Proceedings of the
44th IEEE Conference on Decision and Control, and
the European Control Conference 2005
Seville, Spain, December 12-15, 2005

Disturbance Attenuation for Linear Systems Subject to Actuator Saturation using Output Feedback

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***Abstract*— In this paper, we study the problem of disturbance attenuation by output feedback for linear systems subject to actuator saturation. A nonlinear feedback control law, expressed in the form of a quasi-LPV system with state dependent scheduling parameter, is constructed that leads to the attenuation of the effect of the disturbance on the output of the system. The level of disturbance attenuation is measured in terms of the restricted $\mathcal{L}_2$ gain and the restricted $\mathcal{L}_2$ to $\mathcal{L}_\infty$ gain over a set of bounded disturbances.**

I. INTRODUCTION

The literature on analysis and design of control systems with actuator saturation is growing rapidly (see, for example, [1], [5], [8], [16] and the references there in). This literature covers a wide range of problems, including stabilization, output regulation, and disturbance rejection. Earlier research focuses on open loop systems that are not exponentially stable, for which various control problems have been studied in depth in the global or semi-global framework (see, for example, [2], [8], [9], [10], [14], [15]).

More recent literature has witnessed a shift of focus to exponentially unstable open loop systems. As such a system under actuator saturation is null controllable only in a part of the state space, the objectives here are to characterize the null controllable region [5] and to design feedback laws that work on the entire null controllable region or a large portion of it (see, for example, [3], [4], [5], [6], [11], [12], [13]). Most of this literature on exponentially unstable systems pertains to state feedback. A few exceptions where output feedback are used include [6], [11], [12].

Recently, we have developed a method for the design of output feedback laws that result in large domains of attraction [17]. This method applies to general linear systems including strictly unstable ones. By utilizing the convex hull expression of a saturating linear feedback law [7], a nonlinear output feedback controller is first expressed in the form of a quasi-LPV system. Conditions under which the closed-loop systems is locally asymptotically stable is then established in terms of the coefficient matrices of the controller. The design of the the controller (coefficient matrices) that achieves a large domain of attraction is then formulated and solved as an optimization problem with LMI constraints. Numerical results have demonstrated the effectiveness of the resulting output feedback laws.

In this paper, we will adopt the design approach proposed in [17] to design output feedback laws that attenuate the effect on the system output of the disturbance. The level of disturbance attenuation is measured in terms of the restricted $\mathcal{L}_2$ gain and the restricted $\mathcal{L}_2$ to $\mathcal{L}_\infty$ gain over a bounded region in the state space. As in [17], we will first parameterize the output feedback law in the form of a quasi-LPV system. Conditions on the controller coefficient matrices under which the closed-loop system possesses a certain degree of disturbance attenuation are established in the form of linear matrix inequalities (LMIs). The determination of these controller coefficient matrices is then formulated and solved as LMI optimization problems.

The notation used in this paper is rather standard. $\mathbf{R}$ stands for the set of real numbers and $\mathbf{R}_+$ for the non-negative real numbers. $\mathbf{R}^{m\times n}$ is the set of real $m \times n$ matrices. The transpose of a real matrix M is denoted by M^{T}. We use $\mathbf{S}^{n\times n}$ to denote real, symmetric $n\times n$ matrices, and $\mathbf{S}_+^{n\times n}$ for positive definite matrices. If $M \in \mathbf{S}^{n\times n}$, then $M > 0$ ($M \geq 0$) indicates that M is a positive definite (positive semi-definite) matrix and $M < 0$ ($M \leq 0$) denotes a negative definite (negative semi-definite) matrix. A block diagonal matrix with matrices $X_1, X_2, \cdots, X_p$ on its main diagonal is denoted as $\mathrm{diag}\{X_1, X_2, \cdots, X_p\}$. In large symmetric matrix expressions, terms denoted $\star$ will be induced by symmetry. For two integers k_1, k_2, $k_1 < k_2$, we denote $[k_1, k_2] = \{k_1, k_1 + 1, \cdots, k_2\}$.

The remainder of this paper is organized as follows. Section II provides some preliminary materials and establishes trajectory boundedness conditions under bounded disturbances. In Sections III and IV, we present output feedback laws for disturbance attenuation in terms of restricted $\mathcal{L}_2$ gain and restricted $\mathcal{L}_2$ to $\mathcal{L}_\infty$ gain, respectively. Section V contains numerical examples to illustrate our design procedure and demonstrate the effectiveness of the resulting controllers. Section VI draws the conclusion to the paper. All the proofs will be omitted for lack of space.

II. OUTPUT FEEDBACK CONTROL

Given a linear plant subject to actuator saturation,

$$\left\{\begin{array}{rcl} \dot{x}_{\mathrm{p}} & = & A_{\mathrm{p}}x_{\mathrm{p}} + B_{\mathrm{p}1}w + B_{\mathrm{p}2}\sigma(u), \\ z & = & C_{\mathrm{p}1}x_{\mathrm{p}} + D_{\mathrm{p}11}w + D_{\mathrm{p}12}\sigma(u), \\ y & = & C_{\mathrm{p}2}x_{\mathrm{p}} + D_{\mathrm{p}21}w, \end{array}\right. \tag{1}$$

where $x_{\mathrm{p}} \in \mathbf{R}^{n_x}$ is the state, $u \in \mathbf{R}^{n_u}$ is the control input, $z \in \mathbf{R}^{n_z}$ is the controlled output, $w \in \mathbf{R}^{n_w}$ is

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the disturbance, $y \in \mathbf{R}^{n_y}$ is the measurement output, and $\sigma : \mathbf{R}^{n_u} \rightarrow \mathbf{R}^{n_u}$ is a vector valued standard saturation function, *i.e.*,

$$\sigma(u) = \begin{bmatrix} \sigma(u_1) & \sigma(u_2) & \cdots & \sigma(u_{n_u}) \end{bmatrix}^{\mathrm{T}},$$

with

$$\sigma(u_i) = \operatorname{sgn}(u_i) \ \min\{1, |u_i|\}.$$

Here, we have slightly abused the notation by using σ to denote both the scalar valued and vector valued saturation function. We note that it is without loss of generality to assume a unity saturation level, as the level of a symmetric saturation function can always be scaled to unity by scaling $B_{\mathrm{p}2}$, $D_{\mathrm{p}12}$ and u.

Throughout the paper, we will assume that the triple $(A_{\mathrm{p}}, B_{\mathrm{p}2}, C_{\mathrm{p}2})$ is stabilizable and observable.

A. Problem Formulation

We will consider a dynamic output feedback law of the form,

$$\begin{cases} \dot{x}_{\mathrm{c}} &= A_{\mathrm{c}}(x_{\mathrm{c}}, y)x_{\mathrm{c}} + B_{\mathrm{c}}(x_{\mathrm{c}}, y)y, \\ u &= C_{\mathrm{c}}x_{\mathrm{c}} + D_{\mathrm{c}}y, \end{cases} \tag{2}$$

where $x_{\mathrm{c}} \in \mathbf{R}^{n_{x_{\mathrm{c}}}}$ is the controller state and $n_{x_{\mathrm{c}}}$ is the controller dimension which needs to be determined. C_{c} and D_{c} are constant matrices of appropriate dimensions.

Our objective is to design a dynamic output feedback law of the form (2) that asymptotically stabilizes the plant (1) locally, and minimizes the gain from the disturbance to the controlled output output within a region in the state space. This objective is achieved as follows. We will first parameterize the controller in a quasi-LPV form and then establish conditions under which the zero state response of the closed-loop system under a given set of bounded disturbances remains in an ellipsoid of the form

$$\Omega(X, \eta) = \left\{ x \in \mathbf{R}^{n_x + n_{x_{\mathrm{c}}}} : x^{\mathrm{T}} X x \le \eta \right\}, \quad X > 0, \eta > 0.$$

The controller coefficient matrices are then optimized by solving an optimization problem with LMI constraints. We are particularly interested in two measures of the level of disturbance attenuation:

1) Restricted $\mathcal{L}_2$ gain,
2) Restricted $\mathcal{L}_2$ to $\mathcal{L}_\infty$ gain.

To this end, we will need to use a tool from [5] for expressing the linear saturating feedback $\sigma(C_{\mathrm{c}}x_{\mathrm{c}} + D_{\mathrm{c}}y)$ on a convex hull. For $H_C \in \mathbf{R}^{n_u \times n_{x_{\mathrm{c}}}}$ and $H_D \in \mathbf{R}^{n_u \times n_y}$, define

$$\mathcal{L}(H_C, H_D) = \Big\{ (x_c, y) \in \mathbf{R}^{n_c + n_y} : \\ |H_{C_i} x_c + H_{D_i} y| \le 1, \ i \in [1, n_u] \Big\},$$

where H_{C_i} and H_{D_i} represent the ith row of matrices H_C and H_D respectively. We note that $\mathcal{L}(H_C, H_D)$ represents the region in $\mathbf{R}^{n_{x_{\mathrm{c}}} + n_y}$ where the auxiliary feedback $H_C x_{\mathrm{c}} + H_D y$ does not saturate.

Also, let $\mathcal{V}$ be the set of $n_u \times n_u$ diagonal matrices whose diagonal elements are either 1 or 0. There are 2^{n_u} elements in $\mathcal{V}$. Suppose these elements of $\mathcal{V}$ are labeled as $E_j,\ j \in [0, 2^{n_u} - 1]$ and denote $E_j^- = I - E_j$. Clearly, $E_j^- \in \mathcal{V}$ if $E_j \in \mathcal{V}$. The following lemma is adopted from [5].

Lemma 1: For a given $u = C_{\mathrm{c}}x_{\mathrm{c}} + D_{\mathrm{c}}y$, if $(x_c, y) \in \mathcal{L}(H_C, H_D)$, then

$$\sigma(u) \in \operatorname{co}\Big\{ E_j(C_{\mathrm{c}}x_{\mathrm{c}} + D_{\mathrm{c}}y) + E_j^-(H_C x_{\mathrm{c}} + H_D y), \\ j \in [0, 2^{n_u} - 1] \Big\},$$

where co stands for the convex hull.

We recall from [5] that, when there is only one saturation ($n_u = 1$), the convex covering in the above lemma will lead to non-conservative results when it is used to detect invariant ellipsoids.

By Lemma 1, for any H_C, H_D such that $(x_c, y) \in \mathcal{L}(H_C, H_D)$, the saturated linear feedback law $u = C_{\mathrm{c}}x_{\mathrm{c}} + D_{\mathrm{c}}y$ can be represented as

$$\sigma(u) = \sum_{j=0}^{2^{n_u}-1} \rho_j \left[E_j(C_{\mathrm{c}}x_{\mathrm{c}} + D_{\mathrm{c}}y) + E_j^-(H_C x_{\mathrm{c}} + H_D y) \right],$$

for some scalars $0 \le \rho_j \le 1,\ j \in [0, 2^{n_u} - 1]$ such that $\sum_{j=0}^{2^{n_u}-1} \rho_j = 1$. We note here that the values of the parameters $\rho = \begin{bmatrix} \rho_0 & \rho_1 & \cdots & \rho_{2^{n_u}-1} \end{bmatrix}^{\mathrm{T}}$ are dependent on x_{c} and y and are available for real-time use in gain-scheduling control. A formula for computing these values can be found in [17].

We will use the functions $\rho_j(x_c, y)$'s to parameterize the output feedback control (2) into the following quasi-LPV system

$$\begin{bmatrix} \dot{x}_{\mathrm{c}} \\ u \end{bmatrix} = \begin{bmatrix} \sum_{j=0}^{2^{n_u}-1} \rho_j A_{\mathrm{c}j} & \sum_{j=0}^{2^{n_u}-1} \rho_j B_{\mathrm{c}j} \\ C_{\mathrm{c}} & D_{\mathrm{c}} \end{bmatrix} \begin{bmatrix} x_{\mathrm{c}} \\ y \end{bmatrix}, \tag{3}$$

such that the effect of the disturbance on the controlled output is attenuated to a large degree within a region of the state space around the origin.

Similarly, the original plant can be written in a quasi-LPV form

$$\dot{x}_{\mathrm{p}} = A_{\mathrm{p}}x_{\mathrm{p}} + B_{\mathrm{p}1}w + B_{\mathrm{p}2}\Bigg\{ \sum_{j=0}^{2^{n_u}-1} \rho_j \Big[E_j(C_{\mathrm{c}}x_{\mathrm{c}} + D_{\mathrm{c}}y) \\ + E_j^-(H_C x_{\mathrm{c}} + H_D y) \Big] \Bigg\}, \tag{4}$$

$$z = C_{\mathrm{p}1}x_{\mathrm{p}} + D_{\mathrm{p}11}w + D_{\mathrm{p}12}\Bigg\{ \sum_{j=0}^{2^{n_u}-1} \rho_j \Big[E_j(C_{\mathrm{c}}x_c + D_{\mathrm{c}}y) \\ + E_j^-(H_C x_{\mathrm{c}} + H_D y) \Big] \Bigg\}, \tag{5}$$

$$y = C_{\mathrm{p}2}x + D_{\mathrm{p}21}w. \tag{6}$$

Combining the dynamics of the plant and the output feedback controller, we obtain,

$$\begin{bmatrix}\dot{x}\\ z\end{bmatrix} = \begin{bmatrix}A(\rho) & B(\rho)\\ C(\rho) & D(\rho)\end{bmatrix}\begin{bmatrix}x\\ w\end{bmatrix}, \qquad \rho \in \Gamma, \tag{7}$$

where

$$x = \begin{bmatrix}x_{\mathrm{p}}^{\mathrm{T}} & x_{\mathrm{c}}^{\mathrm{T}}\end{bmatrix}^{\mathrm{T}},$$
$$\Gamma = \left\{\rho \in \mathbf{R}^{2^{n_u}} : \sum_{j=0}^{2^{n_u}-1} \rho_j = 1,\ 0 \le \rho_j \le 1,\ j \in [0, 2^{n_u}-1]\right\}.$$

Note that the quasi-LPV form is a valid representation of the original nonlinear system when $(x_{\mathrm{c}}, y) \in \mathcal{L}(H_C, H_D)$. The state-space data of the closed-loop system is given by

$$\begin{bmatrix}A(\rho) & B(\rho)\\ C(\rho) & D(\rho)\end{bmatrix} = \left[\begin{array}{cc|c} A_{\mathrm{p}} & 0 & B_{\mathrm{p}1}\\ 0 & 0 & 0\\ \hline C_{\mathrm{p}1} & 0 & D_{\mathrm{p}11}\end{array}\right] + \left[\begin{array}{cc} 0 & B_{\mathrm{p}2}\\ I & 0\\ \hline 0 & D_{\mathrm{p}12}\end{array}\right]$$
$$\times\left(\sum_{j=0}^{2^{n_u}-1}\rho_j\begin{bmatrix}A_{\mathrm{c}j} & B_{\mathrm{c}j}\\ E_jC_{\mathrm{c}} + E_j^-H_C & E_jD_{\mathrm{c}} + E_j^-H_D\end{bmatrix}\right)$$
$$\times\left[\begin{array}{cc|c} 0 & I & 0\\ C_{\mathrm{p}2} & 0 & D_{\mathrm{p}21}\end{array}\right]$$
$$= \sum_{j=0}^{2^{n_u}-1}\rho_j\begin{bmatrix}A_j & B_j\\ C_j & D_j\end{bmatrix}.$$

Note that the closed-loop state-space matrices are affine functions of scheduling parameter ρ.

B. Disturbance Tolerance

We will define the set of energy and magnitude bounded disturbances as

$$\mathcal{W}_\eta = \left\{w : \mathbf{R}_+ \to \mathbf{R}^{n_w},\ \int_0^\infty w^{\mathrm{T}}(t)w(t)dt \le \eta,\ |w_i| \le \bar{w}_i, i \in [1, n_w]\right\},$$

where $\eta > 0$ is known and $\bar{w}_i$'s are also given. For energy bounded disturbances, it was argued in [3] that the trajectories of saturated systems cannot be invariant within an ellipsoid. Nevertheless, one can find a larger ellipsoid to accommodate the effect of the disturbance.

For a given $\Lambda > 0$, with $\lambda_{\max}(\Lambda) \le \alpha$, we define

$$\Omega\left(X, \Lambda, \eta + \alpha\sum_{i=1}^{n_w}\bar{w}_i^2\right) = \left\{(x, w) : x^{\mathrm{T}}Xx + w^{\mathrm{T}}\Lambda w \le \eta + \alpha\sum_{i=1}^{n_w}\bar{w}_i^2\right\}. \tag{8}$$

The purpose of introducing $\Omega(X, \Lambda, \eta + \alpha\sum_{i=1}^{n_w}\bar{w}_i^2)$ is to overbound the set $\Omega(X, \eta)$. In particular, we have the following relation between these two sets.

Lemma 2: If $x \in \Omega(X, \eta)$, then, for any $w \in \mathcal{W}_\eta$, $(x, w) \in \Omega(X, \Lambda, \eta + \alpha\sum_{i=1}^{n_w}\bar{w}_i^2)$.

Note that

$$\begin{aligned}&\mathcal{L}(H_C, H_D)\\ &= \{(x_{\mathrm{c}}, y) : |H_{Ci}x_{\mathrm{c}} + H_{Di}y| \le 1,\ i \in [1, n_u]\}\\ &= \{(x_{\mathrm{p}}, x_{\mathrm{c}}, w) : |H_{Di}C_{\mathrm{p}2}x_{\mathrm{p}} + H_{Ci}x_{\mathrm{c}} + H_{Di}D_{\mathrm{p}21}w| \le 1,\ i \in [1, n_u]\}\\ &:= \mathcal{L}(H_DC_2, H_C, H_DD_{\mathrm{p}21}).\end{aligned}$$

In the sequel, whenever necessary, we will replace the following relation

$$(x_{\mathrm{c}}, y) \in \mathcal{L}(H_C, H_D), \qquad \forall (x, w) \in \Omega(X, \eta) \times \mathcal{W}_\eta,$$

by a more stringent condition

$$\Omega\left(X, \Lambda, \eta + \alpha\sum_{i=1}^{n_w}\bar{w}_i^2\right) \subset \mathcal{L}(H_DC_{\mathrm{p}2}, H_C, H_DD_{\mathrm{p}21}).$$

Although this will introduce some conservatism, the latter condition can be readily converted into an LMI constraint. The following theorem provides a quantification of the disturbance tolerance ability of the closed-loop system.

Theorem 1: If there exist positive definite matrices $X \in \mathcal{S}_+^{(n_x+n_{x_{\mathrm{c}}})\times(n_x+n_{x_{\mathrm{c}}})}$, $\Lambda \in \mathcal{S}_+^{n_w\times n_w}$, and matrices $(A_{\mathrm{c}j}, B_{\mathrm{c}j}) \in \mathbf{R}^{n_x\times n_x}\times\mathbf{R}^{n_x\times n_y}$, $j \in [0, 2^{n_u}-1]$, $(C_{\mathrm{c}}, D_{\mathrm{c}}) \in \mathbf{R}^{n_u\times n_x}\times\mathbf{R}^{n_u\times n_y}$ and $(H_C, H_D) \in \mathbf{R}^{n_u\times n_x}\times\mathbf{R}^{n_u\times n_y}$ such that

$$\begin{bmatrix}A_j^{\mathrm{T}}X + XA_j & XB_j\\ B_j^{\mathrm{T}}X & -I\end{bmatrix} < 0, \quad j \in [0, 2^{n_u}-1], \tag{9}$$

$$\Omega\left(X, \Lambda, \eta + \alpha\sum_{i=1}^{n_w}\bar{w}_i^2\right) \subset \mathcal{L}(H_DC_{\mathrm{p}2}, H_C, H_DD_{\mathrm{p}21}), \tag{10}$$

$$X > 0, \tag{11}$$

$$\Lambda \le \alpha I, \tag{12}$$

then the trajectory of the closed-loop system that starts from the origin will remain inside the ellipsoid $\Omega(X, \eta)$ for every $w \in \mathcal{W}_\eta$.

Theorem 1 provides a bounded state region $\Omega(X, \eta)$ for the closed-loop system under an output feedback law of the form (2). Moreover, one can also find the maximum disturbance tolerance level $\eta_{\max}$ by solving an LMI optimization problem.

III. Restricted $\mathcal{L}_2$ Gain Synthesis

Under the actuator saturation, the $\mathcal{L}_2$ gain may not be well defined for sufficiently large disturbances. With such large disturbances, the state may go unbounded under any control input. This is what motivated us to consider only energy bounded disturbances, and determine the minimum energy amplification from disturbance to output. In this section, we will design an output feedback law that minimizes the $\mathcal{L}_2$ gain

$$\max_{x(0)=0, w\in\mathcal{W}_\eta}\frac{\|z\|_2}{\|w\|_2}.$$

A. Full Order Output Feedback Law

The synthesis condition for a full order output feedback law is given in the following theorem.

Theorem 2: Given scalars $\eta < \eta_{\max}$ and α, the restricted $\mathcal{L}_2$ gain of the closed-loop system is rendered less than or equal to γ if there exist positive-definite matrices $R, S \in \mathcal{S}_+^{n_x \times n_x}$, $\Lambda \in \mathcal{S}_+^{n_w \times n_w}$, and matrices $(\bar{A}_{\mathrm{c}j}, \bar{B}_{\mathrm{c}j}) \in \mathbf{R}^{n_x \times n_x} \times \mathbf{R}^{n_x \times n_y}, j \in [0, 2^{n_u}-1]$, $(\bar{C}_{\mathrm{c}}, \bar{D}_{\mathrm{c}}) \in \mathbf{R}^{n_u \times n_x} \times \mathbf{R}^{n_u \times n_y}$ and $(\bar{H}_1, \bar{H}_2) \in \mathbf{R}^{n_u \times n_x} \times \mathbf{R}^{n_u \times n_y}$ such that

$$\begin{bmatrix} \left\{ \begin{array}{c} A_{\mathrm{p}}R + B_{\mathrm{p}2}\bar{U}_{\mathrm{c}j} \\ +RA_{\mathrm{p}}^{\mathrm{T}} + \bar{U}_{\mathrm{c}j}^{\mathrm{T}}B_{\mathrm{p}2}^{\mathrm{T}} \end{array} \right\} & \star & \star & \star \\ \bar{A}_c^j + A_{\mathrm{p}}^{\mathrm{T}} + C_{\mathrm{p}2}^{\mathrm{T}}\bar{V}_{\mathrm{c}j}^{\mathrm{T}}B_{\mathrm{p}2}^{\mathrm{T}} & \left\{ \begin{array}{c} A_{\mathrm{p}}^{\mathrm{T}}S + C_{\mathrm{p}2}^{\mathrm{T}}\bar{B}_{\mathrm{c}j}^{\mathrm{T}} \\ +SA_{\mathrm{p}} + \bar{B}_{\mathrm{c}j}C_{\mathrm{p}2} \end{array} \right\} & \star & \star \\ B_{\mathrm{p}1}^{\mathrm{T}} + D_{\mathrm{p}21}^{\mathrm{T}}\bar{V}_{\mathrm{c}j}^{\mathrm{T}}B_{\mathrm{p}2}^{\mathrm{T}} & B_{\mathrm{p}1}^{\mathrm{T}}S + D_{\mathrm{p}21}^{\mathrm{T}}\bar{B}_{\mathrm{c}j}^{\mathrm{T}} & -I & \star \\ C_{\mathrm{p}1}R + D_{\mathrm{p}12}\bar{U}_{\mathrm{c}j} & C_{\mathrm{p}1} + D_{\mathrm{p}12}\bar{V}_{\mathrm{c}j}C_{\mathrm{p}2} & D_{\mathrm{p}11} + D_{\mathrm{p}12}\bar{V}_{\mathrm{c}j}D_{\mathrm{p}21} & -\gamma^2 I \end{bmatrix} < 0,\ \ j \in [0, 2^{n_u}-1], \tag{13}$$

$$\begin{bmatrix} \frac{1}{\eta + \alpha\sum_{j=1}^{n_w}\bar{w}_j^2} & \star & \star & \star \\ \bar{H}_{C_i}^{\mathrm{T}} & R & \star & \star \\ C_{\mathrm{p}2}^{\mathrm{T}}\bar{H}_{D_i}^{\mathrm{T}} & I & S & \star \\ D_{\mathrm{p}21}^{\mathrm{T}}\bar{H}_{D_i}^{\mathrm{T}} & 0 & 0 & \Lambda \end{bmatrix} \geq 0, \quad i = [1, n_u], \tag{14}$$

$$\begin{bmatrix} R & I \\ I & S \end{bmatrix} > 0, \tag{15}$$

$$\Lambda \leq \alpha I, \tag{16}$$

where $\bar{U}_{\mathrm{c}j} = E_j\bar{C}_{\mathrm{c}} + E_j^-\bar{H}_C, \bar{V}_{\mathrm{c}j} = E_j\bar{D}_{\mathrm{c}} + E_j^-\bar{H}_D$. Moreover, the coefficient matrices of an nth-order output feedback LPV controller and its associated linear subspace are given by

$$\begin{bmatrix} A_{\mathrm{c}j} & B_{\mathrm{c}j} \\ C_{\mathrm{c}} & D_{\mathrm{c}} \\ H_C & H_D \end{bmatrix} = \begin{bmatrix} N & SB_{\mathrm{p}2}E_j & SB_{\mathrm{p}2}E_j^- \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}^{-1} \times \left\{ \begin{bmatrix} \bar{A}_{\mathrm{c}j} & \bar{B}_{\mathrm{c}j} \\ \bar{C}_{\mathrm{c}} & \bar{D}_{\mathrm{c}} \\ \bar{H}_C & \bar{H}_D \end{bmatrix} - \begin{bmatrix} SA_{\mathrm{p}}R & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \right\} \begin{bmatrix} M^{\mathrm{T}} & 0 \\ C_{\mathrm{p}2}R & I \end{bmatrix}^{-1}, \tag{17}$$

where $M, N \in \mathbf{R}^{n_x \times n_x}$ are such that $MN^{\mathrm{T}} = I_{n_x} - RS$.

B. Reduced Order Output Feedback Law

Note that in Theorem 2, the order of output feedback control is the same as the plant. It is possible to design output feedback law of a lower order. The problem of synthesizing a reduced order output feedback law is generally non-convex. However, if part of the state is measurable, then one can synthesize a lower order output feedback law by solving a convex optimization problem.

Let us consider the open-loop system with partial state measurement as

$$\left\{ \begin{array}{rcl} \dot{x}_{\mathrm{p}} & = & A_{\mathrm{p}}x_{\mathrm{p}} + B_{\mathrm{p}1}w + B_{\mathrm{p}2}\sigma(u) \\ e & = & C_{\mathrm{p}1}x_{\mathrm{p}} + D_{\mathrm{p}11}w + D_{\mathrm{p}12}\sigma(u) \\ y & = & \begin{bmatrix} \bar{C}_{\mathrm{p}2} \\ 0_{r \times (n_x - r)} \quad I_r \end{bmatrix} x_{\mathrm{p}} + \begin{bmatrix} \bar{D}_{\mathrm{p}21} \\ 0 \end{bmatrix} w, \end{array} \right. \tag{18}$$

The special form of output matrix implies that the last r states are measurable. In this case, one can construct a nonlinear output feedback controller in the form of (2) with its order no higher than $(n_x - r)$.

Theorem 3: If there exist positive definite matrices $R, S > 0, \Lambda > 0$, and matrices $(\bar{A}_{\mathrm{c}j}, \bar{B}_{\mathrm{c}j}), j \in [0, 2^{n_u}-1]$, $(\bar{C}_{\mathrm{c}}, \bar{D}_{\mathrm{c}})$ and $(\bar{H}_C, \bar{H}_D)$ such that

$$\begin{bmatrix} \left\{ \begin{array}{c} A_{\mathrm{p}}R + B_{\mathrm{p}2}\bar{U}_{\mathrm{c}j} \\ +RA_{\mathrm{p}}^{\mathrm{T}} + \bar{U}_{\mathrm{c}j}^{\mathrm{T}}B_{\mathrm{p}2}^{\mathrm{T}} \end{array} \right\} & \star & \star & \star \\ \bar{A}_{\mathrm{c}j} + T^{\mathrm{T}}A_{\mathrm{p}}^{\mathrm{T}} + T^{\mathrm{T}}\bar{C}_{\mathrm{p}2}^{\mathrm{T}}\bar{V}_{\mathrm{c}j}^{\mathrm{T}}B_{\mathrm{p}2}^{\mathrm{T}} & \left\{ \begin{array}{c} T^{\mathrm{T}}A_{\mathrm{p}}^{\mathrm{T}}ST + T^{\mathrm{T}}\bar{C}_{\mathrm{p}2}^{\mathrm{T}}\bar{B}_{\mathrm{c}j}^{\mathrm{T}} \\ +T^{\mathrm{T}}SA_{\mathrm{p}}T + \bar{B}_{\mathrm{c}j}\bar{C}_{\mathrm{p}2}T \end{array} \right\} & \star & \star \\ B_{\mathrm{p}1}^{\mathrm{T}} + \bar{D}_{\mathrm{p}21}^{\mathrm{T}}\bar{V}_{\mathrm{c}j}^{\mathrm{T}}B_{\mathrm{p}2}^{\mathrm{T}} & B_{\mathrm{p}1}^{\mathrm{T}}ST + \bar{D}_{\mathrm{p}21}^{\mathrm{T}}\bar{B}_{\mathrm{c}j}^{\mathrm{T}} & -I & \star \\ C_{\mathrm{p}1}R + D_{\mathrm{p}12}\bar{U}_{\mathrm{c}j} & (C_{\mathrm{p}1} + D_{\mathrm{p}12}\bar{V}_{\mathrm{c}j}\bar{C}_{\mathrm{p}2})T & D_{\mathrm{p}11} + D_{\mathrm{p}12}\bar{V}_{\mathrm{c}j}\bar{D}_{\mathrm{p}21} & -\gamma^2 I \end{bmatrix} < 0, \quad j \in [0, 2^{n_u}-1], \tag{19}$$

$$\begin{bmatrix} \frac{1}{\eta + \alpha\sum_{j=1}^{n_w}\bar{w}_j^2} & \star & \star & \star \\ \bar{H}_{C_i}^{\mathrm{T}} & R & \star & \star \\ T^{\mathrm{T}}C_{\mathrm{p}2}^{\mathrm{T}}\bar{H}_{D_i}^{\mathrm{T}} & T^{\mathrm{T}} & T^{\mathrm{T}}ST & \star \\ \bar{D}_{\mathrm{p}21}^{\mathrm{T}}\bar{H}_{D_i}^{\mathrm{T}} & 0 & 0 & \Lambda \end{bmatrix} \geq 0, \quad i \in [1, n_u], \tag{20}$$

$$\begin{bmatrix} R & T \\ T^{\mathrm{T}} & T^{\mathrm{T}}ST \end{bmatrix} > 0, \tag{21}$$

$$\Lambda \leq \alpha I, \tag{22}$$

where $T^{\mathrm{T}} = \begin{bmatrix} I_{n_x - r} & 0_{(n_x - r) \times r} \end{bmatrix}$, and $\bar{U}_{\mathrm{c}j}, \bar{V}_{\mathrm{c}j}$ are defined similarly to Theorem 2. Then, with the following controller gains

$$\begin{bmatrix} A_{\mathrm{c}j} & B_{\mathrm{c}j} \\ C_{\mathrm{c}} & D_{\mathrm{c}} \\ H_C & H_D \end{bmatrix} = \begin{bmatrix} T^{\mathrm{T}}N & T^{\mathrm{T}}SB_{\mathrm{p}2}E_j & T^{\mathrm{T}}SB_{\mathrm{p}2}E_j^- \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}^{-1} \times \left\{ \begin{bmatrix} \bar{A}_{\mathrm{c}j} & \bar{B}_{\mathrm{c}j} \\ \bar{C}_{\mathrm{c}} & \bar{D}_{\mathrm{c}} \\ \bar{H}_1 & \bar{H}_2 \end{bmatrix} - \begin{bmatrix} T^{\mathrm{T}}SA_{\mathrm{p}}R & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \right\} \left[\begin{array}{c|c} M^{\mathrm{T}} & 0 \\ \hline C_{\mathrm{p}2}R & I_{n_y - r} \\ \begin{bmatrix} 0 & I_r \end{bmatrix} & 0 \end{array} \right]^{-1},$$

where $MN^{\mathrm{T}} = I_{n_x} - RS, M, N \in \mathbf{R}^{n_x \times (n-r)}$, the closed-loop system composing of the plant (1) and the $(n_x - r)$th order output feedback law (2) has its restricted $\mathcal{L}_2$ gain less than γ.

IV. Restricted $\mathcal{L}_2$ to $\mathcal{L}_\infty$ Gain Synthesis

We next consider a different measure of the level of disturbance attenuation, $\mathcal{L}_2$ to $\mathcal{L}_\infty$ gain, which is defined as

$$\max_{x(0)=0, w \in \mathcal{W}_\eta} \frac{\|z\|_\infty}{\|w\|_2}.$$

For simplicity, we further assumed that $D_{\mathrm{p}11} = 0$, either $D_{\mathrm{p}12}$ or $D_{\mathrm{p}21}$ is also zero in plant dynamics (1). This assumption renders $D(\rho) = 0$ as desired. The control synthesis condition under this performance measure is then given below:

Theorem 4: For given scalars $\eta < \eta_{\max}$ and α, the restricted $\mathcal{L}_2$ to $\mathcal{L}_\infty$ gain of the closed-loop system is less than or equal to γ if there exist positive definite matrices $R, S \in \mathcal{S}_+^{n_x \times n_x}$, $\Lambda \in \mathcal{S}_+^{n_w \times n_w}$, and matrices $(\bar{A}_{\mathrm{c}j}, \bar{B}_{\mathrm{c}j}) \in \mathbf{R}^{n_x \times n_x} \times \mathbf{R}^{n_x \times n_y}$, $j \in [0, 2^{n_u} - 1]$, $(\bar{C}_{\mathrm{c}}, \bar{D}_{\mathrm{c}}) \in \mathbf{R}^{n_u \times n_x} \times \mathbf{R}^{n_u \times n_y}$ and $(\bar{H}_C, \bar{H}_D) \in \mathbf{R}^{n_u \times n_x} \times \mathbf{R}^{n_u \times n_y}$ such that

$$\begin{bmatrix} \left\{ \begin{array}{c} A_{\mathrm{p}} R + B_{\mathrm{p}2} \bar{U}_{\mathrm{c}j} \\ + R A_{\mathrm{p}}^{\mathrm{T}} + \bar{U}_{\mathrm{c}j}^{\mathrm{T}} B_{\mathrm{p}2}^{\mathrm{T}} \end{array} \right\} & \star & \star \\ \bar{A}_{\mathrm{c}j} + A_{\mathrm{p}}^{\mathrm{T}} + C_{\mathrm{p}2}^{\mathrm{T}} \bar{V}_{\mathrm{c}j}^{\mathrm{T}} B_{\mathrm{p}2}^{\mathrm{T}} & \left\{ \begin{array}{c} A_{\mathrm{p}}^{\mathrm{T}} S + C_{\mathrm{p}2}^{\mathrm{T}} \bar{B}_{\mathrm{c}j}^{\mathrm{T}} \\ + S A_{\mathrm{p}} + \bar{B}_{\mathrm{c}j} C_{\mathrm{p}2} \end{array} \right\} & \star \\ B_{\mathrm{p}1}^{\mathrm{T}} + D_{\mathrm{p}21}^{\mathrm{T}} \bar{V}_{\mathrm{c}j}^{\mathrm{T}} B_{\mathrm{p}2}^{\mathrm{T}} & B_{\mathrm{p}1}^{\mathrm{T}} S + D_{\mathrm{p}21}^{\mathrm{T}} \bar{B}_{\mathrm{c}j}^{\mathrm{T}} & -I \end{bmatrix} < 0,$$
$$j \in [0, 2^{n_u} - 1], \quad (23)$$

$$\begin{bmatrix} \frac{1}{\eta + \alpha \sum_{j=1}^{n_w} \bar{w}_j^2} & \star & \star & \star \\ \bar{H}_{C_i}^{\mathrm{T}} & R & \star & \star \\ C_{\mathrm{p}2}^{\mathrm{T}} \bar{H}_{D_i}^{\mathrm{T}} & I & S & \star \\ D_{\mathrm{p}21}^{\mathrm{T}} \bar{H}_{D_i}^{\mathrm{T}} & 0 & 0 & \Lambda \end{bmatrix} > 0, \quad i \in [1, n_u], \quad (24)$$

$$\begin{bmatrix} R & I \\ I & S \end{bmatrix} > 0, \quad (25)$$

$$\Lambda \le \alpha I, \quad (26)$$

$$\begin{bmatrix} \gamma^2 I & \star & \star \\ R C_{\mathrm{p}1}^{\mathrm{T}} + \bar{U}_{\mathrm{c}j}^{\mathrm{T}} D_{\mathrm{p}12}^{\mathrm{T}} & R & \star \\ C_{\mathrm{p}1}^{\mathrm{T}} + C_{\mathrm{p}2}^{\mathrm{T}} \bar{V}_{\mathrm{c}j}^{\mathrm{T}} D_{\mathrm{p}12}^{\mathrm{T}} & I & S \end{bmatrix} \ge 0, \quad j \in [0, 2^{n_u} - 1],$$
$$(27)$$

where $\bar{U}_{\mathrm{c}j} = E_j \bar{C}_{\mathrm{c}} + E_j^- \bar{H}_C, \bar{V}_{\mathrm{c}j} = E_j \bar{D}_{\mathrm{c}} + E_j^- \bar{H}_D$. Moreover, the coefficient matrices of an nth-order output feedback controller (2) and its associated linear subspace are determined by (17).

Paralleling to $\mathcal{L}_2$ gain result, one can also derive reduced order output feedback control synthesis condition for $\mathcal{L}_2$ to $\mathcal{L}_\infty$ gain. This will not be included here due to space limitation.

V. Examples

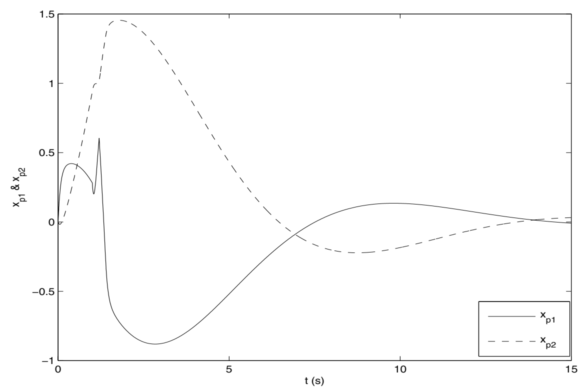
The first example is to optimize the $\mathcal{L}_2$ gain of closed-loop system system. For this purpose, we consider the second-order system (1) with

$$A_{\mathrm{p}} = \begin{bmatrix} 0.6 & -0.8 \\ 0.8 & 0.6 \end{bmatrix}, \quad B_{\mathrm{p}1} = \begin{bmatrix} 0.2 \\ 0.1 \end{bmatrix}, \quad B_{\mathrm{p}2} = \begin{bmatrix} -3.8 \\ 0.9 \end{bmatrix},$$
$$C_{\mathrm{p}1} = \begin{bmatrix} 0.1 & 0.2 \end{bmatrix}, \quad D_{\mathrm{p}11} = 0.1, \quad D_{\mathrm{p}12} = 0.01,$$
$$C_{\mathrm{p}2} = \begin{bmatrix} -2.2 & 2 \end{bmatrix}, \quad D_{\mathrm{p}21} = -0.2.$$

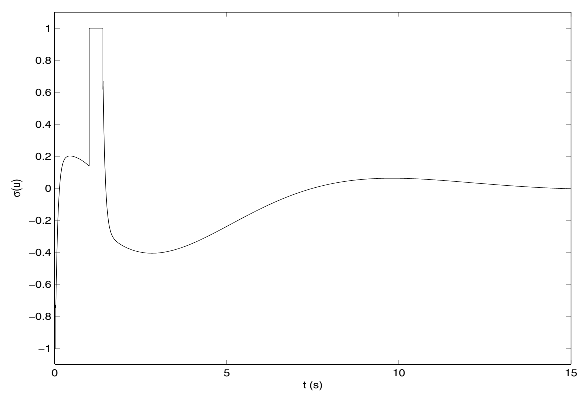
We set the magnitude bound of the disturbance as 3.464, and choose the value of α to be 0.1. Applying Theorem 1, we get $\eta_{\max} = 12.0$. This is the maximal disturbance energy the system can tolerate. Using a smaller $\eta = 11.997$, we solve the $\mathcal{L}_2$ gain control synthesis by Theorem 2 and the resulting restricted $\mathcal{L}_2$ gain is $\gamma^\star = 0.433$. Therefore we obtain an output feedback controller that guarantees the system stability and performance within a feasible region. The disturbance w used for simulation study is

$$w(t) = \begin{cases} 3.464, & 0 \le t \le 1 \text{ sec}, \\ 0, & t > 1 \text{ sec}. \end{cases}$$

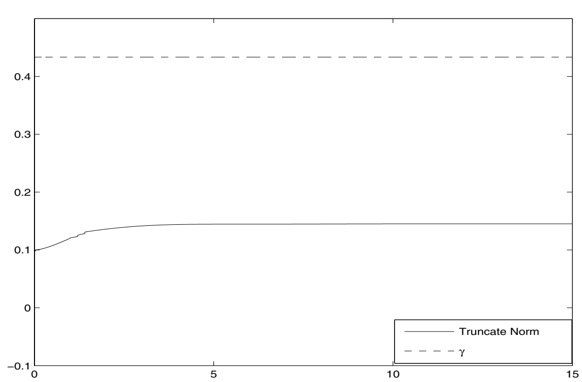
Fig. 1 provides the simulation results of the closed-loop system. The first two subplots show the state trajectory and control input for the given bounded disturbance. As can be seen in the third subplot, the truncated $\mathcal{L}_2$ gain $\frac{\|z\|_{2,T}}{\|w\|_{2,T}}$ is always less than $\gamma^\star$.



(a) state trajectory



(b) control input



(c) truncated $\mathcal{L}_2$ gain

Fig. 1. State trajectory and control input with optimized $\mathcal{L}_2$ gain.

In the second example, we consider the $\mathcal{L}_2$ to $\mathcal{L}_\infty$ gain as the measure of disturbance attenuation. The open-loop system (1) has its state-space matrices given by

$$A_{\mathrm{p}} = \begin{bmatrix} 0.6 & -0.8 \\ 0.8 & 0.6 \end{bmatrix}, \quad B_{\mathrm{p}1} = \begin{bmatrix} 0.2 \\ 0.1 \end{bmatrix}, \quad B_{\mathrm{p}2} = \begin{bmatrix} 1 \\ 3.8 \end{bmatrix},$$
$$C_{\mathrm{p}1} = \begin{bmatrix} 0.1 & 0.2 \end{bmatrix}, \quad D_{\mathrm{p}11} = D_{\mathrm{p}12} = 0$$
$$C_{\mathrm{p}2} = \begin{bmatrix} -6 & 6 \end{bmatrix}, \quad D_{\mathrm{p}21} = 0.1.$$

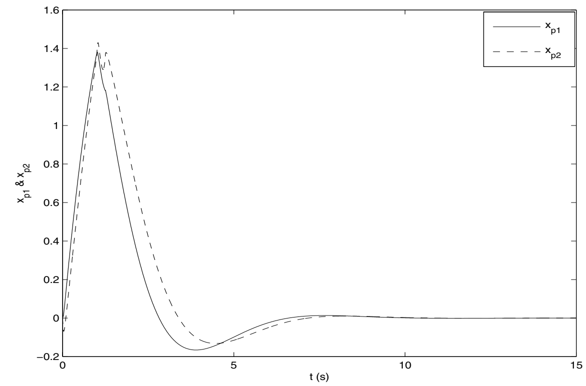
Selecting the magnitude bound of the disturbance as 7.7492 andα as 0.1, we obtain $\eta_{\max} = 60.061$. For $\eta = 60.051 < \eta_{\max}$, the optimized $\mathcal{L}_2$ to $\mathcal{L}_\infty$ gain is $\gamma^\star = 0.1$ by solving the synthesis condition in Theorem 4. Then we conduct the simulation using an admissible disturbance trajectory

$$w(t) = \begin{cases} 7.7492, & 0 \le t \le 1 \text{ sec}, \\ 0, & t > 1 \text{ sec}, \end{cases}$$

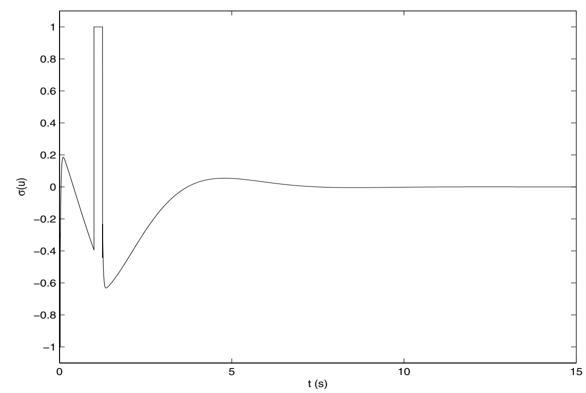
and obtain simulation results in Fig. 2. It is easy to observe that the magnitude of output z is always less than $\gamma^\star \|w\|_2 = 0.775$, as expected.

VI. Conclusions

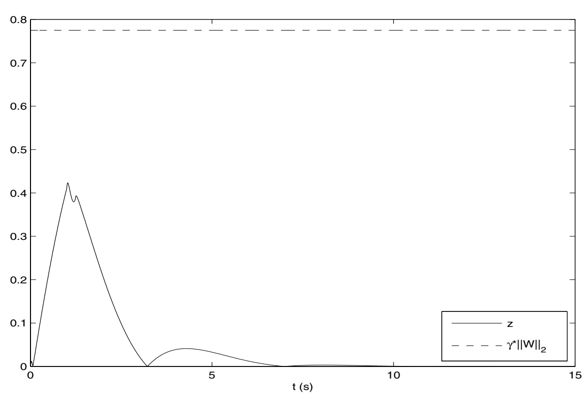
This paper presents a method to synthesize nonlinear output feedback controllers with optimal $\mathcal{L}_2$ gain and $\mathcal{L}_2$ to $\mathcal{L}_\infty$ gain performance. The resulting control law is nonlinear in nature, and was parameterized in quasi-LPV form. The control synthesis conditions were formulated as LMI optimization problems, and can be solved using efficient interior-point algorithms. Several examples have been worked out to demonstrate the proposed design approach.



(a) state trajectory



(b) control input



(c) boundedness of trajectory

Fig. 2. State trajectory and control input with optimized $\mathcal{L}_2$ to $\mathcal{L}_\infty$ gain.

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