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Applications of the Joint Spectral Radius to Some Problems of Functional Analysis, Probability and Combinatorics

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Abstract— **In this paper we discuss applications of the joint spectral characteristics of finite dimensional linear operators such as joint spectral radius, lower spectral radius, p-radius, Lyapunov exponent etc. to some problems of functional analysis, fractal geometry, probability theory and combinatorial number theory.**

I. INTRODUCTION

The joint spectral radius and the lower spectral radius of liner operators have found a lot of applications in various areas of mathematics. For the sake of simplicity in this paper we consider the case of two operators, although most of results can be easily generalized to an arbitrary finite family of linear operators.

***Definition** 1: The joint spectral radius (JSR) of finite-dimensional linear operators A_0, A_1 is the value*

$$\hat{\rho} \;=\; \lim_{m\to\infty} \max_{d_1,\ldots,\,d_m\in\{0,1\}} \|A_{d_1}\cdots A_{d_m}\|^{1/m}.$$

The lower spectral radius is

$$\check{\rho} \;=\; \lim_{m\to\infty} \min_{d_1,\ldots,\,d_m\in\{0,1\}} \|A_{d_1}\ldots A_{d_m}\|^{1/m}$$

Both these limits exist and do not depend on the norm (see, for instance, [1]). Many problems is reduced to computing or estimating JSR or LSR of suitable linear operators. Although the numerical computation of these values is hard, in some practical cases it is possible to find them precisely. In this paper we discuss mostly those applications, where JSR or LSR can be found explicitly.

In nearly all cases when it is possible to compute the values $\hat{\rho}$ and $\check{\rho}$ precisely it is done by the same approach using the following two simple statements. The first one is well known:

***Proposition** 1: For any operators A_0, A_1 and for any their product $\Pi_m = A_{d_1}\cdots A_{d_m}$ we have*

$$\check{\rho} \;\le\; \bigl(\rho(\Pi_m)\bigr)^{1/m} \;\le\; \hat{\rho}\,.$$

(By $\rho(A)$ we denote the usual spectral radius of the operator A, which is the largest modulo of its eigenvalues).

Proof: For every k we have

$$\min_{d_1,\ldots,d_{km}} \|\Pi_{km}\| \;\le\; \bigl\|(\Pi_m)^k\bigr\| \;\le\; \max_{d_1,\ldots,d_{km}} \|\Pi_{km}\|.$$

It remains to take the power $1/km$ of these three values and to take the limit as $k\to\infty$. ∎

The second statement is less known, but is also simple.

***Proposition** 2: Suppose that linear operators A_0, A_1 act in $\mathbb{R}^d$, and λ is a positive value; then* :

a) *if there is a convex body (a convex compact set with a nonempty interior) $M\subset\mathbb{R}^d$ symmetric with respect to the origin and such that for some integer $m\ge 1$ we have*

$$\Pi_m M \;\subset\; \lambda^m M,$$

for all $\Pi_m = A_{d_1}\cdots A_{d_m}$, then $\hat{\rho}(A_0,A_1)\le\lambda$

b) if there is a closed set $Q\subset\mathbb{R}^d$ not containing the origin such that for some integer $m\ge 1$ we have

$$\Pi_m Q \;\subset\; \lambda^m Q,$$

for all $\Pi_m = A_{d_1}\cdots A_{d_m}$, then $\check{\rho}(A_0,A_1)\ge\lambda$.

Proof: a) Denote by $\|\cdot\|_M$ the Minkowski norm, corresponding to the convex body M. Since $\|\Pi_m\|_M\le\lambda^m$, it follows that the norm of each product $A_{d_1}\ldots A_{d_{km}}$ is at most λ^{mk}. Whence, $\hat{\rho}\le\lambda$.

b) Let $h=\inf\{\|u\|,\,u\in Q\}$. Clearly, $h>0$. Take an arbitrary point $a\in Q$. For any $d_1,\ldots,d_{mk}$ we obtain

$$A_{d_1}\ldots A_{d_{mk}}a\in\lambda^{mk}Q.$$

Consequently $\|A_{d_1}\ldots A_{d_{mk}}a\|\ge h\lambda^{mn}$, therefore $\check{\rho}\ge\lambda$. ∎

The idea of computing of JSR and LSR is the following: Proposition 2 gives an upper bound for $\hat{\rho}$ and a lower bound for $\check{\rho}$. To obtain this we need only to present an appropriate convex body M and, respectively, a closed set Q. On the other hand, Proposition 1 provides converse estimates: for each product Π_m the value $\bigl(\rho(\Pi_m)\bigr)^{1/m}$ estimates $\hat{\rho}$ from below and $\check{\rho}$ from above. If we are "lucky", then an upper bound will coincide with the lower one, and the precise values of LSR or JSR will be found. For this it suffices to find a suitable product Π_m and a set M (or Q).

In the next section we present a detailed analysis of applications of JSR and LSR to the computing of the global and local regularity of fractal curves. For this we involve some other spectral characteristics of linear operators such as the p-radius, the Lyapunov exponent and the JSR along a sequence. These results is applied to study wavelets and scaling functions (section III), the distribution of random power series (chapter IV) and the asymptotic behavior of the Euler partition function (section V).

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II. FRACTAL CURVES

In this section we analyse the regularity of fractal curves. We devote much attention to this topic, because the further applications (section III - V) will be based on the results of this section.

Let $\tilde{A}_0, \tilde{A}_1$ be affine operators acting in the d-dimensional Euclidian space $\mathbb{R}^d$. By A_i we denote the linear part of the operator $\tilde{A}_i,\ i = 0, 1$. So, A_i is a linear operator. We always assume that this pair of affine operators is irreducible (the operators have no common invariant affine subspaces) otherwise one can consider the restriction of these operators on the invariant subspace.

***Definition** 2: Fractal (self-similar set) of two affine operators $\tilde{A}_0, A_1$ is a compact set $K \subset \mathbb{R}^d$ such that*

$$\tilde{A}_0 K \cup \tilde{A}_1 K = K.$$

According to the classical results of J.Hutchinson [2] if the operators $\tilde{A}_i$ are both contraction, i.e., $\|A_i\| < 1,\ i = 0, 1$, then this pair of operators possesses a unique fractal K. This sufficient condition for the existence of a fractal can be sharpened to almost a criterion by means of JSR, which was proved in [3]:

***Proposition** 3: Let $\tilde{A}_0, \tilde{A}_1$ be an irreducible pair of affine operators acting in $\mathbb{R}^d$. If $\hat{\rho}(A_0, A_1) < 1$, then this pair possesses a unique fractal. Conversely, it the pair $\tilde{A}_0, A_1$ possess a fractal, then $\hat{\rho}(A_0, A_1) \le 1$.*

Everywhere below we assume $\hat{\rho}(A_0, A_1) < 1$. This, in particular, yields $\rho(A_i) < 1,\ i = 0, 1$, where ρ is the usual spectral radius. This, in turn, implies that each affine operator $\tilde{A}_i$ is contraction in a suitable norm, hence it possesses a unique fixed point v_i. Thus, $\tilde{A}_i v_i = v_i\,, \quad i = 0, 1$.

***Theorem** 1: Let an irreducible pair of affine operators $\tilde{A}_0, \tilde{A}_1$ satisfy the following two assumptions:*
1) $\hat{\rho}(A_0, A_1) < 1\,$;
2) $\tilde{A}_0 v_1 = \tilde{A}_1 v_0$ *(v_i is the fixed point of the operator $\tilde{A}_i$), then the fractal K of these operators is an image of a continuous curve in $\mathbb{R}^d$. There is a unique continuous function $v : [0, 1] \to \mathbb{R}^d$ (a fractal curve) such that*

$$v(x) \ = \ \tilde{A}_i\, v(2x - 1), \qquad x \in \Big[\frac{i}{2}, \frac{i+1}{2}\Big],\ i = 0, 1, \quad (1)$$

and therefore $v\big([0, 1]\big) = K$. The values of the function are given by the formula

$$v(x) \ = \ \lim_{m\to\infty} \tilde{A}_{d_1} \cdots \tilde{A}_{d_m} v_0, \qquad (2)$$

where $d_1, \ldots, d_m, \ldots$ are digits in the binary expansion of the number x, so $x = 0.d_1 \ldots d_m \ldots$. In particular, at diadic points $x = 0.d_1 \ldots d_m$ one has

$$v(x) \ = \ \tilde{A}_{d_1} \cdots \tilde{A}_{d_m} v_0, \qquad (3)$$

Conversely, if for irreducible pair of affine operators A_0, A_1 equation (1) possesses a continuous solution, then $\hat{\rho}(A_0, A_1) < 1$, the both operators $\tilde{A}_i$ has fixed points v_i and $\tilde{A}_0 v_1 = \tilde{A}_1 v_0$.

It turns out that the JSR not only gives the criterion of existence for continuous fractal curves, but also allows us to express precisely their regularity.

Let us recall some notation. The modulus of continuity of a function $v(x)$ is the value

$$\omega(v, t) \ = \ \sup_{x\in[0,1],\,|h|\le t} \|v(x) - v(h + h)\|$$

The Hölder exponent of a function $v(x)$ is

$$\alpha_v(x) \ = \ \sup\Big\{\alpha \ge 0,\ \omega(v, t) \le Ct^{\alpha}\Big\}.$$

***Theorem** 2: Under the assumptions of Theorem 1 we have*

$$\alpha_v \ = \ -\log_2 \hat{\rho}(A_0, A_1). \qquad (4)$$

Moreover, if the pair of linear operators A_0, A_1 is irreducible (they do not have a nontrivial common invariant linear *subspace), then*

$$C_1 t^{\alpha_v} \ \le \ \omega(v, t) \ \le C_2 t^{\alpha_v}, \qquad (5)$$

where C_1, C_2 are positive constants.

***Remark** 1:* The expression for the Hölder exponent of fractal curves was first derived (under some stricter conditions) in [4], [5]. Theorem 2 gives in addition the explicit asymptotic for the moduli of continuity. Moreover, the constants C_1, C_2 can be effectively estimated for every irreducible pair of operators [3].

We give the common proof of Theorems 1 and 2.
Proof: We begin with proving the convergence of the limit (2). Simultaneously we establish that the function given by that formula is continuous and $\alpha_v \ge -\log_2 \hat{\rho}$.

First we define the function $v(x)$ at diadic points x by formula (3). Let $x < y$ be dyadic points, and $(x, y) \ne (0, 1)$. Denote by z the dyadic number of the smallest order (that has the form $z = k2^{-q}$ with the smallest possible q) such that $x \le z \le y$. Such z is clearly unique. We have

$$z = 0.d_1 \ldots d_q\,, \quad y = 0.d_1 \ldots d_q d_{q+1} \ldots$$

Let us denote by r the smallest number such that $r > q$ and $d_r = 1$. It is easy to see that $|y - z| > 2^{-r}$. Let $P \subset \mathbb{N}$ be the set of all indices p such that the pth digits after the dyadic point (in the expansions of y and z) are different. Clearly, r is the smallest element of the set P. We have

$$\big\|v(y) - v(z)\big\| = \Big\|\sum_{p\in P} v(0, d_1 \ldots d_{p-1} 1) - v(0, d_1 \ldots d_{p-1} 0)\Big\|$$

$$\le \Big\|\sum_{p\ge r} v(0, d_1 \ldots d_{p-1} 1) - v(0, d_1 \ldots d_{p-1} 0)\Big\|$$

$$= \sum_{p\ge r} \Big\|A_{d_1} \cdots A_{d_{p-1}}(v_1 - v_0)\Big\| \le \sum_{p\ge r} C_{\varepsilon}(\hat{\rho} + \varepsilon)^{p-1}$$

$$= \frac{C_{\varepsilon}(\hat{\rho} + \varepsilon)^r}{(\hat{\rho} + \varepsilon)(1 - \hat{\rho} - \varepsilon)} \le \frac{C_{\varepsilon}}{(\hat{\rho} + \varepsilon)(1 - \hat{\rho} - \varepsilon)} \cdot |y - z|^{-\log_2(\hat{\rho}+\varepsilon)}$$

where the constant C_ε depends only on $\hat\rho$ and ε. Having estimated the value $\|v(z)-v(x)\|$ in the same way we get

$$\|v(y)-v(x)\| \ \le \ \tilde C_\varepsilon |y-x|^{-\log_2(\hat\rho+\varepsilon)}, \qquad (6)$$

where $\tilde C_\varepsilon$ depends only on $\hat\rho$ and ε. This inequality holds for all dyadic x, y, therefore the function $v(x)$ is uniformly continuous on the set of dyadic numbers. Hence this function is continuously extended by formula (2) onto the whole segment $[0,1]$, and for all x, y inequality (6) holds. Thus,

$$\alpha_v \ge -\log_2 \hat\rho.$$

If the pair of operators A_0, A_1 is irreducible, then we use the inequality $C_1\hat\rho^m \le \max\limits_{d_1,\ldots,d_m} \|A_{d_1}\cdots A_{d_m}\| \le C_2\hat\rho^m$ [3]. Repeating our proof and setting $\varepsilon = 0, C_\varepsilon = C_2$, we obtain

$$\|v(y)-v(x)\| \ \le \ C_2 |y-x|^{-\log_2 \hat\rho}.$$

It remains to establish te inverse inequality $\alpha_v \le -\log_2\hat\rho$. Consider the set L of all points $u\in\mathbb{R}^d$ such that

$$\max_{d_1\ldots,d_m} \|\Pi_m u\| = o(1)\,\hat\rho^m\,, \ \text{as } m\to\infty, \qquad (7)$$

where $\Pi_m = A_{d_1}\cdots A_{d_m},\ o(1)\to 0$. Clearly, L is a linear subspace in $\mathbb{R}^d$ invariant with respect to both A_0, A_1. Let us denote $a = v_1 - v_0$. If $a \in L$, then $L = \mathbb{R}^d$, otherwise the affine plane $v_0 + L$ would be a nontrivial common invariant subspace of A_0, A_1. Thus, (7) holds for all $u \in \mathbb{R}^d$. Taking an orthonormal basis $\{u_j\}_{j=1}^d$ in $\mathbb{R}^d$ and applying (7) to its elements, we get

$$\max_{d_1,\ldots,d_m} \|\Pi_m u_j\| \ \le \ r_m\,\hat\rho\,, \quad m\in\mathbb{N},\ j=1,\ldots,d,$$

where $r_m \to 0$ as $m\to\infty$. Therefore, for any element of the unit sphere $u = \sum\limits_{j=1}^d \beta_j u_j\,,\ \sum\limits_{j=1}^d \beta_j^2 = 1$ we have

$$\max_{d_1,\ldots,d_m} \|\Pi_m u\| \ \le \ \sum_{j=1}^d \beta_j \|\Pi_m u_j\| \le r_m \sum_{j=1}^d \beta_j \ \le \ \sqrt{d}\, r_m\,\hat\rho.$$

Thus, $\max\limits_{d_1,\ldots,d_m} \|\Pi_m\|^{1/m} \ \le \ [\sqrt{d}\,r_m\,\hat\rho]^{1/m}$, which becomes less than $\hat\rho$ as $m\to\infty$. This contradiction shows that $a \notin L$. Whence there exists a constant $C_1 > 0$ and arbitrarily long sequences $d_1,\ldots,d_m$, for which

$$\|A_{d_1}\cdots A_{d_m}(v_1 - v_0)\| \ge C_1\,\hat\rho^m.$$

Therefore, for the points $x = 0.d_1\ldots d_m 0$ and $y = 0.d_1\ldots d_m 1$ we have

$$\|v(y)-v(x)\| \ \ge \ C_1\,\hat\rho^m \ = \ C_1\,(y-x)^{-\log_2\hat\rho}, \qquad (8)$$

and hence $\alpha_v \le -\log_2\hat\rho$.

Conversely, if equation (1) possesses a continuous solution $v(x)$, then the left hand side of (8) tends to zero, and therefore $\hat\rho < 1$. Furthermore, it follows from (1) that $v_i = v(i)$ is a fixed point of the operator $\tilde A_i\,,\ i = 0,1$ and

$$v\big(1/2\big) \ = \ A_0 v(1) \ = \ A_1 v(0).$$

■

The spectral characteristics of the operators A_0, A_1 express not only the global regularity of the fractal curves on the whole segment $[0,1]$, but also a local behavior at each point x. For given $x\in[0,1]$ the local Hölder exponent of the function v at the point x is defined as

$$\alpha_v(x) \ = \ \sup\Big\{\alpha\ge 0,\ \|v(x+h)-v(x)\| \le Ch^\alpha\Big\}. \quad (9)$$

In contrast to the global exponent of regularity α_v, the local exponent can take arbitrary large values, including $+\infty$. The local exponent is expressed in terms of the so-called JSR along a sequence.

***Definition** 3: Let A_0, A_1 be two linear operators and $(x) = d_1, d_2, \ldots$ be an infinite sequence of zeros and ones. Then the joint spectral radius along the sequence (x) is*

$$\hat\rho_x(A_0,A_1) \ = \ \limsup_{m\to\infty} \|A_{d_1}\cdots A_{d_m}\|^{1/m}$$

We call a number $x\in[0,1]$, *normal* if for any $\varepsilon > 0$ one can find a number $n(\varepsilon)$ such that for every $m \ge n(\varepsilon)$ the binary expansion of $x = 0.d_1d_2\ldots$ contains two different digits $d_k \ne d_l$ with $m \le k < l \le m(1+\varepsilon)$. In short, the normal numbers can not be approximated too good by dyadic rationals. Almost all (in Lebesgue measure) points of the segment $[0,1]$ are normal. All rational numbers are normal except for dyadic ones. The proof of the next theorem is similar to that of Theorems 1 and 2, and we omit it.

***Theorem** 3: For any point $x = 0.d_1d_2\ldots$ one has*

$$\alpha_v(x) \ \le \ -\log_2\hat\rho_x.$$

If, moreover, x is normal, then

$$\alpha_v(x) \ = \ -\log_2\hat\rho_x.$$

This theorem allows us to make comprehensive conclusions on the distribution of points with a given exponents of local regularity. First of all, if the operators A_0, A_1 are both nondegenerate, then the value $\hat\rho_x$ does not depend on any finite number of digits in the binary expansion of x. So, the local regularity at normal points depends entirely on the "tail" of the sequence $x = 0.d_1d_2\ldots$. Therefore, by the so-called "low of zero and one" (see, for instance, [6]) for almost all points x the exponents $\alpha_v(x)$ are the same. This average local regularity (we denote it by $\alpha_{\rm av}$) is expressed by the formula

$$\alpha_{\rm av} \ = \ -\log_2\bar\rho\,,$$

where

$$\bar\rho(A_0,A_1) \ = \ \lim_{m\to\infty}\Big(\prod_{d_1,\ldots,d_m}\|\Pi_m\|\Big)^{1/m2^m}$$

(the geometric mean of the norms of all possible 2^m operator products of length m consisting of A_0 and A_1) is called the Lyapunov exponent of these operators. The proof of this result can be found in [7]. The average regularity has a close relation with the multyfractal dimension and applied in ergodic theory and dynamical systems [8].

Another conclusion of Theorem 3 is that for every point x we have

$$-\log_2 \hat{\rho} \ \le\ \alpha_v(x) \ \le\ -\log_2 \check{\rho},$$

therefore the JSR and LSR provide us with the bounds of local regularity. A nontrivial result is that for any fractal curve both these bounds are sharp and achieved at some points x. Moreover, if A_0, A_1 are both nondegenerate, then the values of local regularity cover the whole segment between $-\log_2 \hat{\rho}$ and $-\log_2 \check{\rho}$. For any point α from this segment the set of points x, for which $\alpha_v(x) = \alpha$ is dense everywhere on $[0,1]$ and has zero measure, whenever $\alpha \ne \alpha_{\rm av}$ [7].

III. SCALING FUNCTIONS AND WAVELETS

Wavelets are orthonormal systems of functions that can be obtained from one function by scaling and integer translates. They have a lot of applications in functional analysis, signal processing, approximation theory etc. Compactly supported wavelets play a special role due to their convenience in the implementations. The system of compactly supported wavelets on the real line is a complete orthonormal system in $L_2(\mathbb{R})$ that has the form

$$\psi_{jk}(x) \ =\ 2^{j/2}\psi(2^j x - k), \qquad j,k \in \mathbb{Z}\,,$$

where $\psi_{00} = \psi$ is a compactly supported function in $L_2(\mathbb{R})$. The construction of this system is always reduced to solving the refinement equation

$$\varphi(x) \ =\ \sum_{k=0}^{N} c_k \varphi(2x-k), \tag{10}$$

where c_k are complex coefficients and $\sum_k c_k = 2$. This is well known that this equation always has a unique, up to normalization, compactly supported solution in distributions, which, moreover, has its support on the segment $[0, N]$. This solution is called scaling function or refinable function. Having this solution φ one obtains the function ψ by a simple formula [4]. The question, whether the solution φ will be continuous and what is its smoothness can be solved explicitly in terms of JSR of special linear operators. To do this one has to consider the operators acting in $\mathbb{R}^N$ and given by their matrices T_0, T_1 as follows:

$$(T_i)_{jk} \ =\ c_{2j-k+i-1}, \quad 1 \le j,k \le N,\ i = 0,1, \tag{11}$$

where c_r is the corresponding coefficient of the equation (10), whenever $1 \le r \le N$ and $c_r = 0$ otherwise. It is easily can be checked that the refinement equation is equivalent to equation (1) for the operators $\tilde{A}_i = T_i$ and for the function

$$v(x) \ =\ \big(\varphi(x), \dots, \varphi(x+N-1)\big)^T. \tag{12}$$

The operators T_i are restricted to the smallest (by inclusion) common invariant affine subspace $\tilde{V}$ of T_0, T_1 containing the vector $v(0)$. Thus, function (12) is a fractal curve as well. Therefore the analysis of its global and local regularity can be realized using Theorems 1 - 3. In particular, the function φ is continuous iff $\hat{\rho}(A_0, A_1) < 1$ and, moreover, $\alpha_\varphi = -\log_2 \hat{\rho}$. (we have denoted $A_i = T_i|_V$, where V is the linear part of the space $\tilde{V}$, i.e., the smallest common invariant linear subspace of these operators containing the vector $v(1) - v(0)$). Thus, the problem of regularity of wavelets is reduced to the computing of the corresponding JSR. The question is complicated by the fact that the space V may be different for various equations, and the structure of the operators T_i on this space is not a priory clear. This problem was solved in [9], where the space V and the structure of operators A_0, A_1 were found explicitly. Moreover, it was shown that both operators are nondegenerate, which makes it possible to apply Theorem 3 for the analysis of the local regularity of wavelets and for computing their moduli of continuity.

***Example** 1:* [The third wavelet of Daubechies ψ_3]. This function is supported on the segment $[0, 5]$ and obtained from a refinement equation with six coefficients. In this case we have $L = 3$ and

$$m(z) = \frac{1}{2}\Big((0.398538)z^2 - (2.162272)z + (3.763736)\Big).$$

It is easy to check that the 2×2-matrices A_0, A_1 can be simultaneously symmetrized, therefore

$$\hat{\rho} = \max\{\rho(A_0),\, \rho(A_1)\} =$$

$$\max\{0.398538,\, 2.162272,\, 3.763736\} = 3.763736$$

Therefore $l = 1$ (i.e., $\psi \in C(\mathbb{R})$) and

$$\alpha_\psi = L - 1 - \log_2 \hat{\rho} = 3 - \log_2(3.763736) = 0.087833\dots.$$

Thus,

$$\omega(\psi', t) \ \asymp\ t^{0.087833}.$$

On the other hand, $\check{\rho} = \rho(A_1) = 2.162272\dots$. To prove this we use Proposition 2. As the set Q we take the union of the two angles: the first one has its vertex at the point $(1,0)^T$ and has its sides with the directions of the vectors $(5,1)^T$ and $(5,-1)^T$; the second one is obtained from the first one by the reflection w.r.t the origin.

IV. THE DISTRIBUTION OF A RANDOM SERIES

We consider a random series

$$\eta = \sum_{l=0}^{\infty} \eta_k t^k, \tag{13}$$

where $t \in (0,1)$, and the random variables $\eta_k,\ k = 0,1,\cdots$ are mutually independent and equally distributed with distribution function F_{η_0} (for a given random variable ν we denote by F_ν and f_ν the distribution function and the density function respectively). Under very general assumptions on the distribution of η_0 the series (13) converges with probability 1 for every $t \in (0,1)$. Moreover, the distribution function of this series is continuous and it is of pure type, i.e., is either absolutely continuous or purely singular (its derivative is zero almost everywhere) [10]. The main problem is to separate the cases of absolutely continuity and singularity of F_{η_0}.

This problem for various distributions t and η_0 have been studied in many works. The papers [11] and [12] analyzed a special case of discrete distribution of η_0. Namely, $t = 1/n$, where $n \geq 2$ is an integer, and η_0 is an arbitrary integer-valued random variable with finite expectation of modulo. So we consider the series

$$\eta = \sum_{k=0}^{N} \frac{\eta_k}{n^k}, \tag{14}$$

where η_0 takes integer values k with probabilities p_k and $\sum p_k = 1$. The sharp criterion of absolutely continuity of F_η was obtained in [12]. That criterion was formulated in terms of zeros of the characteristic function

$$m(z) \ = \ \sum_k p_k z^k.$$

Moreover, it was shown that the density function satisfies

$$\sum_{k\in\mathbb{Z}} f_\eta(x-kn) \ = 1 \quad \text{almost everywhere.} \tag{15}$$

Let us now show how to use the notion of JSR for more detailed analysis of the density function f_η. We restrict ourselves to the case, when only finitely many coefficients p_k are nonzero. So, $p_k = 0$, whenever $k < 0$ or $k > N$. For the sake of simplicity we consider the case $n = 2$, although the case of general n is considered in the same way (as it was done in [12]).

***Theorem** 4: Suppose $N \geq 2$ and all the probabilities $p_0, \ldots, p_N$ are positive; then if the random value η has a summable density $f_\eta \in L_1(\mathbb{R})$, then this density is continuous on $\mathbb{R}$.*

Before giving a proof let us make some observations. The key one is that the density f_η is expressed by the solution of the refinement equation (10) with $c_k = 2p_k$. Therefore we can consider the vector-function $v(x)$ by formula (12) and apply Theorem 1 for $v(x)$ and for the operators (11) with $c_k = 2p_k$. To establish the continuity of $v(x)$ it suffices to show that there is an affine plane $\tilde{V} \subset \mathbb{R}^N$ not containing the origin such that

1) $T_i\tilde{V} \ \subset \tilde{V}\,, \quad i = 0, 1;$
2) $\hat{\rho}\bigl(T_0|_V\,,\, T_1|_V\bigr) \ < \ 1,$

where V is the linear part of $\tilde{V}$. Theorem 1 then guarantees that $v(x)$ is continuous and its image is contained on $\tilde{V}$. We denote by $\mathbb{R}^N_+$ the positive octant in $\mathbb{R}^N$ (the set of points with nonnegative coordinates) and establish the following auxiliary result.

***Proposition** 4: Let T_0, T_1 be arbitrary $N \times N$-matrices with nonnegative entries. Suppose $\tilde{V} \subset \mathbb{R}^N$ is an affine plane not containing zero and such that*

$$\tilde{V} \cap \mathbb{R}^N_+ \neq \emptyset, \qquad T_i\tilde{V} \subset \tilde{V}, \ i = 0, 1.$$

Let $A_i = T_i|_V$. If there exists an $l \geq 1$ such that the matrix product $A_{d_1} \cdots A_{d_l}$ has at least $N-1$ positive rows for any sequence $d_1, \cdots, d_l$, then $\hat{\rho}(A_0, A_1) < 1$.

In the proof of Proposition 4 we use the following simple lemma (the reader will easily establish it).

***Lemma** 1: Let $\tilde{V} \subset \mathbb{R}^N$ be an affine plane. Suppose a point $x \in \tilde{V}$ has nonnegative coordinates and at least $N-1$ of them are positive; then $\dim(\tilde{\mathrm{V}} \cap \overline{\mathbb{R}}^{\mathrm{N}}_+) = \dim \tilde{\mathrm{V}}$.*

Let us now prove Proposition 4

Proof: Let $\dim \tilde{\mathrm{V}} = \mathrm{r}$. First let us show that the set $G = \tilde{V} \cap \mathbb{R}^N_+$ is r-dimensional. Indeed, for any point $y \in G$ and for any $\Pi_l = T_{d_1} \cdots T_{d_l}$ the vector $x = \Pi_l y$ has at least $N-1$ positive coordinates, because Π_l has at least $N-1$ positive rows. Hence, by Lemma 1, we have $\dim \mathrm{G} = \mathrm{r}$. This means that G possesses a nonempty interior $\mathrm{int}\mathrm{G}$ in $\tilde{V}$. Thus, G is a convex polytope in $\tilde{V}$ with nonempty interior. It is obvious that $\Pi_l G \subset G$. Moreover, since Π_l has $N-1$ positive rows, it follows that the polytope $\Pi_l G$ can intersect only one $(N-1)$-dimensional face of the positive octant $\mathbb{R}^N_+$. Consequently, $\Pi_l G$ has common points with at most one $(r-1)$-dimensional face of G. Hence there is an $h \in V$ such that

$$(A_{d_1} \cdots A_{d_l}\, G + h) \subset \mathrm{int}\,\mathrm{G}.$$

This yields that for some $\lambda \in (0, 1)$ one has

$$A_{d_1} \cdots A_{d_l}\, M \ \subset \ \lambda M,$$

where $M = \frac{1}{2}(G + (-G))$ is a centrally symmetric polytope in the space V. Applying Proposition 2 we conclude the proof. ∎

Now we are ready to prove Theorem 4.

Proof: Consider the affine span $\tilde{V}$ of the set

$$\Bigl\{ \ 2^q \int_x^{x+2^{-q}} v(x)dx \ \Bigm| \ x = 2^{-q}l, \ l = 0, \cdots, 2^q - 1, \ q \geq 0 \Bigr\},$$

where $v(x)$ is the L^1-solution (of equation (1)), which exists by the assumption. Clearly, $\tilde{V}$ is invariant w.r.t. the operators T_0, T_1. Furthermore, equality (15) yields that $\tilde{V}$ is contained in the affine hyperplane $\{y \in \mathbb{R}^N, \ \sum_k y_k \ = \ 1\}$, and therefore, does not contain the origin. Let us finally note that the point $\int_0^1 v(x)\,dx$, which belongs to $\tilde{V}$, possesses a positive coordinate (otherwise $\varphi \equiv 0$ a.e.). Thus $\tilde{V}$ satisfies the assumptions of Proposition 4. Now observe that the kth row of the matrix $\Pi_l = T_{d_1} \cdots T_{d_l}$ has the entries

$$(\Pi_l)_{kj} = c_{2^l(k-1)+s-j+1}, \tag{16}$$

where $s = \sum_{r=0}^{l-1} d_r 2^{l-r-1}$, and c_r are the coefficients of the polynomial

$$P_l(z) \ = \ m(z)\, m(z^2) \cdots \ m(z^{2^{l-1}}).$$

Since all the coefficients of $m(z)$ are positive, we see that $c_r > 0$ for $r = 0, \ldots, 2^{l-1}N$. Now this is easy to verify that if $2^l > N$, then for any $d_1, \ldots, d_l$ either the first $N-1$ rows or the last $N-1$ rows of the matrix (16) are positive. ∎

V. ASYMPTOTICS OF THE PARTITION FUNCTION

For an arbitrary $d \in \mathbb{N} \cup \{\infty\}$, $d \geq 2$ the binary partition function $b(k) = b(d,k)$ is defined on the set on nonnegative integers k as the total number of different binary expansions $k = \sum_{j=0}^{\infty} d_j 2^j$, where the "digits" d_j take values from the set $0, \ldots, d-1$. Leonard Euler in [1] studied the partition function $b(\infty, k)$ in connection with some power series. The asymptotic behavior of $b(d,k)$ as $k \to \infty$ was studied in various interpretations by K.Mahler, N.G. de Bruijn, D. E. Knuth, B.Reznick and others (see [13] for many references). Clearly, for $d = 2$ we have $b(k) \equiv 1$. For $d \geq 3$ such a binary expansion is no more unique, and the problem arises to characterize the asymptotic behavior of the function $b(k)$ as $k \to \infty$. For even d this problem was solved by B.Reznick in [13]. For odd values of d the asymptotic behavior of $b(k)$ is more complicated, it was studied in [13] and [14]. Denote

$$p_1 = \liminf_{k\to\infty} \log b(k)/\log k; \quad p_2 = \limsup_{k\to\infty} \log b(k)/\log k.$$

If d is even, then we have $p_1 = p_2$, but for odd d this is not always the case. Already for $d = 3$ one has $p_1 < p_2$. In [13] these exponents were computed explicitly for $d = 3$, the problem for the other odd values of d were left open. In [14] this problem was attacked by using the JSR. The exponents p_1, p_2 were found for $d = 5, 7, 9, 11$ and 13, for other odd d an explicit formula for them was conjectured.

It was shown that

$$p_1 = \log_2 \check{\rho}\,, \qquad p_2 = \log_2 \hat{\rho}$$

where $\hat{\rho}$ and $\check{\rho}$ are the LSR and JSR of the operators T_0, T_1, whose matrices are given by formulas (11) with $N = d-1$ and $c_0 = \ldots = c_d = 1$.

***Conjecture** 1: If d is an odd integer, then*

$$\check{\rho} = \min\bigl\{\rho(T_0), \sqrt{\rho(T_0T_1)}\bigr\}, \quad \hat{\rho} = \max\bigl\{\rho(T_0), \sqrt{\rho(T_0T_1)}\bigr\}.$$

where ρ denotes the (usual) spectral radius, i.e., the largest modulo of the eigenvalues.

In [14] this conjecture was proved for $d = 3, 5, \ldots, 13$ that made it possible to compute explicitly the growth exponents p_1, p_2 for these values of d. To prove this we used Proposition 2 and constructed the sets M and Q as suitable polytopes in $\mathbb{R}^{d-1}$. That construction is easily extended for all dimensions d. However, we have proved the inclusions $T_iM \subset \hat{\rho}M$ and $T_iQ \subset \check{\rho}Q$ only for the dimensions $d \leq 13$.

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