-off between the stabilizability and the improvement of the closed loop solution.

C. Periodic Control by Raising Frequency

In order to achieve the additional freedom to shape the closed loop solution, let us continue with the periodic control

$$u = 2k\sum_{l=1}^{L}\sin\left((2l-1)\frac{2\pi}{\omega}t\right)x \tag{6}$$

where L is the positive integer. The closed loop solution is given by

$$x(t) = \exp\left((a+k)t - \frac{\omega k}{4\pi L}\sin\left(\frac{4\pi L}{\omega}t\right)\right)x(0) \tag{7}$$

by the use of the following identity

$$2(\sin p)\sum_{l=1}^{L}\sin((2l-1)p) = 1 - \cos(2Lp).$$

The closed loop system is stabilized by choosing k satisfying $k < -a$. In addition, the closed loop solution (7) converges to the solution of its average system (5) by choosing sufficiently large L.

III. Main Results

In this section we extend the idea of the feedback control by raising frequency in (6) to a multidimensional case. Since the closed loop solution cannot be explicitly derived for this case, the averaging method is applied to prove the closed loop stability as well as to show the asymptotic approximation by its average system.

Firstly we introduce the key lemma.

Lemma 1: Given a positive integer N and a periodic function $g(t)$ defined in (2). Factor rational numbers ϕ_l contained in $g(t)$ as the ratio of coprime integers ν_l and $\delta_l\,(\geq \nu_l)$ for each $l = 1, \cdots, L$, *i.e.*

$$\phi_l =: \frac{\nu_l}{\delta_l}.$$

There exists a function $f(t, N)$ which is ω-periodic and continuous for t, and satisfies

$$g(t)f(t,N) = 1 - \cos\frac{4\pi NG}{\omega}t, \tag{8}$$

where G is the least common multiple of δ_l.

Proof: ϕ_l is represented by the following fractional form

$$\phi_l = \frac{\psi_l}{2NG}$$
$$\psi_l := \frac{2\nu_l NG}{\delta_l}$$

for each l. We note that ψ_l are nonnegative integers by their construction. Then there exist nonnegative integers φ_m $(m = 1, \cdots, 2NG-L)$ such that the set $\{\varphi_m\}$ is a complementary of $\{\psi_l\}$ in $\{0, \cdots, 2NG-1\}$. Define a function

$$f(t,N) := \frac{2^{2NG-1}}{(-1)^{NG}} \prod_{m=1}^{2NG-L} \sin\left(\frac{2\pi}{\omega}\left(t - \frac{\varphi_m \omega}{2NG}\right)\right),$$

then it is clear that $f(t,N)$ is ω-periodic and continuous for t. Since the identity

$$\prod_{r=1}^{s} \sin\left(z + \frac{2(r-1)\pi}{n}\right) = (-1)^{\frac{n}{2}} 2^{-(n-1)}(1 - \cos nz)$$

is satisfied for even $s > 0$, it can be shown that $f(t,N)$ satisfies the identity (8). ■

Remark : For even G, it can be shown that there exists a function $f(t,N)$ which is ω-periodic and continuous for t, and satisfies the identity

$$g(t)f(t,N) = 1 - \cos\frac{2\pi NG}{\omega}t. \tag{9}$$

■

Next the periodic feedback control law is derived for the linear periodic system (1). The candidate of the periodic feedback control law is given by

$$u = f(t,N)\,k\,x, \tag{10}$$

where $k \in \mathbf{R}^{1\times n}$ is the constant matrix and f is the scalar continuous function satisfying (8) for a given positive integer N. The choice of k and N will be clear in the subsequent of this section. The closed loop system consisted of (1) and (10) is given by

$$\dot{x} = \left(A + bk - bk\cos\frac{4\pi NG}{\omega}t\right)x. \tag{11}$$

Define the parameter

$$\varepsilon := \frac{\omega}{4\pi NG}$$

and transform the time scale as follows

$$\tau = \frac{t}{\varepsilon}.$$

Then the closed loop system is transformed to be

$$\frac{dx}{d\tau} = \varepsilon(A + bk - bk\cos\tau)x. \tag{12}$$

Now we choose k and N so that the linear periodic system (12) becomes exponentially stable. In the first step, we choose k such that $A+bk$ is stable, *i.e.* the real parts of all eigenvalues of $A+bk$ are negative. Then the average system

$$\frac{dx}{d\tau} = \varepsilon(A + bk)x. \tag{13}$$

becomes exponentially stable. In the second step, we choose sufficiently large N such that the closed loop system (11) becomes exponentially stable. Since we have chosen k such that the average system (13) is exponentially stable, it follows from the averaging method that the linear periodic system (12) is exponentially stable for sufficiently small ε [4]. Transforming the time scale into the original time scale t, the closed loop system (11) becomes exponentially stable for sufficiently large N.

The aim of this paper to propose a linear periodic state feedback control law satisfying the properties i')–iii'). In order to prove those properties for (10), we need to qualitatively evaluate the statements of the averaging method (see Appendix). Based on this result, the closed loop stability is proved and the requirement iii') is shown to be satisfied as follows:

Theorem 1: Assume that $A+bk$ is stable. Let P and Q be the solutions of the Lyapnov equation

$$(A+bk)^T P + P(A+bk) = -Q. \tag{14}$$

Let M be a scalar constant satisfying

$$M > \frac{\omega}{4\pi Gc} \tag{15}$$
$$c := \|bk\|(2(2\|A+bk\| + \|bk\|)P_{\min}Q_{\min}^{-1} + 1, \tag{16}$$

where $P_{\max}$, $P_{\min}$, $Q_{\min}$ are the maximum, minimum eigenvalues of P and the minimum eigenvalue of Q respectively. Then the closed loop system (11) is exponentially stable for all integer $N \geq M$. Furthermore if the initial condition satisfy

$$\|x(0,N) - x_{ave}(0)\| \leq \frac{\rho_1}{N} \tag{17}$$
$$\|x(0,N)\| \leq \rho_2, \tag{18}$$

where x is the solution of (11) and x_{ave} is the solution of its average system

$$\dot{x}_{ave} = (A+bk)x_{ave}, \tag{19}$$

the initial response of (11) is approximated by

$$\|x(t,N) - x_{ave}(t)\| \leq \frac{\kappa(\rho_1,\rho_2)}{N} \tag{20}$$
$$\kappa(\rho_1,\rho_2) := \frac{P_{\max}^{\frac{1}{2}}}{P_{\min}^{\frac{1}{2}}}(\rho_1 + \rho_3\rho_2) \tag{21}$$
$$\rho_3 := \|bk\| + \frac{cP_{\max}}{P_{\min}} \tag{22}$$

for all $t \geq 0$ and for all integer $N \geq M$. ■

In this method, two independent designing parameters k and N are available. Firstly k is used to stabilize the average system. Then N is used to shape the closed loop solution by generating the sufficiently high frequency trigonometric function in the coefficient of the closed loop system. Both procedures are easily carried out via simple symbolic computations and the requirement ii') is satisfied.

Lastly we show that the requirement i') is satisfied. Let $\Phi(t,N)$ denotes the fundamental matrix of the closed loop system (11), *i.e.* Φ is the solution of the following initial value problem:

$$\dot{\Phi} = \left(A + bk - bk \cos \frac{4\pi NG}{\omega} t \right) \Phi, \quad \Phi(0, N) = I.$$

Consider the same initial condition $x(0, N) = x_{ave}(0) =: x_0$ for (11) and (19). Substitute $t = \omega$ into (20), we have

$$\|(\Phi(\omega, N) - e^{(A+bk)\omega})x_0\| \le \frac{\rho_3 P_{\max}^{\frac{1}{2}}}{N P_{\min}^{\frac{1}{2}}} \|x_0\|,$$

then it follows that $\Phi(\omega, N)$ uniformly converges to $e^{(A+bk)\omega}$, *i.e.*

$$\lim_{N \to \infty} \|\Phi(\omega, N) - e^{(a+bk)\omega}\| = 0.$$

If the system (1) is controllable, (A, b) becomes controllable and the characteristic multipliers of (19) can be specified to the desired place. Then the characteristic multipliers of the closed loop systems asymptotically converge to those specified ones, therefore the requirement i') is satisfied.

Corollary 1: Suppose that the system (1) is controllable. Choose the constant matrix k such that the characteristic multipliers of (19) are specified to the desired place. Then, as the integer $N \to \infty$, the characteristic multipliers of the closed loop system consisted of (1) and (10) converge to the desired ones. ■

IV. Illustrative Example

In this section we study the problem of attitude stabilization for small satellites with magnetic actuators. Observing the periodic nature of the geomagnetic field, the Euler's equation linearlized around the yaw axis is given by

$$\dot{x} = Ax + bg(t)u$$
$$A = \begin{bmatrix} 0 & 1 \\ 28.8 & 0 \end{bmatrix}, \; b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; g(t) = \cos(2\pi t)$$

where $x := [\, \gamma \; \dot{\gamma} \,]^T$ consists of the yaw angle γ and the yaw rate $\dot{\gamma}$. The period ω is normalized to be $\omega = 1$. The control objective is to design a linear periodic state feedback control law such that the satellite is stabilized within a half period. So the requirement iii') have importance in this application. In addition, the magnitude of the control input is restricted to be less than 10 [A-m^2] from the practical reason [6].

The 1-periodic function g is represented by the form (2)

$$g(t) = \sin\left(2\pi \left(t - \frac{3\pi}{4}\right)\right),$$

the least common divisor G is chosen to be 4.

Since (A, b) is controllable, a linear periodic state feedback control law is designed based on Corollary 1. In the first step, we design the state feedback gain to be $k = [\, 8.52 \quad 1 \,]$ such that the average system (almost) converges to 0 at $t = 0.5$ and the magnitude of the input for the average system to be less than 10 [A-m^2]. In the second step, we compute the periodic coefficient $f(t, N)$ based on (9), *e.g.*

$$f(t, 1) = 2(\cos 2\pi t - \cos 6\pi t)$$
$$f(t, 4) = 2(\cos 2\pi t - \cos 6\pi t + \cos 10\pi t - \cos 14\pi t + \cos 18\pi t - \cos 22\pi t + \cos 26\pi t - \cos 30\pi t).$$

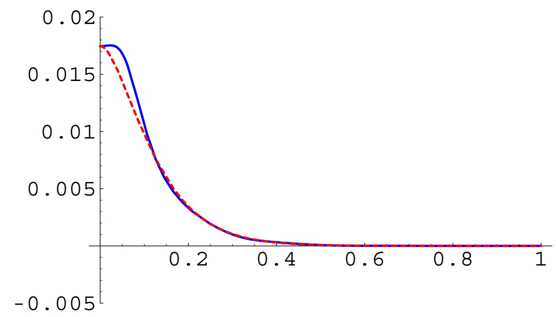

Fig. 1. Yaw Angle γ $(N = 1)$

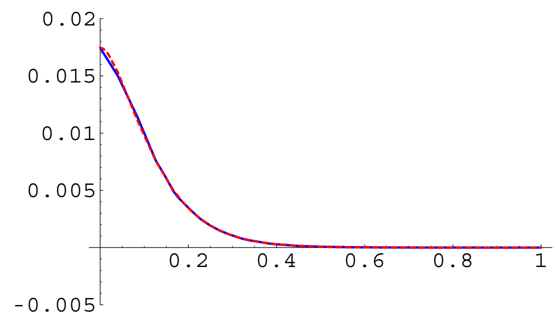

Fig. 2. Yaw Angle γ $(N = 4)$

A set of simulation is carried out to evaluate the effectiveness of the proposed method. The initial response is computed for $N = 1$ and $N = 4$ with the initial value $\gamma(0) = 2\pi/360$ [rad], $\dot{\gamma}(0) = 0$ [rad/period]. The initial responses of $\gamma, \dot{\gamma}$ (blue solid line) converge to those of the average system (red dot line) by raising frequency (see Fig. 1–4). The magnitude of the input u is bounded by 10 [A-m^2], therefore the control objective is satisfied (see Fig. 5–6).

V. Conclusion

The novel periodically time-varying control law was proposed for a class of continuous-time linear periodic systems. The designing procedure consists of two steps. A constant matrix is selected to specify the desired closed loop solution at first. Then a scalar periodic coefficient is selected to stabilize the closed loop systems as well as to make the closed loop solution to asymptotically converge to the solution of the average system. Both procedures are computed via symbolic computations, therefore the proposed controller can be easily computed.

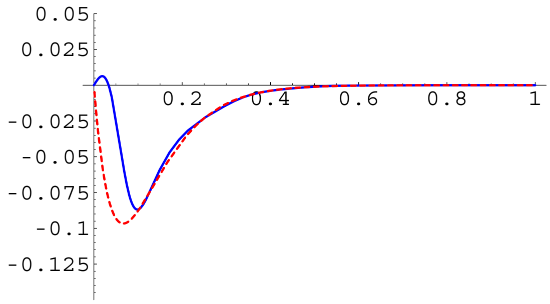

Fig. 3. Yaw Rate $\dot{\gamma}$ $(N = 1)$

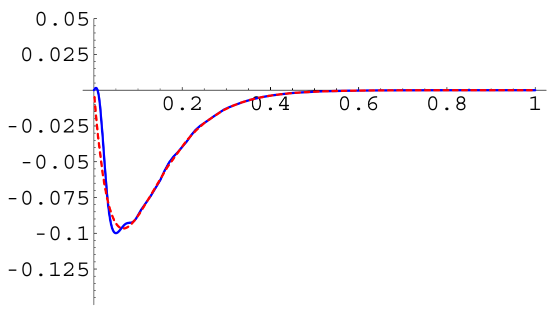

Fig. 4. Yaw Rate $\dot{\gamma}$ $(N = 4)$

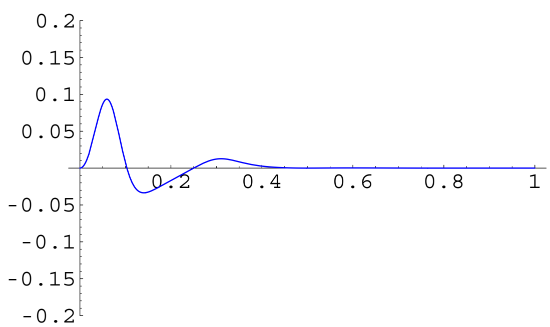

Fig. 5. Input u $(N = 1)$

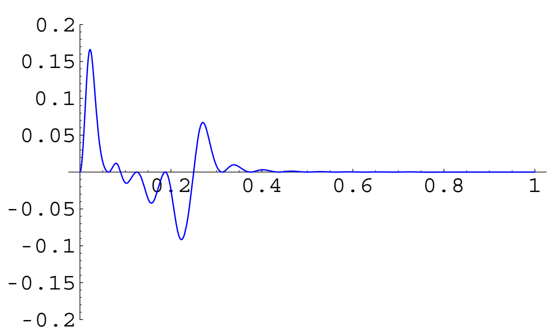

Fig. 6. Input u $(N = 4)$

Appendix: Averaging Method

The averaging method was proved for general periodically time-varying nonlinear systems [4]. But it is described quantitatively and cannot be applied to prove the required property i')–iii'). Therefore we give an alternative proof for linear periodic systems and make a quantitative evaluation.

Lemma 2: Let A_{ave} denotes the stable matrix and $A_p(t)$ denotes the continuous ω-periodic function satisfying

$$\int_0^{\omega} A_p(t)dt = 0. \tag{23}$$

Let $P = P^T > 0$ and $Q = Q^T > 0$ denote the solution of the Lyapnov equation

$$A_{ave}^T P + P A_{ave} = -Q. \tag{24}$$

Choose a scalar $\varepsilon^* > 0$ satisfying

$$0 < \varepsilon^* < c_4^{-1} \tag{25}$$

where c_4 is defined by

$$c_1 := \sup_{0 \le t} \|F(t)\| \tag{26}$$

$$c_2 := \|A_{ave}\| \tag{27}$$

$$c_3 = \sup_{0 \le t} \|A_p(t)\| \tag{28}$$

$$c_4 := c_1(2(2c_2 + c_3)P_{\max}Q_{\min}^{-1} + 1) \tag{29}$$

$$F(t) := \int_0^t A_p(\tau)d\tau \tag{30}$$

and $P_{\max}, P_{\min}, Q_{\max}$ are the maximum, minimum eigenvalues of P and the maximum eigenvalue of Q respectively. Then the linear periodic system

$$\dot{x}(t,\varepsilon) = \varepsilon A(t)x(t,\varepsilon) \tag{31}$$

$$A(t) := A_{ave} + A_p(t) \tag{32}$$

is exponentially stable for

$$0 < \varepsilon \le \varepsilon^*. \tag{33}$$

In addition, if the initial condition satisfies

$$\|x(0,\varepsilon) - x_{ave}(0,\varepsilon)\| \le \varepsilon\rho_1 \tag{34}$$

$$\|x(0,\varepsilon)\| \le \rho_2, \tag{35}$$

where x_{ave} is the solution of the average system

$$\dot{x}_{ave}(t,\varepsilon) = \varepsilon A_{ave} x_{ave}(t,\varepsilon), \tag{36}$$

the initial response of (31) is approximated by

$$\|x(t,\varepsilon) - x_{ave}(t,\varepsilon)\| \le \varepsilon\kappa(\rho_1,\rho_2) \tag{37}$$

where κ is defined by

$$\kappa(\rho_1,\rho_2) := \frac{P_{\max}^{\frac{1}{2}}}{P_{\min}^{\frac{1}{2}}}(\rho_1 + \rho_2\rho_3) \tag{38}$$

$$\rho_3 := c_1 + \frac{c_4 P_{\max}}{P_{\min}}. \tag{39}$$

Proof: Firstly we prove that (31) is exponentially stable for ε satisfying (33). Consider the change of variables

$$x(t,\varepsilon) = (I + \varepsilon F(t))y(t,\varepsilon).$$

Differentiating both sides with respect to t, we obtain

$$\dot{y}(t,\varepsilon) = (\varepsilon A_{ave} + \varepsilon^2 B(t,\varepsilon))y(t,\varepsilon) \quad (40)$$

$$B(t,\varepsilon) = \frac{1}{\varepsilon}((I + \varepsilon F(t))^{-1} - I)A_{ave} + (I + \varepsilon F(t))^{-1}A(t)F(t).$$

Then it can be shown that $B(t,\varepsilon)$ is uniformly bounded

$$\sup_{0\le t, 0<\varepsilon\le\varepsilon^*} \|B(t,\varepsilon)\| \le \frac{c_1(2c_2 + c_3)}{1 - \varepsilon^* c_1}.$$

Let

$$V(y) := y^T P y$$

be the Lyapnov function candidate for (40). The derivative of V along the trajectories of (40) satisfies

$$\dot{V} \le \varepsilon c_5 \|y\|^2$$

$$c_5 := -Q_{\min} + \frac{2\varepsilon^* P_{\max} c_1(2c_2 + c_3)}{1 - \varepsilon^* c_1}.$$

Note that $c_5 < 0$. By the comparison lemma, it can be shown that $y(t,\varepsilon)$ satisfies the inequality

$$\|y(t,\varepsilon)\| \le \frac{P_{\max}^{\frac{1}{2}}}{P_{\min}^{\frac{1}{2}}} \exp\left(\frac{c_5\varepsilon}{2P_{\max}}t\right) \|y(0,\varepsilon)\|. \quad (41)$$

Hence (31) is shown to be exponentially stable

$$\|x(t,\varepsilon)\| \le c_6 \exp\left(\frac{c_5\varepsilon}{2P_{\max}}t\right) \|x(0,\varepsilon)\|.$$

$$c_6 := (1 + \varepsilon^* c_1)\frac{P_{\max}^{\frac{1}{2}}}{P_{\min}^{\frac{1}{2}}}$$

Next we prove the asymptotic convergence of the initial response. Approximate the linear periodic system (40) by the average system (36) and define the approximation error

$$e(t,\varepsilon) := y(t,\varepsilon) - x_{ave}(t,\varepsilon).$$

Differentiating both sides with respect to t, we obtain

$$\dot{e}(t,\varepsilon) = \varepsilon A_{ave} e(t,\varepsilon) + \varepsilon^2 B(t,\varepsilon)y(t,\varepsilon). \quad (42)$$

Let

$$U(e) := e^T P e$$

be the Lyapnov function candidate for (42). Differentiate U along the trajectories of (42), we obtain

$$\dot{U} \le -\frac{Q_{\min}\varepsilon}{P_{\max}}U + \varepsilon^2 c_7 \sqrt{U}\|y\|.$$

$$c_7 := \frac{2c_1(2c_2 + c_3)P_{\max}}{(1 - \varepsilon^* c_1)P_{\min}^{\frac{1}{2}}}$$

To obtain a linear differential inequality, we take

$$W(e(t,\varepsilon)) := \sqrt{U(e(t,\varepsilon))},$$

then it can be shown that

$$D^+W \le -\frac{Q_{\min}\varepsilon}{2P_{\max}}W + \frac{\varepsilon^2 c_7}{2}\|y\|$$

where D^+W is the upper right-hand derivative of W with respect to t [4]. Substitute (41) into the right hand side of the above equation, it can be shown that

$$\begin{aligned} &W(e(t,\varepsilon)) \\ &\le \exp\left(-\frac{Q_{\min}\varepsilon}{2P_{\max}}t\right) W(e(0,\varepsilon)) + \varepsilon\eta(t,\varepsilon)\|y(0,\varepsilon)\| \\ &\eta(t,\varepsilon) \\ &:= \frac{\varepsilon c_7}{2}\int_0^t \exp\left(-\frac{Q_{\min}\varepsilon}{2P_{\max}}(t-\tau)\right)\frac{\|y(\tau,\varepsilon)\|}{\|y(0,\varepsilon)\|}d\tau \\ &= \frac{P_{\max}^{\frac{3}{2}}}{\varepsilon^* P_{\min}}\left(\exp\left(\frac{c_5\varepsilon}{2P_{\max}}t\right) - \exp\left(-\frac{Q_{\min}\varepsilon}{2P_{\max}}t\right)\right) \end{aligned}$$

by the comparison lemma. Since $\eta(t,\varepsilon)$ is uniformly bounded on $[0,\infty) \times (0,\varepsilon^*]$, we obtain

$$\|e(t,\varepsilon)\| \le \frac{P_{\max}^{\frac{1}{2}}}{P_{\min}^{\frac{1}{2}}} \exp\left(-\frac{Q_{\min}\varepsilon}{2P_{\max}}t\right) \|e(0,\varepsilon)\| + c_8\varepsilon\|y(0,\varepsilon)\|$$

$$c_8 := \frac{P_{\max}^{\frac{3}{2}}}{\varepsilon^* P_{\min}^{\frac{3}{2}}}$$

Approximate the linear periodic system (31) by its average system (36), we obtain

$$\begin{aligned} &\|x(t,\varepsilon) - x_{ave}(t,\varepsilon)\| \\ &\le \|e(t,\varepsilon)\| + \varepsilon\|F(t)\|\|y(t,\varepsilon)\| \\ &\le \frac{P_{\max}^{\frac{1}{2}}}{P_{\min}^{\frac{1}{2}}}(\|x(0,\varepsilon) - x_{ave}(0,\varepsilon)\| + \varepsilon\rho_3\|x(0,\varepsilon)\|). \end{aligned}$$

Hence the initial response of (31) is shown to asymptotically converge to that of the average system (36). ■

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