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Constrained Optimal Trajectory Tracking on the Group of Rigid Body Motions

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Abstract— **In this paper we study optimality conditions for the finite time horizon, constrained optimal trajectory tracking problem on the group of rigid body motions SE(3). We treat SE(3) as a differentiable manifold and use a geometric approach to derive the necessary optimality conditions. To do so, we begin by studying simple optimal control problems on SE(3), including deriving the equations of motion of a rigid body (i.e., Euler's equations) by formulating the dynamics as a constrained variational optimal control problem. The main contribution of the paper is the derivation of the necessary optimality conditions for constrained optimal trajectory tracking on SE(3) and SO(3). A simple example on SO(2) is given.**

I. Introduction

Motivated by classical linear feedback control theory, in this paper we use Lagrange's method for constrained problems in the calculus of variations to study a general optimal trajectory tracking problem on the group of rigid body motions SE(3). Applications for the present work include, in particular, spacecraft trajectory (position and velocity) tracking. We approach the problem from a geometric mechanical viewpoint and rely on methods in differential and Riemannian geometry since SE(3), as a group, is itself a Riemannian manifold.

Optimal control problems for systems evolving on Riemannian manifolds have been addressed in the past. In [1], the authors study a second order calculus of variations problem on Riemannian manifolds with a specialized result for compact semi-simple Lie groups. P. Crouch and collaborators have considered extensions of this problem to second order systems evolving on Riemannian manifolds, in particular semi-simple Lie groups, including interpolation constraints and the sub-Riemannian problem [2], [3]. While all these results rely on variational (Lagrangian) approaches, of relevance is the maximum principle approach adopted in [4], [5].

In this paper we pursue a Lagrange method variational approach (see, for example, [6]) to studying a finite time horizon optimal trajectory tracking problem on the group of rigid body motions SE(3). The cost functional considered aims at minimizing the deviation of the configuration and velocity trajectories from a desired value, as well as minimizing the applied control effort. The constraints are simply the dynamics satisfied by the rigid body, expressed over a Lie group and its algebra, as well as any holonomic constraints and initial and terminal boundary conditions. To deal with the non-compact nature of SE(3) we introduce the useful notation of the double bracket $[[\cdot,\cdot]]$. Moreover, we compute the curvature tensor, which naturally appears in the necessary conditions for second order problems, for SE(3). In this paper we restrict our attention to normal extremals.

In the next section, we state some basic mathematical facts for systems evolving on SE(3) and its subgroups and later use these results to study optimal control problems on SE(3) and its subgroups.

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II. Mathematical Background

A. Basic Definitions and Facts

The material discussed in this section can be found, for example, in Section 2 of [7] as well as [8]. In this section we state some properties of dynamical systems evolving on proper subgroups, denoted by $\mathcal{G}$ with Lie algebra denoted by $\mathfrak{g}$, of SE(3). We often specialize the result to SE(3). The equations of motion for a dynamical system with configuration $g \in \mathcal{G}$ are given by:

$$\dot{g} = g\mathbf{V}, \tag{2.1}$$

where $\mathbf{V} \in \mathfrak{g}$ is the velocity in the body frame. The system $\dot{g} = g\mathbf{V}$ is said to be *left-invariant* since it is invariant under left multiplication by constant matrices.

For all $g \in \mathcal{G}$ and all $\mathbf{X}, \mathbf{Y} \in \mathfrak{g}$ the adjoint map Ad_g and the matrix commutator $\mathrm{ad}_{\mathbf{X}}$ are given by $\mathrm{Ad}_g\mathbf{Y} = g\mathbf{Y}g^{-1}$ and $\mathrm{ad}_{\mathbf{X}}\mathbf{Y} = [\mathbf{X}, \mathbf{Y}] = \mathbf{XY} - \mathbf{YX}$, where $[\cdot,\cdot]$ is the matrix Lie bracket.

On SE(3), a group element g is represented as a pair $g = (R, p) \in \mathrm{SO}(3) \times \mathbb{R}^3$ and velocity by the pair $\mathbf{V} = (\hat{\boldsymbol{\omega}}, \mathbf{v}) \in \mathfrak{so}(3) \times \mathbb{R}^3$ using homogeneous coordinates. (SO(3) is the special orthogonal group of rotations and $\mathfrak{so}(3)$ is its Lie algebra.) In matrix form, these are given by

$$g = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{V} = \begin{bmatrix} \hat{\boldsymbol{\omega}} & \mathbf{v} \\ 0 & 0 \end{bmatrix}. \tag{2.2}$$

The operator $\hat{\cdot} : \mathbb{R}^3 \to \mathfrak{so}(3)$ is such that $\hat{\boldsymbol{\omega}} y = \omega \times y$, $\times$ being the vector cross product.

With the kinematics given by equation (2.1), the dynamic equation of motion is given by

$$\dot{\mathbf{V}} = \mathbf{f}(g, \mathbf{V}) + \mathbf{U}. \tag{2.3}$$

The vector field $\mathbf{f}(g, \mathbf{V}) \in \mathfrak{se}(3)$ represents the system's internal drift and $\mathbf{U} \in \mathfrak{se}(3)$ is the control input. The drift term has the general form:

$$\mathbf{f}(g, \mathbf{V}) = \begin{bmatrix} \hat{\mathbf{f}}_r(g, \mathbf{V}) & \mathbf{f}_t(g, \mathbf{V}) \\ 0 & 0 \end{bmatrix}, \tag{2.4}$$

where $\hat{\mathbf{f}}_r \in \mathfrak{so}(3)$ corresponds to rotational drift and $\mathbf{f}_t \in \mathbb{R}^3$ corresponds to translational drift. Likewise, we have

$$\mathbf{U} = \begin{bmatrix} \hat{\boldsymbol{\tau}} & \mathbf{u} \\ 0 & 0 \end{bmatrix}, \tag{2.5}$$

for the control input, where $\hat{\boldsymbol{\tau}} \in \mathfrak{so}(3)$ is the input torque acting on the body and $\mathbf{u} \in \mathbb{R}^3$ is the input force assumed acting at the body's center of mass. The vector fields $\hat{\mathbf{f}}_r$, $\mathbf{f}_t$, $\hat{\boldsymbol{\tau}}$ and $\mathbf{u}$ take up the following forms:

$$\begin{aligned} \hat{\mathbf{f}}_r &= \begin{bmatrix} 0 & -f_{r3} & f_{r2} \\ f_{r3} & 0 & -f_{r1} \\ -f_{r2} & f_{r1} & 0 \end{bmatrix}, \quad \mathbf{f}_t = \begin{bmatrix} f_{t1} \\ f_{t2} \\ f_{t3} \end{bmatrix}, \\ \hat{\boldsymbol{\tau}} &= \begin{bmatrix} 0 & -\tau_3 & \tau_2 \\ \tau_3 & 0 & -\tau_1 \\ -\tau_2 & \tau_1 & 0 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}. \end{aligned} \tag{2.6}$$

B. Metric on $\mathfrak{se}(3)$

On SE(3), an inner product on the Lie algebra $\mathfrak{se}(3)$ can be extended to a Riemannian metric over the manifold using left (or right) translations as follows. Let $\mathbf{G}_e$ be a positive-definite matrix and $\mathbf{V}_1,\ \mathbf{V}_2 \in \mathfrak{se}(3)$. Let the inner product at the identity e be given by:

$$\langle \mathbf{V}_1, \mathbf{V}_2 \rangle_e = \mathbf{v}_1^T \mathbf{G}_e \mathbf{v}_2,$$

where $\mathbf{V}_i = \hat{\mathbf{v}}_i$ with $\mathbf{v}_i \in \mathbb{R}^6$ being the matrix representation of $\mathbf{V}_i$, $i = 1, 2$. If $\tilde{\mathbf{V}}_1$ and $\tilde{\mathbf{V}}_2$ are arbitrary vector fields at an arbitrary group element $g \in$ SE(3), then the inner product $\left\langle \tilde{\mathbf{V}}_1, \tilde{\mathbf{V}}_2 \right\rangle_g$ on the tangent space T_gSE(3) can be defined by a left-invariant metric given by:

$$\left\langle \tilde{\mathbf{V}}_1, \tilde{\mathbf{V}}_2 \right\rangle_g = \left\langle g^{-1}\tilde{\mathbf{V}}_1, g^{-1}\tilde{\mathbf{V}}_2 \right\rangle_e = \langle \mathbf{V}_1, \mathbf{V}_2 \rangle_e .$$

A right-invariant metric is defined analogously.

For SE(3), however, there does not exist a bilinear form on $\mathfrak{se}(3)$ that is both positive definite and Ad-invariant [9]. In general, one may consider a class of left invariant metrics which may be specialized to the Killing form, the Klein form, a linear combination of the Klein and the Killing forms or the decoupled Park [9] form. For more on this general class of metrics on SE(3), see [10]. Since positive definiteness is crucial in an optimal control context (we must have a positive definite cost function), we elect to work with the standard inner product on $\mathbb{R}^6$, where we discard the Lie algebra structure of $\mathfrak{se}(3)$ and set

$$\begin{aligned} \langle \mathbf{V}_1, \mathbf{V}_2 \rangle_{\mathbb{R}^6} &= \langle \boldsymbol{\omega}_1, \boldsymbol{\omega}_2 \rangle_{\mathbb{R}^3} + \langle \mathbf{v}_1, \mathbf{v}_2 \rangle_{\mathbb{R}^3} \\ &= \langle \boldsymbol{\Omega}_1, \boldsymbol{\Omega}_2 \rangle + \langle \mathbf{v}_1, \mathbf{v}_2 \rangle_{\mathbb{R}^3} , \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ without a subscript denotes the Killing form on $\mathfrak{so}(3)$: $\langle \boldsymbol{\Omega}_1, \boldsymbol{\Omega}_2 \rangle = \langle \boldsymbol{\omega}_1, \boldsymbol{\omega}_1 \rangle_{\mathbb{R}^3}$ with $\boldsymbol{\Omega} = \hat{\boldsymbol{\omega}}$.

III. Optimal Control of a Rigid Body on SE(3)

A. Free Rigid Body Equations of Motion as a Constrained Variational Problem

Before giving the main result of this paper, we study two simpler problems to illustrate the approach. In this section we start by deriving the rigid body equations of motion in the body-fixed frame using Lagrange's method for constrained problems in the calculus of variations. We begin with the kinematic equations of motion:

$$\dot{g} = g\mathbf{V} \in \mathrm{T}_g\mathrm{SE}(3). \tag{3.1}$$

Equation (3.1) can be re-written as an expression over $\mathfrak{se}(3)$ as: $g^{-1}\dot{g} - \mathbf{V} = 0$. The inverse g^{-1} is given by:

$$g^{-1} = \begin{bmatrix} R^T & -R^T p \\ 0 & 1 \end{bmatrix} \in \mathrm{SE}(3). \tag{3.2}$$

Hence, we have:

$$g^{-1}\dot{g} = \begin{bmatrix} R^T\dot{R} & R^T\dot{p} \\ 0 & 0 \end{bmatrix} \in \mathfrak{se}(3), \tag{3.3}$$

such that the following kinematic equations hold:

$$\dot{R} = R\boldsymbol{\Omega},\ \dot{p} = R\mathbf{v}. \tag{3.4}$$

Finally, for a perturbed element $g_\epsilon(t, \epsilon) = (R_\epsilon(t, \epsilon), p_\epsilon(t, \epsilon))$ that satisfies $g_\epsilon(t, 0) = g(t)$, we have an analogous expression to (3.1) as follows (see [11], page 41):

$$\left.\frac{\partial g_\epsilon}{\partial \epsilon}\right|_{\epsilon=0} = g\mathbf{W} = \begin{bmatrix} R\mathbf{W}_1 & R\mathbf{w}_2 \\ 0 & 0 \end{bmatrix} \in \mathrm{T}_g\mathrm{SE}(3) \tag{3.5}$$

where

$$\mathbf{W} = \begin{bmatrix} \mathbf{W}_1 & \mathbf{w}_2 \\ 0 & 0 \end{bmatrix} \in \mathfrak{se}(3) \tag{3.6}$$

and $\mathbf{W}_1 \in \mathfrak{so}(3)$ and $\mathbf{w}_1 \in \mathbb{R}^3$ such that

$$\left.\frac{\partial R_\epsilon}{\partial \epsilon}\right|_{\epsilon=0} = R\mathbf{W}_1, \quad \left.\frac{\partial p_\epsilon}{\partial \epsilon}\right|_{\epsilon=0} = R\mathbf{w}_2. \tag{3.7}$$

Here $\mathbf{W} = (\mathbf{W}_1, \mathbf{w}_2) \in \mathfrak{se}(3)$ is the *variation vector field* expressed in the body fixed frame.

Lemma III.1. *Let $g_\epsilon \in$ SE(3) and $\mathbf{W} \in \mathfrak{se}(3)$ be defined as above, then we have*

$$\left.\frac{\partial g_\epsilon^{-1}}{\partial \epsilon}\right|_{\epsilon=0} = -\mathbf{W}g^{-1}.$$

For SE(3), it is important to note that, unlike SO(3), $\langle [\mathbf{A}, \mathbf{B}], \mathbf{C} \rangle \neq \langle \mathbf{A}, [\mathbf{B}, \mathbf{C}] \rangle$, $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathfrak{se}(3)$. Instead we have the following lemma.

Lemma III.2. *Let $\mathbf{A} = (\boldsymbol{\Omega}_a, \mathbf{v}_a), \mathbf{B} = (\boldsymbol{\Omega}_b, \mathbf{v}_b), \mathbf{C} = (\boldsymbol{\Omega}_c, \mathbf{v}_c) \in \mathfrak{se}(3)$. Then we have*

$$\langle [\mathbf{A}, \mathbf{B}], \mathbf{C} \rangle_{\mathbb{R}^6} = \langle \mathbf{A}, [\mathbf{B}, \mathbf{C}] + [[\mathbf{B}, \mathbf{C}]] \rangle_{\mathbb{R}^6} ,$$

where $[[\mathbf{B}, \mathbf{C}]]$ is given by

$$[[\mathbf{B}, \mathbf{C}]] = \begin{bmatrix} \widehat{\mathbf{v}_b \times \mathbf{v}_c} & \boldsymbol{\Omega}_c\mathbf{v}_b \\ 0 & 0 \end{bmatrix} \in \mathfrak{se}(3).$$

Proof

$$\begin{aligned} &\langle [\mathbf{A}, \mathbf{B}], \mathbf{C} \rangle_{\mathbb{R}^6} = \langle [\boldsymbol{\Omega}_a, \boldsymbol{\Omega}_b], \boldsymbol{\Omega}_c \rangle + \langle \boldsymbol{\Omega}_a\mathbf{v}_b - \boldsymbol{\Omega}_b\mathbf{v}_a, \mathbf{v}_c \rangle_{\mathbb{R}^3} \\ &= \langle \boldsymbol{\Omega}_a, [\boldsymbol{\Omega}_b, \boldsymbol{\Omega}_c] \rangle + \left\langle \boldsymbol{\Omega}_a, \widehat{\mathbf{v}_b \times \mathbf{v}_c} \right\rangle + \langle \mathbf{v}_a, \boldsymbol{\Omega}_b\mathbf{v}_c \rangle_{\mathbb{R}^3} \\ &= \langle \mathbf{A}, [\mathbf{B}, \mathbf{C}] + [[\mathbf{B}, \mathbf{C}]] \rangle_{\mathbb{R}^6} , \end{aligned}$$

where we have used the facts that: $\langle \boldsymbol{\Omega}_a\mathbf{v}_b, \mathbf{v}_c \rangle_{\mathbb{R}^3} = \mathbf{v}_c \cdot (\boldsymbol{\omega}_a \times \mathbf{v}_b) = \boldsymbol{\omega}_a \cdot (\mathbf{v}_b \times \mathbf{v}_c) = \langle \boldsymbol{\omega}_a, \mathbf{v}_b \times \mathbf{v}_c \rangle_{\mathbb{R}^3} = \left\langle \boldsymbol{\Omega}_a, \widehat{\mathbf{v}_b \times \mathbf{v}_c} \right\rangle$ and $-\langle \boldsymbol{\Omega}_b\mathbf{v}_a, \mathbf{v}_c \rangle_{\mathbb{R}^3} = -\mathbf{v}_c \cdot (\boldsymbol{\omega}_b \times \mathbf{v}_a) = -\mathbf{v}_a \cdot (\mathbf{v}_c \times \boldsymbol{\omega}_b) = \mathbf{v}_a \cdot (\boldsymbol{\omega}_b \times \mathbf{v}_c) = \langle \mathbf{v}_a, \boldsymbol{\Omega}_b\mathbf{v}_c \rangle_{\mathbb{R}^3}$. ■

For SO(3), Lemma (III.2) reduces to the standard: $\langle [\mathbf{A}, \mathbf{B}], \mathbf{C} \rangle = \langle \mathbf{A}, [\mathbf{B}, \mathbf{C}] \rangle$.

To derive a rigid body's equations of motion, we minimize the kinetic energy:

$$\mathcal{J} = \int_0^T \frac{1}{2} \left\langle \mathbf{V}, \tilde{\mathbf{J}}(\mathbf{V}) \right\rangle_{\mathbb{R}^6} \mathrm{d}t \tag{3.8}$$

subject to the constraint given by equation (3.1) and the boundary conditions:

$$\begin{aligned} &g(0) = g_0 = (R_0, p_0),\ \ \mathbf{V}(0) = \mathbf{V}_0 = (\boldsymbol{\omega}_0, \mathbf{v}_0), \\ &g(T) = g_T = (R_T, p_T),\ \ \mathbf{V}(T) = \mathbf{V}_T = (\boldsymbol{\omega}_T, \mathbf{v}_T), \end{aligned} \tag{3.9}$$

where $\tilde{\mathbf{J}} : \mathfrak{se}(3) \to \mathfrak{se}(3)$ is the symmetric, positive definite, and, hence, invertible operator defined by:

$$\tilde{\mathbf{J}}(\mathbf{V}) = \begin{bmatrix} \mathbf{J}(\boldsymbol{\Omega}) & m\mathbf{v} \\ 0 & 0 \end{bmatrix}, \ \forall \mathbf{V} = (\boldsymbol{\omega}, \mathbf{v}) \in \mathfrak{se}(3) \quad (3.10)$$

where m is the mass of the body and $\mathbf{J} : \mathfrak{so}(3) \to \mathfrak{so}(3)$ is the symmetric[1], positive definite, and, hence, invertible inertia operator (see [12], page 349.) By our definition of inner product on $\mathbb{R}^6$ in Section (II-B), the integrand in the cost functional (3.8) corresponds to the total kinetic energy: $\frac{1}{2}\langle \boldsymbol{\Omega}, \mathbf{J}(\boldsymbol{\Omega})\rangle + \frac{1}{2}\langle \mathbf{v}, m\mathbf{v}\rangle_{\mathbb{R}^3}$.

First, we form the modified cost functional:

$$\mathcal{J} = \int_0^T \frac{1}{2}\left\langle \mathbf{V}, \tilde{\mathbf{J}}(\mathbf{V})\right\rangle_{\mathbb{R}^6} + \left\langle \boldsymbol{\Lambda}, g^{-1}\dot{g} - \mathbf{V}\right\rangle_{\mathbb{R}^6} \mathrm{d}t$$

where $\boldsymbol{\Lambda} = (\boldsymbol{\Lambda}_1, \boldsymbol{\lambda}_2) \in \mathfrak{se}(3)$ is the Lagrange multiplier with $\boldsymbol{\Lambda}_1 = \hat{\boldsymbol{\lambda}}_1 \in \mathfrak{so}(3)$ and $\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2 \in \mathbb{R}^3$. Then we have:

$$\begin{aligned} \left.\frac{\partial \mathcal{J}}{\partial \epsilon}\right|_{\epsilon=0} &= \int_0^T \left\langle \left.\frac{\mathrm{D}\mathbf{V}_\epsilon}{\partial \epsilon}\right|_{\epsilon=0}, \tilde{\mathbf{J}}(\mathbf{V}) - \boldsymbol{\Lambda}\right\rangle \\ &\quad + \left\langle \boldsymbol{\Lambda}, [\mathbf{V}, \mathbf{W}] + \frac{\mathrm{D}\mathbf{W}}{\mathrm{d}t}\right\rangle \mathrm{d}t. \end{aligned}$$

Integrating $\left\langle \boldsymbol{\Lambda}, [\mathbf{V}, \mathbf{W}] + \frac{\mathrm{D}\mathbf{W}}{\mathrm{d}t}\right\rangle$ by parts and using Lemma (III.2), we find that

$$\begin{aligned} \left.\frac{\partial \mathcal{J}}{\partial \epsilon}\right|_{\epsilon=0} &= \int_0^T \left\langle \left.\frac{\mathrm{D}\mathbf{V}_\epsilon}{\partial \epsilon}\right|_{\epsilon=0}, \tilde{\mathbf{J}}(\mathbf{V}) - \boldsymbol{\Lambda}\right\rangle \\ &\quad - \left\langle \mathbf{W}, [\mathbf{V}, \boldsymbol{\Lambda}] + [[\mathbf{V}, \boldsymbol{\Lambda}]] + \frac{\mathrm{D}\boldsymbol{\Lambda}}{\mathrm{d}t}\right\rangle \mathrm{d}t. \end{aligned}$$

Hence, the necessary optimality conditions are given by

$$\frac{\mathrm{D}\boldsymbol{\Lambda}}{\mathrm{d}t} = [\boldsymbol{\Lambda}, \mathbf{V}] - [[\mathbf{V}, \boldsymbol{\Lambda}]], \text{ where } \boldsymbol{\Lambda} = \tilde{\mathbf{J}}(\mathbf{V}).$$

Expanding this expression, we get:

$$\begin{aligned} \frac{\mathrm{D}\mathbf{M}}{\mathrm{d}t} &= [\mathbf{M}, \boldsymbol{\Omega}], \ \mathbf{M} = \boldsymbol{\Lambda}_1 = \mathbf{J}(\boldsymbol{\Omega}) \\ \frac{\mathrm{D}\mathbf{l}}{\mathrm{d}t} &= \mathbf{l} \times \boldsymbol{\omega}, \ \mathbf{l} = \boldsymbol{\lambda}_2 = m\mathbf{v}, \end{aligned} \quad (3.11)$$

which are the equations of motion for a rigid body in a body fixed frame, where $\mathbf{l}$ is the linear momentum and $\mathbf{M}$ is the angular momentum, both expressed in a body fixed frame.

Recall the relation between the time derivative of an arbitrary variable $\mathbf{s}(t)$ in a body fixed frame, $\frac{\mathrm{d}'\mathbf{s}}{\mathrm{d}t}$, with the space fixed frame time derivative $\frac{\mathrm{d}\mathbf{s}}{\mathrm{d}t}$: $\frac{\mathrm{d}\mathbf{s}}{\mathrm{d}t} = \frac{\mathrm{d}'\mathbf{s}}{\mathrm{d}t} + \boldsymbol{\omega} \times \mathbf{s}$. Equation (3.11) then implies that the rate of change of the linear momentum in the space fixed frame is zero: $\frac{\mathrm{d}\mathbf{l}_2^s}{\mathrm{d}t} = \frac{\mathrm{d}'\mathbf{l}}{\mathrm{d}t} + \boldsymbol{\omega} \times \mathbf{l} = \mathbf{l} \times \boldsymbol{\omega} + \boldsymbol{\omega} \times \mathbf{l} = 0$ as one expects since no external forces are applied at the center of mass of the rigid body. Similarly, no external torques are applied on the body and, hence, we have $\frac{\mathrm{d}\mathbf{m}^s}{\mathrm{d}t} = \frac{\mathrm{d}'\mathbf{m}}{\mathrm{d}t} + \boldsymbol{\omega} \times \mathbf{m} = 0$, where again $\mathbf{M} = \hat{\mathbf{m}}$.

B. Second Order Optimal Control Problem on SE(3)

We now study the minimum control problem in body-fixed variables again using Lagrange's method for constrained problems in the calculus of variations. We wish to minimize

$$\mathcal{J} = \int_0^T \frac{1}{2}\langle \mathbf{U}, \mathbf{U}\rangle_{\mathbb{R}^6} \mathrm{d}t \quad (3.12)$$

subject to the second order dynamics:

$$\begin{aligned} \dot{g} &= g\mathbf{V} \quad (3.13) \\ \frac{\mathrm{D}\tilde{\mathbf{J}}(\mathbf{V})}{\mathrm{d}t} &= \left[\tilde{\mathbf{J}}(\mathbf{V}), \mathbf{V}\right] - \left[\left[\mathbf{V}, \tilde{\mathbf{J}}(\mathbf{V})\right]\right] + \mathbf{U} \end{aligned}$$

[1] $\mathbf{J}$ is symmetric with respect to the inner product $\langle\cdot,\cdot\rangle$ defined by the Killing form.

and the boundary conditions:

$$g(0) = g_0, \ \mathbf{V}(0) = \mathbf{V}_0, \ g(T) = g_T, \ \mathbf{V}(T) = \mathbf{V}_T, \ (3.14)$$

where $\mathbf{U} = (\hat{\boldsymbol{\tau}}, \mathbf{u}) \in \mathfrak{se}(3)$ is the control vector field in body fixed coordinates.

We first form the modified cost functional:

$$\begin{aligned} \mathcal{J} = \int_0^T & \frac{1}{2}\langle \mathbf{U}, \mathbf{U}\rangle + \langle \boldsymbol{\Lambda}_1, g^{-1}\dot{g} - \mathbf{V}\rangle + \left\langle \boldsymbol{\Lambda}_2, \frac{\mathrm{D}\tilde{\mathbf{J}}(\mathbf{V})}{\mathrm{d}t} \right. \\ & \left. - \left[\tilde{\mathbf{J}}(\mathbf{V}), \mathbf{V}\right] + \left[\left[\mathbf{V}, \tilde{\mathbf{J}}(\mathbf{V})\right]\right] - \mathbf{U}\right\rangle \mathrm{d}t, \end{aligned}$$

where $\boldsymbol{\Lambda}_1 = (\boldsymbol{\Lambda}_{11}, \boldsymbol{\lambda}_{12})$ and $\boldsymbol{\Lambda}_2 = (\boldsymbol{\Lambda}_{21}, \boldsymbol{\lambda}_{22})$, with $\boldsymbol{\Lambda}_{11}, \boldsymbol{\Lambda}_{21} \in \mathfrak{so}(3)$ and $\boldsymbol{\lambda}_{12}, \boldsymbol{\lambda}_{22} \in \mathbb{R}^3$, are Lagrange multipliers. After a lengthy computation, we find that

$$\begin{aligned} \left.\frac{\partial \mathcal{J}}{\partial \epsilon}\right|_{\epsilon=0} &= \int_0^T \left\langle \left.\frac{\mathrm{D}\mathbf{U}}{\partial \epsilon}\right|_{\epsilon=0}, \mathbf{U} - \boldsymbol{\Lambda}_2\right\rangle \\ &+ \left\langle \mathbf{W}, \tilde{\mathbf{R}}\left(\tilde{\mathbf{J}}(\boldsymbol{\Lambda}_2), \mathbf{V}\right)\mathbf{V} - [\mathbf{V}, \boldsymbol{\Lambda}_1] - [[\mathbf{V}, \boldsymbol{\Lambda}_1]] \right. \\ &\left. - \frac{\mathrm{D}\boldsymbol{\Lambda}_1}{\mathrm{d}t}\right\rangle + \left\langle \left.\frac{\mathrm{D}\mathbf{V}}{\partial \epsilon}\right|_{\epsilon=0}, \tilde{\mathbf{J}}([\boldsymbol{\Lambda}_2, \mathbf{V}]) - \left[\boldsymbol{\Lambda}_2, \tilde{\mathbf{J}}(\mathbf{V})\right] \right. \\ &\left. - \left[\left[\boldsymbol{\Lambda}_2, \tilde{\mathbf{J}}(\mathbf{V})\right]\right] - \frac{\mathrm{D}\tilde{\mathbf{J}}(\boldsymbol{\Lambda}_2)}{\mathrm{d}t} - \boldsymbol{\Lambda}_1\right\rangle \mathrm{d}t, \end{aligned}$$

where $\tilde{\mathbf{R}}$ is the curvature tensor associated with SE(3). The curvature tensor $\tilde{\mathbf{R}}$ arises due to the identity (see [13], page 52):

$$\frac{\partial}{\partial \epsilon}\frac{\partial}{\partial t}\mathbf{Y} - \frac{\partial}{\partial t}\frac{\partial}{\partial \epsilon}\mathbf{Y} = \tilde{\mathbf{R}}(\mathbf{W}, \mathbf{Y})\mathbf{V},$$

where $\mathbf{W}$ is the variation vector field associated with a curve $\mathbf{c}(t)$ on a manifold M, with $\mathbf{V} = \mathrm{d}\mathbf{c}(t)/\mathrm{d}t$ being the velocity vector field and $\mathbf{Y} \in \mathrm{T}_{\mathbf{c}(t)}M$ being any vector field along the curve $\mathbf{c}(t) \in M$. Setting $\partial\mathcal{J}/\partial\epsilon\big|_{\epsilon=0} = 0$, we obtain the following theorem.

Theorem III.1. *The necessary optimality conditions for the problem of minimizing (3.12) subject to the dynamics (3.13) and the boundary conditions (3.14) are given by*

$$\begin{aligned} \boldsymbol{\Lambda}_2 &= \mathbf{U} \\ \frac{\mathrm{D}\boldsymbol{\Lambda}_1}{\mathrm{d}t} &= \tilde{\mathbf{R}}\left(\tilde{\mathbf{J}}(\boldsymbol{\Lambda}_2), \mathbf{V}\right)\mathbf{V} - [\mathbf{V}, \boldsymbol{\Lambda}_1] - [[\mathbf{V}, \boldsymbol{\Lambda}_1]] \\ \frac{\mathrm{D}\tilde{\mathbf{J}}(\boldsymbol{\Lambda}_2)}{\mathrm{d}t} &= \tilde{\mathbf{J}}([\boldsymbol{\Lambda}_2, \mathbf{V}]) - \left[\boldsymbol{\Lambda}_2, \tilde{\mathbf{J}}(\mathbf{V})\right] - \left[\left[\boldsymbol{\Lambda}_2, \tilde{\mathbf{J}}(\mathbf{V})\right]\right] \\ &\quad - \boldsymbol{\Lambda}_1. \end{aligned}$$

In obtaining the above result we used the fact that the vector fields $\mathbf{V}$ and $\mathbf{W}$ are left-invariant vector fields. The curvature tensor is evaluated at a point $g_\epsilon(t) \neq \mathrm{Id}$, that is we get $\tilde{\mathbf{R}}_{g_\epsilon}\left(\frac{\partial g_\epsilon}{\partial \epsilon}, \frac{\partial g_\epsilon}{\partial t}\right)\mathbf{V}$. Evaluating this at $\epsilon = 0$ we get: $\tilde{\mathbf{R}}_g(g\mathbf{W}, g\mathbf{V})\mathbf{V}$. Since $g\mathbf{W}$ and $g\mathbf{V}$ are left-invariant vector fields at the group element $g(t)$, by the identification $\mathrm{T}_g\mathrm{SE}(3) \simeq \mathfrak{se}(3)$, we have $\tilde{\mathbf{R}}_g(g\mathbf{W}, g\mathbf{V})\mathbf{V} = \tilde{\mathbf{R}}(\mathbf{W}, \mathbf{V})\mathbf{V}$, which is the curvature tensor evaluated at the identity element. The result directly follows using the properties of the curvature tensor and Lemma (III.2).

Although the curvature tensor $\mathbf{R}$ for a compact semi-simple Lie group is well known [13], the curvature tensor $\tilde{\mathbf{R}}$ for SE(3) is not. For a compact semi-simple Lie group $\mathcal{G}$ with Lie algebra $\mathfrak{g}$, the curvature tensor, with respect to a bi-invariant metric, is given by:

$$\mathbf{R}\left(\mathbf{X},\mathbf{Y}\right)\mathbf{Z}=\frac{1}{4}\left[\left[\mathbf{X},\mathbf{Y}\right],\mathbf{Z}\right],\ \forall\mathbf{X},\mathbf{Y},\mathbf{Z}\in\mathfrak{g} \qquad (3.15)$$

We now compute the curvature tensor on SE(3) with respect to the metric $\langle\cdot,\cdot\rangle_{\mathbb{R}^6}$. We first state the following theorem, whose proof can be found in [14], pages 273-4.

Theorem III.2. *For a Lie group $\mathcal{G}$ with Lie algebra $\mathfrak{g}$ and an inner product $\mathbb{I}$ on $\mathfrak{g}$ whose associated left-invariant Riemannian metric is $\mathbb{G}_{\mathbb{I}}$ on $\mathrm{T}_g\mathcal{G}$, the Levi-Civita connection induced by $\mathbb{G}_{\mathbb{I}}$ is left-invariant and the corresponding bilinear map, denoted by $\nabla:\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$, is given by*

$$\nabla_{\mathbf{X}}\mathbf{Y}=\frac{1}{2}\left[\mathbf{X},\mathbf{Y}\right]-\frac{1}{2}\mathbb{I}^{\sharp}\left(\mathrm{ad}^*_{\mathbf{X}}\mathbb{I}^{\flat}(\mathbf{Y})+\mathrm{ad}^*_{\mathbf{Y}}\mathbb{I}^{\flat}(\mathbf{X})\right). \qquad (3.16)$$

Theorem (III.2) is stated in terms of notation used by the authors in [14]. The inner product $\mathbb{I}$ corresponds to $\langle\cdot,\cdot\rangle_e$ and the metric $\mathbb{G}_{\mathbb{I}}$ corresponds to $\langle\cdot,\cdot\rangle_g$, both defined earlier in Section (II-B). The map $\mathrm{ad}^*_{\mathbf{X}}:\mathfrak{g}^*\to\mathfrak{g}^*$ is the dual to the adjoint map $\mathrm{ad}_{\mathbf{X}}:\mathfrak{g}\to\mathfrak{g}$ (see [12], page 133, and [14], pages 264-5.) If we let $\mathbf{X}=(\hat{\mathbf{x}}_\omega,\mathbf{x}_v)\in\mathfrak{se}(3)$ and $\mathbf{Y}=(\hat{\mathbf{y}}_\omega,\mathbf{y}_v)\in\mathfrak{se}(3)$, then we have

$$\mathrm{ad}_{\mathbf{X}}=\begin{bmatrix}\hat{\mathbf{x}}_\omega & \mathbf{0}_{3\times3}\\ \hat{\mathbf{x}}_v & \hat{\mathbf{x}}_\omega\end{bmatrix},\ \mathrm{ad}^*_{\mathbf{X}}=\begin{bmatrix}-\hat{\mathbf{x}}_\omega & -\hat{\mathbf{x}}_v\\ \mathbf{0}_{3\times3} & -\hat{\mathbf{x}}_\omega\end{bmatrix}. \qquad (3.17)$$

Note that in this matrix form, these two operators act on elements of $\mathfrak{se}(3)$ which are viewed as elements of $\mathbb{R}^6$ in the form $(\mathbf{x}_\omega,\mathbf{x}_v)\in\mathbb{R}^6$ as opposed to $(\hat{\mathbf{x}}_\omega,\mathbf{x}_v)\in\mathfrak{se}(3)$. Also note that the matrix representation of $\mathrm{ad}^*_{\mathbf{X}}$ is the transpose of that of $\mathrm{ad}_{\mathbf{X}}$. Finally, in the theorem, $\flat$ and $\sharp$ denote the flat and sharp operators, respectively (see [12], [14].) In this paper, we use the standard inner product on $\mathbb{R}^6$ and, hance, $\mathbb{I}^{\flat}(\mathbf{X})=\mathbb{I}^{\sharp}(\mathbf{X})=\mathbf{X}$. A simple computation gives:

$$\nabla_{\mathbf{X}}\mathbf{Y}=\begin{bmatrix}\frac{1}{2}\left[\hat{\mathbf{x}}_\omega,\hat{\mathbf{y}}_\omega\right] & \hat{\mathbf{x}}_\omega\mathbf{y}_v\\ 0 & 0\end{bmatrix}.$$

Using the identity (see [14], page 132)

$$\tilde{\mathbf{R}}\left(\mathbf{X},\mathbf{Y}\right)\mathbf{Z}=-\nabla_{\mathbf{X}}\nabla_{\mathbf{Y}}\mathbf{Z}+\nabla_{\mathbf{Y}}\nabla_{\mathbf{X}}\mathbf{Z}+\nabla_{[\mathbf{X},\mathbf{Y}]}\mathbf{Z}, \qquad (3.18)$$

and after a straightforward computation, we find that:

$$\tilde{\mathbf{R}}\left(\mathbf{X},\mathbf{Y}\right)\mathbf{Z}=\begin{bmatrix}\frac{1}{4}\left[\left[\mathbf{X},\mathbf{Y}\right],\mathbf{Z}\right] & 0\\ 0 & 0\end{bmatrix}. \qquad (3.19)$$

This result is not surprising in light of the fact that SE(3) is homeomorphic to the product space SO(3)$\times\mathbb{R}^3$, with the curvature of $\mathbb{R}^3$ being identically zero.

IV. Trajectory Tracking on SE(3)

In this section we again use Lagrange's method for constrained problems in the calculus of variations to study a generic constrained optimal trajectory tracking problem on SE(3). Moreover, we will use $\langle\cdot,\cdot\rangle$ in this section to denote $\langle\cdot,\cdot\rangle_{\mathbb{R}^6}$, our choice of inner product on SE(3). We first make a few definitions.

Let $g(t)=(R(t),p(t))\in$ SE(3) denote the trajectory and $g_d(t)=(R_d(t),p_d(t))\in$ SE(3) be the desired configuration to be tracked on SE(3). Define the natural error [7] as

$$e=g_d^{-1}g=\begin{bmatrix}R_d^TR & R_d^T(p-p_d)\\ 0 & 1\end{bmatrix}\in\mathrm{SE}(3).$$

Then the error $e=\mathrm{Id}$ whenever $g(t)=g_d(t)$, where Id is the identity element on SE(3).

While $\dot{g}=g\mathbf{V}$ defines a left invariant control system, we will let the desired trajectory satisfy a right control system differential equation: $\dot{g}_d=\bar{\mathbf{V}}_dg_d$. The reason we do this is that the inverse of g_d appears in our definition for the error function. To make the error differential equation satisfy a left control system, it is then essential, as will become more obvious below, to have $\dot{g}_d^{-1}$ be a left invariant vector field. This is done by having g_d satisfy a right control system equation. Note that if $\dot{g}_d=\bar{\mathbf{V}}_dg_d$ and since $g_dg_d^{-1}=\mathrm{Id}$, then $\dot{g}_dg_d^{-1}+g_d\dot{g}_d^{-1}=0$ implies that $\dot{g}_d^{-1}=-g_d^{-1}\bar{\mathbf{V}}_d$. Hence, $\dot{g}_d^{-1}$ is a left-invariant vector field. A simple calculation gives $\dot{e}=e\left(\mathbf{V}-\mathrm{Ad}_{g^{-1}}\bar{\mathbf{V}}_d\right)$. Note that

$$\left.\frac{\partial e_\epsilon}{\partial\epsilon}\right|_{\epsilon=0}=\left.\frac{\partial}{\partial\epsilon}g_d^{-1}g_\epsilon\right|_{\epsilon=0}=g_d^{-1}g\mathbf{W}=e\mathbf{W}. \qquad (4.1)$$

We will use weighting matrices to penalize deviations of both position and velocity from some desired values. This allows us to penalize certain components of position and velocity errors. We do this as follows. As in linear feedback control, we set up our optimal control cost functional to minimize the deviation of the actual velocity from a nominal desired value. Let $\mathbf{V}^e=\mathbf{V}_d-\mathbf{V}\in\mathfrak{se}(3)$ be the velocity error. Let the measure of velocity error be given by $\left\langle\mathbf{V}^e,\tilde{\mathbf{K}}_{\mathbf{v}}(\mathbf{V}^e)\right\rangle$, where $\tilde{\mathbf{K}}_{\mathbf{v}}:\mathfrak{se}(3)\to\mathfrak{se}(3)$ be a symmetric, positive semi-definite operator defined by

$$\tilde{\mathbf{K}}_{\mathbf{v}}(\mathbf{V})=\begin{bmatrix}\mathbf{K}^r_{\mathbf{v}}(\mathbf{\Omega}) & \mathbf{K}^t_{\mathbf{v}}(\mathbf{v})\\ 0 & 0\end{bmatrix}$$

for an arbitrary $\mathbf{V}=(\mathbf{\Omega},\mathbf{v})\in\mathfrak{se}(3)$, where $\mathbf{K}^r_{\mathbf{v}}(\mathbf{\Omega})=K^r_{\mathbf{v}}\mathbf{\Omega}+\mathbf{\Omega}K^r_{\mathbf{v}}$ is the rotational gain operator for the error in angular velocity, $K^r_{\mathbf{v}}$ is a diagonal 3×3 positive semi-definite matrix, $\mathbf{K}^t_{\mathbf{v}}(\mathbf{v})=K^t_{\mathbf{v}}\mathbf{v}$ is the translational gain operator for the error in translational velocity, and $K^t_{\mathbf{v}}$ is a diagonal 3×3 positive definite matrix. Similar definitions apply to the control cost: $\left\langle\mathbf{U},\tilde{\mathbf{K}}_{\mathbf{u}}(\mathbf{U})\right\rangle$, where $\tilde{\mathbf{K}}_{\mathbf{u}}:\mathfrak{se}(3)\to\mathfrak{se}(3)$ is a symmetric, positive definite operator (positive definiteness is required for the control weighting operator.)

For position error we use a more general definition of the logarithmic map than that used in [7] to define the configuration error metric by $\left\langle\log\left(e\right),\tilde{\mathbf{K}}_p\left(\log\left(e\right)\right)\right\rangle$, where, for an arbitrary $g\in$ SE(3), $\mathbf{X}=\log(g)\in\mathfrak{g}$ is the exponential coordinates of the group element g in an open neighborhood of the origin of SE(3). The logarithmic map is regarded as a local chart of the manifold $\mathcal{G}$. See [7] and [8] for analytic expressions for the logarithmic map log. In the above expression, $\tilde{\mathbf{K}}_p:\mathfrak{se}(3)\to\mathfrak{se}(3)$ is a symmetric, positive semi-definite operator defined by

$$\tilde{\mathbf{K}}_p(\boldsymbol{\chi})=\begin{bmatrix}\mathbf{K}^r_p(\boldsymbol{\psi}) & \mathbf{K}^t_p(\boldsymbol{\xi})\\ 0 & 0\end{bmatrix}$$

for an arbitrary $\boldsymbol{\chi}=(\boldsymbol{\psi},\boldsymbol{\xi})\in\mathfrak{se}(3)$, where $\mathbf{K}^r_p(\boldsymbol{\psi})=K^r_p\boldsymbol{\psi}+\boldsymbol{\psi}K^r_p$ is the rotational gain operator for the error in attitude, K^r_p is a diagonal 3×3 positive semi-definite matrix, $\mathbf{K}^t_p(\boldsymbol{\xi})=K^t_p\boldsymbol{\xi}$ is the translational gain operator for the error in translational position, and K^t_p is a diagonal 3×3 positive semi-definite matrix. In this case, $\boldsymbol{\chi}=\log\left(g_d^{-1}g\right)\in\mathfrak{se}(3)$ is viewed as the exponential coordinates of the error e.

The above definitions allow us to penalize components of translational position, attitude, translational velocity and angular velocity errors independently. For example, in dual spacecraft interferometric imaging, we may wish to minimize the magnitude of the relative velocity in the observation x-

y plane without minimizing the out-of-plane z component [15]. In this case, $K^t_{\mathbf{v}} = \mathrm{diag}(a, b, 0)$ for some $a, b > 0$.

Problem IV.1. *Minimize*

$$\begin{aligned}\mathcal{J} &= \frac{1}{2}\int_0^T \left\langle \mathbf{U}, \tilde{\mathbf{K}}_{\mathbf{u}}(\mathbf{U})\right\rangle + \left\langle \mathbf{V}^e, \tilde{\mathbf{K}}_{\mathbf{v}}\left(\mathbf{V}^e\right)\right\rangle \\ &\quad + \left\langle \log\left(e\right), \tilde{\mathbf{K}}_p\left(\log\left(e\right)\right)\right\rangle \mathrm{d}t \qquad (4.2)\end{aligned}$$

subject to

Dynamics

$$\dot{g} = g\mathbf{V} \qquad (4.3)$$

$$\frac{\mathrm{D}\tilde{\mathbf{J}}\left(\mathbf{V}\right)}{\mathrm{d}t} = \left[\tilde{\mathbf{J}}\left(\mathbf{V}\right), \mathbf{V}\right] - \left[\left[\mathbf{V}, \tilde{\mathbf{J}}\left(\mathbf{V}\right)\right]\right] + \mathbf{U}$$

Holonomic Constraints

$$\langle \mathbf{V}, \mathbf{X}_i\rangle = 0,\ i = 1, \ldots, n,\ n < 6, \qquad (4.4)$$

Boundary Conditions

$$g(0) = g_0,\ \mathbf{V}(0) = \mathbf{V}_0,\ g(T) = g_T,\ \mathbf{V}(T) = \mathbf{V}_T, \quad (4.5)$$

where $\mathbf{X}_i \in \mathfrak{se}(3)$, $i = 1, \ldots, n$, are independent vector fields associated with the imposed constraints.

Note that the constraints (4.4) can only be as many as five. Since SE(3) is a six dimensional manifold, having 6 constraints ($n = 6$) completely specifies the motion. In the supplement [16], we show how the constraint vector fields $\mathbf{X}_i$ are derived from holonomic constraints. By the independence of $\mathbf{X}_i$ we can combine the n constraints by introducing the lagrange multipliers ζ_i and the vector field (expressed as an element in the Lie algebra, as opposed to elements in its dual space $\mathfrak{se}^*(3)$) $\mathbf{Z} = \sum_{i=1}^n \zeta_i \mathbf{X}_i$. Hence, the set of n conditions (4.4) can alternatively be expressed as

$$\langle \mathbf{V}, \mathbf{Z}\rangle = 0. \qquad (4.6)$$

In the supplement [16], we use Lagrange's method for constrained variational optimal control to obtain the following theorem.

Theorem IV.1 (Necessary Conditions for Constrained Optimal Trajectory Tracking on SE(3))**.** *The necessary conditions satisfied by an optimal trajectory $(g(t), \mathbf{V}(t), \mathbf{U}(t))$ of the Problem (IV.1) are given by*

$$\begin{aligned}\dot{g} &= g\mathbf{V} \\ \frac{\mathrm{D}\tilde{\mathbf{J}}\left(\mathbf{V}\right)}{\mathrm{d}t} &= -\left[\mathbf{V}, \tilde{\mathbf{J}}\left(\mathbf{V}\right)\right] - \left[\left[\mathbf{V}, \tilde{\mathbf{J}}\left(\mathbf{V}\right)\right]\right] + \tilde{\mathbf{K}}_{\mathbf{u}}^{-1}\left(\mathbf{\Lambda}_2\right) \\ \frac{\mathrm{D}\mathbf{\Lambda}_1}{\mathrm{d}t} &= \tilde{\mathbf{R}}\left(\tilde{\mathbf{J}}\left(\mathbf{\Lambda}_2\right), \mathbf{V}\right)\mathbf{V} - \left[\mathbf{V}, \mathbf{\Lambda}_1\right] - \left[\left[\mathbf{V}, \mathbf{\Lambda}_1\right]\right] \\ &\quad -\nabla_{\mathbf{Z}}\mathbf{V} - \left[\mathbf{Z}, \mathbf{V}\right] - \left[\left[\mathbf{Z}, \mathbf{V}\right]\right] + B(e) \\ \frac{\mathrm{D}\tilde{\mathbf{J}}\left(\mathbf{\Lambda}_2\right)}{\mathrm{d}t} &= \tilde{\mathbf{J}}\left(\left[\mathbf{\Lambda}_2, \mathbf{V}\right]\right) - \left[\mathbf{\Lambda}_2, \tilde{\mathbf{J}}\left(\mathbf{V}\right)\right] - \left[\left[\mathbf{\Lambda}_2, \tilde{\mathbf{J}}\left(\mathbf{V}\right)\right]\right] \\ &\quad -\mathbf{\Lambda}_1 + \mathbf{Z} - \tilde{\mathbf{K}}_{\mathbf{v}}\left(\mathbf{V}^e\right) \\ \langle \mathbf{V}, \mathbf{Z}\rangle &= 0, \qquad \mathbf{U} = \tilde{\mathbf{K}}_{\mathbf{u}}^{-1}\left(\mathbf{\Lambda}_2\right).\end{aligned}$$

Note that $\tilde{\mathbf{K}}_{\mathbf{u}}$ must be positive definite. Moreover, if we set $\tilde{\mathbf{K}}_{\mathbf{u}}$ to be the identity operator, set $\tilde{\mathbf{K}}_p$ and $\tilde{\mathbf{K}}_{\mathbf{v}}$ to be zero and $\mathbf{Z} = 0$ (no constraints) we get back Theorem (III.1).

V. Trajectory Tracking on SO(3)

Now consider the problem (IV.1) for the SO(3) case. In this section $\langle\cdot,\cdot\rangle$ denotes the Killing form on SO(3) as discussed in Section (II-B). The problem is the same as that defined in Problem IV.1 except that now we drop all translational components of the configuration, velocity and applied forces in $\mathbb{R}^3$.

On SO(3), $R^e = R_d^T R$ is the attitude error and $\mathbf{\Omega}^e = \mathbf{\Omega}_d - \mathbf{\Omega}$ is the velocity error. For SO(3), we simply project the necessary conditions for SE(3) onto its SO(3) subgroup.

Theorem V.1 (Necessary Conditions for Constrained Optimal Trajectory Tracking on SE(3))**.** *The necessary conditions satisfied by an optimal trajectory $(R(t), \mathbf{\Omega}(t), \boldsymbol{\tau}(t))$ of the Problem (IV.1) restricted to SO(3) are given by*

$$\begin{aligned}\dot{R} &= R\mathbf{\Omega} \\ \frac{\mathrm{D}\mathbf{J}\left(\mathbf{\Omega}\right)}{\mathrm{d}t} &= \left[\mathbf{J}\left(\mathbf{\Omega}\right), \mathbf{\Omega}\right] + \left(\mathbf{K}^r_{\mathbf{u}}\right)^{-1}\left(\mathbf{\Lambda}_{21}\right) \\ \frac{\mathrm{D}\mathbf{\Lambda}_{11}}{\mathrm{d}t} &= \mathbf{R}\left(\mathbf{J}\left(\mathbf{\Lambda}_{21}\right), \mathbf{\Omega}\right)\mathbf{\Omega} - \left[\mathbf{\Omega}, \mathbf{\Lambda}_{11}\right] \\ &\quad -\nabla_{\mathbf{Z}}\mathbf{\Omega} - \left[\mathbf{Z}, \mathbf{\Omega}\right] + \mathbf{K}^r_p\left(\log_{\mathrm{SO}(3)}\left(R^e\right)\right) \\ \frac{\mathrm{D}\mathbf{J}\left(\mathbf{\Lambda}_{21}\right)}{\mathrm{d}t} &= \mathbf{J}\left(\left[\mathbf{\Lambda}_{21}, \mathbf{\Omega}\right]\right) - \left[\mathbf{\Lambda}_{21}, \mathbf{J}\left(\mathbf{\Omega}\right)\right] \\ &\quad -\mathbf{\Lambda}_{11} + \mathbf{Z} - \mathbf{K}^r_{\mathbf{v}}\left(\mathbf{\Omega}^e\right) \\ \langle \mathbf{\Omega}, \mathbf{Z}\rangle &= 0, \qquad \boldsymbol{\tau} = \left(\mathbf{K}^r_{\mathbf{u}}\right)^{-1}\left(\mathbf{\Lambda}_{21}\right).\end{aligned}$$

The variables $\mathbf{\Lambda}_{11},\ \mathbf{\Lambda}_{21} \in \mathfrak{so}(3)$ are defined at the end of Section (III-B). Also, recall that the curvature on SO(3) is given by $\mathbf{R}\left(\mathbf{X}, \mathbf{Y}\right)\mathbf{Z} = \frac{1}{4}\left[\left[\mathbf{X}, \mathbf{Y}\right], \mathbf{Z}\right]$ for all $\mathbf{X}, \mathbf{Y}, \mathbf{Z} \in \mathfrak{so}(3)$.

A. Example on SO(2)

We now give a simple example on SO(2). For SO(2), $[\cdot,\cdot] \equiv 0$ and $\mathbf{R} \equiv 0$. The attitude of the planar body is specified by a single variable θ. We do not impose constraints on the body (doing so completely determines the motion) and attempt to minimize the deviation of θ from a given desired value θ_d. In the cases of SE(3) or SO(3), we have more degrees of freedom and may, for example, constrain yaw and roll angles while minimizing pitch error and letting the position of the center of mass be free in space.

If we let θ be the orientation of the planar body, then

$$\hat{\theta} = \log(R) = \begin{bmatrix} 0 & -\theta \\ \theta & 0 \end{bmatrix}.$$

This corresponds to

$$R = \exp\left(\hat{\theta}\right) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

A similar expression is obtained for the desired rotation matrix R_d. Note that

$$R^e = R_d^T R = \begin{bmatrix} \cos(\theta - \theta_d) & -\sin(\theta - \theta_d) \\ \sin(\theta - \theta_d) & \cos(\theta - \theta_d) \end{bmatrix}.$$

In the two dimensional case, we have

$$\mathbf{J}\left(\mathbf{\Lambda}_{21}\right) = \begin{bmatrix} 0 & -I_3\lambda_{21} \\ I_3\lambda_{21} & 0 \end{bmatrix},\ \mathbf{J}\left(\mathbf{\Omega}\right) = \begin{bmatrix} 0 & -I_3\omega \\ I_3\omega & 0 \end{bmatrix},$$

where $\mathbf{\Lambda}_{21} = \hat{\lambda}_{21}$, $\mathbf{\Omega} = \hat{\omega}$ and I_3 is the moment of inertia about the out-of-plane axis of the body. With the above identifications, the necessary conditions of Theorem (V.1) are given by

$$\begin{aligned}\dot{\theta} &= \omega,\ \dot{\omega} = \frac{1}{I_3 k^r_u}\lambda_{21} \\ \dot{\lambda}_{11} &= k^r_p\left(\theta - \theta_d\right),\ \dot{\lambda}_{21} = \frac{k^r_v}{I_3}\left(\omega - \omega_d\right) - \frac{1}{I_3}\lambda_{11},\end{aligned}$$

where k_p^r and k_v^r are the weighting parameters such that $\mathbf{K}_p^r = \frac{k_p^r}{2} I_{2\times 2}$ and $\mathbf{K}_v^r = \frac{k_v^r}{2} I_{2\times 2}$. Note here that the closed-loop system is unstable with eigenvalues given by

$$\pm \quad \frac{\sqrt{-k_v^r - \sqrt{-4I_3^2 k_p^r k_u^r + (k_v^r)^2}}}{\sqrt{2k_u^r} I_3},$$

$$\pm \quad \frac{\sqrt{-k_v^r + \sqrt{-4I_3^2 k_p^r k_u^r + (k_v^r)^2}}}{\sqrt{2k_u^r} I_3}.$$

In this paper, the feedback control laws are not constrained to be stabilizing. Hence, the closed-loop system is not guaranteed to be stable. Since we consider finite horizon optimal trajectory tracking, as opposed to infinite time horizon problems, an unstable closed-loop system is allowed because we only consider a transfer problem in phase space. In this work we do not consider any stability issues.

The above equations were solved while satisfying the boundary conditions $\theta(0) = 2$ radians, $\omega(0) = 0$ radians/second. No terminal conditions are imposed on θ and ω and, hence, we have $\lambda_{11}(T) = \lambda_{21}(T) = 0$. We set $I_3 = k_v^r = k_u^r = 1$. In the first simulation, we set $T = 20$ seconds. We desire to track a unit step signal for $\theta_d(t)$ and track the angular velocity $\omega_d(t) = 0$ r/s. The result is shown in the top two plots in Figure (1). In the second simulation, we desire to track a sinusoidal input, namely, $\theta_d(t) = \sin 0.1t$ and set $\omega_d(t) = 0$ and $T = 20\pi$ seconds. The result is shown in the bottom two plots of Figure (1). Both simulations are conducted for various values of k_p^r. In the figures, we observe that as the weighting on the attitude error is increased, the tracking error is decreased. However, as k_p^r is decreased the response is more sluggish while for smaller values the response converges to the desired value faster but with more under- and overshoot in the transients.

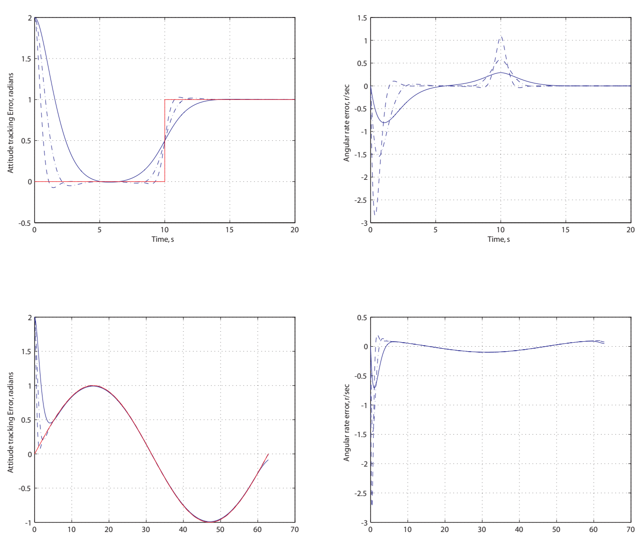

Fig. 1. Attitude and angular velocity error with $k_p^r = 1$ (solid), $k_p^r = 10$ (dash-dotted) and $k_p^r = 100$ (dashed) with fixed $k_v^r = 1$ for unit step desired attitude (top figures) and a sinusoidal desired attitude (bottom figures.) Signals to be tracked are given by solid red.

VI. Conclusion

In this paper we used Lagrange's method in the calculus of variations to study the finite time horizon constrained optimal trajectory tracking problem on the group of rigid body motions SE(3) and its subgroup SO(3). We focused on some of the important properties and background information related to geometric optimal control theory and the group of rigid body motions and its subgroups. We first study a simple optimal control problem on SE(3) and derive Euler's equations by formulating it as a constrained variational optimal control problem. This sets the stage for the main contribution of the paper, which is the derivation of the necessarily optimality conditions for constrained optimal trajectory tracking on SE(3) and SO(3). We concluded the paper with a simple example on SO(2). Future work will study second order optimality conditions and the existence of abnormal extremals.

References

[1] L. Noakes, G. Heinzinger, and B. Paden, "Cubic splines on curved spaces," *IMA Journal of Mathematical Control and Information*, vol. 6, no. 4, pp. 465–473, 1989.

[2] F. Silva-Leite, M. Camarinha, and P. Crouch, "Elastic curves as solutions of Riemannian and sub-Riemannian control problems," *Mathematics of Control, Signals and Systems*, vol. 13, pp. 140–155, 2000.

[3] M. Camarinha, "The geometry of cubic polynomials in Riemannian manifolds," Ph.D. dissertation, Universidade de Coimbra, 1996.

[4] F. Bullo and A. D. Lewis, "Supplementary chapters for *Geometric Control of Mechanical Systems* [14]," 2005. [Online]. Available: http://penelope.mast.queensu.ca/smcs/

[5] A. D. Lewis, "The geometry of the maximum principle for affine connection control systems," 2000, preprint. [Online]. Available: http://penelope.mast.queensu.ca/~andrew/papers/ps/2000d_letter.pdf.

[6] H. Rund, *The Hamilton-Jacobi Theory in the Calculus of Variations*. New York: Krieger, 1966.

[7] F. Bullo and R. M. Murray, *Proportional Derivative (PD) Control on the Euclidean Group*. California Institute of Technology, 1995, CDS Technical Report 95-010.

[8] R. M. Murray, Z. X. Li, and S. S. Sastry, *A Mathematical Introduction to Robotic Manipulation*. Boca Raton, FL: CRC Press, 1994.

[9] F. C. Park, "Distance metrics on the rigid body motions with applications to mechanism design," *ASME Journal of Mechanical Design*, vol. 117, pp. 48–54, 1995.

[10] R. Bhatt, C. P. Tang, and V. Krovi, "Geometric motion planning and formation optimization for a fleet of nonholonomic wheeled mobile robots," *Proceedings of the 2004 IEEE International Conference on Robotics and Automation*, pp. 3276–3281, 2004.

[11] A. L. Onishchik and E. B. Vinberg, *Lie Groups and Lie Algebras I*, ser. Encyclopedia of Mathematical Sciences. Berlin, Germany: Springer-Verlag, 1991, vol. 20, ch. Foundations of Lie Theory.

[12] A. Bloch, J. Baillieul, P. E. Crouch, and J. E. Marsden, *Nonholonomic Mechanics and Control*. New York, NY: Springer-Verlag, 2003.

[13] J. Milnor, *Morse Theory*. Princeton, NJ: Princeton University Press, 1963.

[14] F. Bullo and A. D. Lewis, *Geometric Control of Mechanical Systems*. Springer-Verlag, 2004.

[15] I. I. Hussein, "Motion planning for multi-spacecraft interferometric imaging systems," Ph.D. dissertation, University of Michigan, 2005. [Online]. Available: http://www.umich.edu/~ihussein/DISihussein.pdf.

[16] I. I. Hussein and A. M. Bloch, "Optimal trajectory tracking on the group of rigid body motions: Supplementary material [17]," 2005. [Online]. Available: http://www.umich.edu/~ihussein/HuBl_CDC05.pdf.

[17] ——, "Optimal trajectory tracking on the group of rigid body motions," *2005 IEEE Conference on Decision and Control*, 2005, to appear.