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Compact explicit MPC with guarantee of feasibility for tracking

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***Abstract*— This paper deals with the constrained model predictive control, and focuses on the reference tracking problem. The analysis of the feasibility limitations is done using explicit formulations of the multi-parametric optimizations and invariant sets theory. The enlargement of the set of feasible trajectories by using reference governor schemes is discussed. The main result is a compact piecewise affine law with guarantees of feasibility, constructed by mixing the explicit formulations of the predictive law and the reference adjustment mechanism.**

I. Introduction

Model Predictive Control (MPC) has imposed itself as an optimization based technique with versatile constraints handling capabilities [1], [2]. In the same time, the optimization fundament imposes the feasibility as a crucial demand as long as it represents the main ingredient for the stability [6].

For regulation, the necessary and sufficient conditions of MPC feasibility are based on pseudo-infinite prediction horizons or, similarly, on invariant terminal sets [12], [3]. The reference tracking can take advantage of the prediction capabilities of MPC and deliver excellent control performances but the infeasibility threatening becomes severe, mainly if the reference contradicts with the imposed constraints or the a priori information regarding the set-point is limited.

The infeasibility avoidance strategies can follow two directions: enhance the feasible domain by increasing the prediction horizon, or on-line adjust the trajectory to be followed using a so-called "reference governor". In the first case, the feasible domain is enlarged but remains limited, another disadvantage being the augmentation of the set of decision variables and constraints. The "reference governor" (see [10], [11] and the references therein) implies the existence of solution for an optimization problem apart MPC which will provide the best admissible reference to be followed.

The current paper revisits the main concepts related to the feasibility of MPC and details their definitions for the reference tracking case. The structure of the feasible domain and the reference limitations are analyzed in relation with the MPC parameters. The reference governor schemes will be used to obtain a control law with guarantees of feasibility. Both ingredients (MPC and reference governor) are represented by multiparametric optimization problems ([5], [15]), and their explicit formulations can be gathered in a compact form resulting in a piecewise affine control law, feasible over any set of initial conditions. The conservativeness and the compromise between the memory needs and a fast evaluation mechanism are some of the addressed issues.

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In the following, section II recalls the MPC problem, the explicit formulation and states definitions related to feasibility. Section III deals with the reference tracking problems and the feasibility limitations. In section IV, the infeasibility avoidance mechanism is integrated in the predictive control scheme resulting in the compact MPC with guarantee of feasibility. Finally, section V presents a study case.

II. Model Predictive Control

A. Constrained model predictive control

MPC implies the minimization of a cost index on the predicted plant evolution. For the regulation to origin consider:

$$\Sigma_P : \begin{cases} x_{t+1} = Ax_t + Bu_t, t \geqslant 0, x_0 = x^0 \in X_0 \\ y_t = Hx_t \\ Cx_t + Du_t \leqslant \gamma \end{cases} \quad (1)$$

where $x_t \in \mathbb{R}^n$, $u_t \in \mathbb{R}^m$, $y_t \in \mathbb{R}^p$ are the state, input and output vectors and X_0 the set of initial conditions. It is assumed that the pair (A, B) is stabilizable. The constraints, given by $\gamma \in \mathbb{R}^q$, $D \in \mathbb{R}^{q\times m}$, $C \in \mathbb{R}^{q\times n}$, define a polyhedral region including the origin. At each sampling time, the current state $x_t = x$ (assumed available), is used to find an open-loop control sequence $\mathbf{k}_u^* = [u_{t|t}^T, \ldots, u_{t+N-1|t}^T]^T$:

$$\min_{\mathbf{k}_u} x_{t+N|t}^T P x_{t+N|t} + \sum_{k=0}^{N-1} x_{t+k|t}^T Q x_{t+k|t} + u_{t+k|t}^T R u_{t+k|t}$$
$$\begin{cases} x_{t+k+1|t} = Ax_{t+k|t} + Bu_{t+k|t}, \ k \geqslant 0 \\ Cx_{t+k|t} + Du_{t+k|t} \leqslant \gamma, \ 0 \leqslant k \leqslant N-1 \\ x_N \in X_N \end{cases} \quad (2)$$

where $Q \geqslant 0$ and $R > 0$ are weighting matrices and $(Q^{1/2}, A)$ is detectable. P characterizes the terminal cost and X_N is the associated terminal set. The prediction horizon, together with P, Q and R are the knobs of this optimization.

The first element of $\mathbf{k}_u^*$ is effectively applied and the procedure is restarted following a "receding horizon principle".

B. Multiparametric optimization - explicit solutions

The optimization (2) is tractable and can be rewritten [5]:

$$\begin{aligned} &\mathbf{k}_u^* = \arg\min_{\mathbf{k}_u} \mathbf{k}_u^T \Theta \mathbf{k}_u + \mathbf{k}_u^T \Phi x + x^T \Upsilon x \\ &\text{subject to :} \quad A_{in}\mathbf{k}_u \leqslant b_{in} + B_{in} x \end{aligned} \quad (3)$$

This is a multiparametric quadratic problem (mpQP) and its solution is a piecewise affine function ([5], [9], [8]):

$$\mathbf{k}_u^*(x) = K_i * x + \kappa_i, \text{ for } x \in D_i \,, \quad (4)$$

where D_i are polyhedral regions in $\mathbb{R}^n$. MPC uses only the first component of this optimal solution:

$$u^{MPC}(x) = K_i^{MPC} * x + \kappa_i^{MPC}, \text{ with } x \in D_i \,, \quad (5)$$

and $K_i^{MPC}, \kappa_i^{MPC}$ the first components of K_i, κ_i.

Lately, reliable algorithms exist [13] to develop these explicit solutions and related MPC laws.

C. Feasibility within constrained MPC

Definition: The feasible set for the MPC law (2) is the set of all states for which a control sequence $\mathbf{k}_u^*(x)$ exists:

$$X_f = \bigcup_i D_i \,, \tag{6}$$

with D_i as in (4). X_f is a convex polyhedron and corresponds to the projection of the parameterized polyhedron formed by the constraints in (3), onto the state space [8].

Definition: A set $X \in \mathbb{R}^n$ is positively invariant with respect to the dynamic $x_{t+1} = Ax_t + Bg(x_t)$ with $g(x_t)$ known, if and only if $\forall x_t \in X \Rightarrow x_{t+1} \in X$.

In the literature, one can find exhaustive results regarding the feasibility of MPC laws [12]. In the following we introduce a classification of the infeasibility for MPC based on the relation with the set of initial conditions, X_0:

- $\mathbf{X_0} \setminus \mathbf{X_f} \neq \emptyset$.
  The MPC law is infeasible w.r.t. X_0 because it exists at least one initial condition $x^0 \in X_0$ such that $u^{MPC}(x^0)$ is not well defined.
- $\mathbf{X_0} = \mathbf{X_f}$
  The MPC law is feasible w.r.t. X_0 if and only if X_f is positively invariant w.r.t. $x_{t+1} = Ax_t + Bu^{MPC}(x_t)$.
- $\mathbf{X_0} \subset \mathbf{X_f}$
  The MPC law is feasible w.r.t. X_0 if and only if it exists a set $\Omega \subset \mathbb{R}^n$, positively invariant w.r.t. $x_{t+1} = Ax_t + Bu^{MPC}(x_t)$, which satisfies $X_0 \subset \Omega \subset X_f$.

This classification underlines the role of initial conditions and the threatening represented by an inappropriate MPC tuning leading to self-generated infeasibility. The set X_f depends on MPC parameters and constructive methods do exist for designing feasible laws for the regulation problem (1-2).

III. MPC Feasibility For Tracking Systems

A. Classification of reference tracking problems

Regulating the state to origin using optimal control sequences over receding horizons can be extended to the reference tracking. A classification of these problems [4] is:

1) The model-following problem: The reference signal is the output of a known linear model:

$$\Sigma_R : \begin{cases} z_{t+1} = A_z z_t, z_0 \in Z_0; \\ r_t = H_z z_t \end{cases} \tag{7}$$

with $r_t \in \mathbb{R}^p$, A_z stable, the pair (A_z, H_z) observable and $Z_0 \in \mathbb{R}^{n_z}$ the set of initial conditions. The MPC is supposed to find the optimal control sequence for the system (1) such that the output y tracks the reference r. By recasting:

$$\tilde{x}_{t+1} = \begin{bmatrix} z_{t+1} \\ x_{t+1} \end{bmatrix} = \underbrace{\begin{bmatrix} A_z & 0 \\ 0 & A \end{bmatrix}}_{\tilde{A}} \tilde{x}_t + \underbrace{\begin{bmatrix} 0 \\ B \end{bmatrix}}_{\tilde{B}} u_t, \tag{8}$$
$$\tilde{C}\tilde{x}_t + Du_t \leqslant \tilde{\gamma}; \tilde{C} = \begin{bmatrix} 0 & C \end{bmatrix}; \tilde{D} = D; \tilde{\gamma} = \gamma$$

and $\tilde{Q} = \begin{bmatrix} H_z & -H \end{bmatrix} Q \begin{bmatrix} H_z & -H \end{bmatrix}^T$, the MPC for (1-2) can be applied to construct a feasible law for initial conditions $\tilde{x}_0 \in \tilde{X}_0 = \left\{ \begin{bmatrix} z_0 \\ x_0 \end{bmatrix} \middle| z_0 \in Z_0; x_0 \in X_0 \right\}$.

2) The tracking problem: The desired trajectory is the output of a known linear model (Fig. 1) excited by an exogenous signal, partially known:

$$\Sigma_R : \begin{cases} z_{t+1} = A_z z_t + B_z w_t, \;\; z_t \in \mathbb{R}^r, \;\; z_0 \in Z_0 \\ r_t = H_z z_t; \end{cases} \tag{9}$$

Note that w_t can be the output of a high-order LTV system.

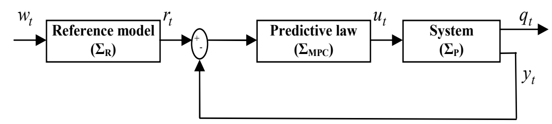

Fig. 1. Classical MPC tracking scheme (q_t - the tracking quality).

All the information on w_t should be incorporated in the reference model such that the MPC law could improve the prediction accuracy. For example if the signal is known in advance over a horizon N_w - $\{w_{t|t}, \ldots, w_{t+N_w-1|t}\}$, then the reference model to be used for the MPC design is:

$$A_z \leftarrow \left[\begin{array}{c|cccc} A_z & B_z & 0 & \cdots & 0 \\ \hline & 0 & 1 & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 0 \\ & \vdots & \vdots & \ddots & 1 \\ & 0 & 0 & \cdots & 0 \end{array}\right]; B_z \leftarrow \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$H_z \leftarrow \begin{bmatrix} H_z \\ 0 \\ \vdots \\ 0 \end{bmatrix}^T; z_k \leftarrow \begin{bmatrix} z_t \\ w_t \\ w_{t+1} \\ \vdots \\ w_{t+N_w-1} \end{bmatrix}; w_t \leftarrow w_{t+N_w}.$$

The following section will examine the influence of the exogenous signal on the feasibility of the control scheme.

B. Infeasibility within reference tracking

Even if the MPC design is reduced to the classical case, the feasibility requirements must be fulfilled with respect to the set of initial conditions in augmented space $\tilde{X}_0$ and not only for X_0. In this respect the terminal set must be based on the model (8) - $\tilde{x}_N \in \tilde{X}_N$ and not only $x_N \in X_N$.

The MPC law will be characterized by a polyhedral set of feasible points in the augmented state space as in (6):

$$\tilde{X}_f = \{\tilde{x} | \Lambda \tilde{x} \leqslant \lambda\} = \left\{ \begin{bmatrix} z^T & x^T \end{bmatrix}^T | \Lambda_z z + \Lambda_x x \leqslant \lambda \right\} \tag{10}$$

The evolution of the reference model is independent of the chosen control action at time t:

$$z_{t+1} = A_z z_t + B_z w_t;$$

and thus at time $t+1$ the infeasibility phenomenon entails that $x_{t+1} \notin X_f(z_{t+1}) = \{x | \Lambda_x x \leqslant \lambda - \Lambda_z z_{t+1}\}$.

But in the same time the MPC law is designed such that in the absence of exogenous signal ($w_t = 0$), the feasibility is preserved (a necessary condition for the constraint fulfilment over the prediction horizon). This means that $x_{t+1} \in X_f(\hat{z}_{t+1}) = \{x | \Lambda_x x \leqslant \lambda - \Lambda_z \hat{z}_{t+1}\}$ with $\hat{z}_{t+1} = A_z z_t$. Based on this observation the infeasibility of the MPC tracking scheme at time t can be classified as:

- $\mathbf{B_z w_t = 0}$
  The MPC law is feasible because $\hat{z}_{k+1} = z_{k+1}$.
- $\mathbf{B_z w_t \neq 0 \wedge D = X_f(z_{t+1}) \cap X_f(\hat{z}_{t+1}) \neq \emptyset}$ (Fig. 2a)
  If $x_{t+1} \in D$ the MPC law is feasible.
  If $x_{t+1} \notin D$ the MPC law is infeasible due to the incompatibility between the current state and the exogenous signal entering the reference model.
- $\mathbf{B_z w_t \neq 0 \wedge D = X_f(z_{t+1}) \cap X_f(\hat{z}_{t+1}) = \emptyset}$ (Fig. 2c)
  Infeasibility due to a jump of the reference model state which overwhelms the closed loop tracking capabilities.

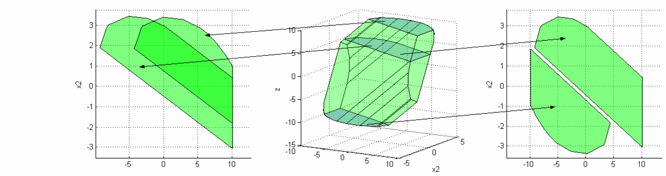

Fig. 2. Example of feasible set $\tilde{X}_f$ (center). $\mathbf{D = X_f(z_{t+1}) \cap X_f(\hat{z}_{t+1}) \neq \emptyset}$ (left). $\mathbf{D = X_f(z_{t+1}) \cap X_f(\hat{z}_{t+1}) = \emptyset}$ (right).

Remark: The only degree of freedom one can dispose within MPC to diminish the infeasibility risk is to enlarge the set $\tilde{X}_f$. This can be done by augmenting the prediction horizon but this implies a more complex optimization problem.

C. Feasible bounds as multiparametric optimizations

The feasible set $\tilde{X}_f$ has been considered as the extended version of a parameterized polyhedron $X_f(z_t)$ with parameters given by the state of the reference model. If the goal is to describe the family of feasible references, then a dual approach has to be considered, the characterization of the feasibility by the fact that $z_{t+1} \notin Z_f(x_{t+1}) = \{z | \Lambda_z z \leqslant \lambda - \Lambda_x x_{t+1}\}$. The feasibility can then be characterized based on the limits of the feasible reference signal r_t or of the feasible exogenous signal w_t as in Table I.

TABLE I

FEASIBILITY CONDITIONS

$r_t^{\min}(x_t) \leqslant r_t \leqslant r_t^{\max}(x_t)$	$w_t^{\min}(x_t, z_t) \leqslant w_t \leqslant w_t^{\max}(x_t, z_t)$
$r_t^{\min}(x_t) = \min_z H_z z$ s.t. $\Lambda_z z \leqslant \lambda - \Lambda_x x_t$	$w_t^{\min} = \min_w w$ s.t. $\Lambda_z B_z w \leqslant \lambda - \Lambda_x x_{t+1} - \Lambda_z A_z z_t$
$r_t^{\max}(x_t) = \max_z H_z z$ s.t. $\Lambda_z z \leqslant \lambda - \Lambda_x x_t$	$w_t^{\max} = \max_w w$ s.t. $\Lambda_z B_z w \leqslant \lambda - \Lambda_x x_{t+1} - \Lambda_z A_z z_t$

The feasible bounds $r_t^{\min}, r_t^{\max}, w_t^{\min}, w_t^{\max}$ are solutions of linear multiparametric optimization problems [14]. Their expression is providing a hint about the way the infeasibility can be avoided by the adjustment of the reference to the corresponding feasible limitation once it becomes infeasible.

Remark: The infeasibility can be seen as a robustness issue for the MPC law with respect to a exogenous signal affecting the model. It is not the approach followed here, our development being based on a reference governor mechanism.

IV. INFEASIBILITY AVOIDANCE

The idea resumed in Table I is that at each sampling time, the feasibility of the MPC scheme depends on whether the reference signal is contained in a safe region:

$$r_t \in X_r(x_t, z_t) \Leftrightarrow H_z z_t \in X_r(x_t, z_t) \tag{11}$$

As long as the exogenous signal is given, and the state of the system is supposed to be known, the feasibility can be obtained through a reference adjustment. This means that the infeasible reference is replaced by its best feasible approximation. This idea has already a wide experience, being known in the literature the "reference governor".

A. Reference governor

A reference governor (RG) is based on the idea that for the reference tracking scheme in (Fig. 1) a MPC law can be designed such that the closed loop performance, feasibility and stability requirements are fulfilled for the model-following problem (in the absence of the exogenous excitation). Once the MPC law description is available, we dispose also of the associate feasible set $\tilde{X}_f$. Based on this assumption the goal of RG is to replace the reference r_t by:

$$\bar{r}_t = \bar{r}_t(x_t, r_t) \tag{12}$$

such that $\bar{r}_t$ is the best approximation of r_t and the MPC law is well defined. Due to the fact that r_t is the output of the reference model, one can rewrite (12) as $\bar{r}_t(z_t, x_t)$. The best approximation must be judged with respect to a cost index. According to this principle the adjusted reference might be:

$$\begin{aligned} &\bar{r}_t = H_z \bar{z}_t^* \\ &\bar{z}_t^*(z_t, x_t) = \arg\min_{\bar{z}_t} (\bar{z}_t - z_t)^T H_z^T S H_z (\bar{z}_t - z_t) \\ &\text{such that} \begin{bmatrix} \bar{z}_t & x_t \end{bmatrix}^T \in \tilde{X}_f \end{aligned} \tag{13}$$

where the matrix S weights the deviation of the references. This formulation underlines the fact that the information needed for the optimization is restricted to the current measurements $\bar{r}_t = \bar{r}_t(z_t, x_t)$, (Fig. 3). In general case, (13) can be stated as an optimization over a wider horizon, N_S:

$$\begin{aligned} &\min_{\bar{w}_t} \sum_{k=1}^{N_S} (\bar{r}_{t+k} - r_{t+k})^T S_1 (\bar{r}_{t+k} - r_{t+k}) + \\ &\qquad + (\bar{w}_{t+k} - w_{t+k})^T S_2 (\bar{w}_{t+k} - w_{t+k}) \\ &\text{such that} \begin{bmatrix} \bar{z}_{t+k} & x_{t+k} \end{bmatrix}^T \in \tilde{X}_f \end{aligned} \tag{14}$$

B. Compact MPC law with guarantee of feasibility

Within the optimization (13) one can observe the dependence of $\bar{z}_t^*$ on the parameters $\{z_t, x_t\}$. The set of constraints depends exclusively on the vector x_t while the dependence on z_t is present in the cost index. If there is no other restriction added to (13), then any $z_t \in \mathbb{R}^{n_z}$ is feasible. By applying the MPC law $u^{MPC}(\bar{z}_t, x_t)$ instead of $u^{MPC}(z_t, x_t)$, one can obtain a predictive law with guarantee of feasibility.

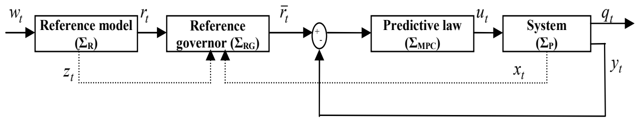

Fig. 3. Reference governor scheme for MPC feasibility

Despite this advantage, the cascaded implementation of the reference governor with the MPC law is quite demanding on-line. A first step is to observe as in [15] that the RG implies in fact a multiparametric quadratic problem (13) for which the explicit solution does exist as in (4).

$$\bar{z}_t(z_t, x_t) = L_i * \begin{bmatrix} z_t \\ x_t \end{bmatrix} + l_i, \text{ for } \begin{bmatrix} z_t \\ x_t \end{bmatrix} \in \mathbb{R}_\mathrm{i}\,, \quad (15)$$

which can be further written componentwise:

$$\bar{z}_t(z_t, x_t) = L_{i_x} * x_t + L_{i_z} * z_t + l_i. \quad (16)$$

The sets R_i are representing the critical regions corresponding to the explicit solution of (13) as in (4). The union $R = \bigcup_i R_i$ represents the domain with guarantees of feasibility. The number of regions R_i is depending on the freedom allowed for the family of references.

Regarding the explicit implementation of the MPC law, as already mentioned (5), it is expressed as a piecewise linear continuous function:

$$u(\bar{z}_t, x_t) = K_i * \begin{bmatrix} \bar{z}_t \\ x_t \end{bmatrix} + \kappa_i, \quad \text{for } \begin{bmatrix} \bar{z}_t \\ x_t \end{bmatrix} \in \mathrm{D_i}\,, \quad (17)$$

or written componentwise:

$$u_t(\bar{z}_t, x_t) = K_{i_x} * x_t + K_{i_z} * \bar{z}_t + \kappa_i. \quad (18)$$

Using these explicit formulations of the RG and MPC blocks, one can conclude that the real time implementation of the predictive scheme with guarantees of feasibility comes to a successive look-up table positioning for the evaluation of the control law at each sampling time (Fig. 4).

Given these two piecewise affine (PWA) functions, and the fact that their evaluations depend on the depth of the binary search tree for each look-up table, the natural question is whether the two functions can be composed in a single control law including both the MPC and the RG mechanism (MPC-RG) and being "everywhere" feasible.

Proposition: Let two PWA and continuous functions:

$$\begin{aligned} f: R \to D;\ R = \bigcup_{i=1}^{r} R_i; \\ f(x) = A_{f_i} x + b_{f_i}\ \forall x \in R_i \\ g: D \to F;\ D = \bigcup_{j=1}^{d} D_j; \\ g(x) = A_{g_j} x + b_{g_j}\ \forall x \in D_j \end{aligned} \quad (19)$$

then a composed PWA continuous function exists such that:

$$\begin{aligned} h: R \to F;\ R = \bigcup_{k=1}^{n_h} DR_k; \\ h(x) = A_{h_k} x + b_{h_k} = g(f(x)),\ \forall x \in DR_k \end{aligned} \quad (20)$$

with D, R, F, D_i, R_j, DR_k convex sets.

Proof: For the existence of the cutting $R = \bigcup_{k=1}^{n_h} DR_i$ it is sufficient to construct for each $R_i, i = 1,..,r$ the subsets:

$$DR_{ij} = \{x | \, x \in R_i \ and \ f(x) \in D_j\},\ j = 1,..,d \quad (21)$$

From hypothesis $f(R_i) \subset D$ and by retaining only the nonempty subsets of this construction one can obtain:

$$R_i = \bigcup_{j}^{DR_{ij} \neq \emptyset} DR_{ij} \quad (22)$$

Now, by associating for each DR_{ij} :

$$A_{h_{ij}} \leftarrow A_{g_j} A_{f_i};\ \ b_{h_{ij}} \leftarrow A_{g_j} b_{f_i} + b_{g_j} \quad (23)$$

a finite n_h is easily obtained such that:

$$\begin{aligned} h: R \to F;\ D = \bigcup_{k=1}^{n_h} DR_k \equiv \bigcup_{i,j}^{DR_{ij} \neq \emptyset} DR_{ij}; \\ h(x) = A_{h_k} x + b_{h_k} = g(f(x))\ \forall x \in DR_k \ \ \square \end{aligned} \quad (24)$$

Remark: For the resulting function $h(.)$, the number of subsets in the definition domain will satisfy $h_h \leqslant d * r$ and due to the fact that the evaluation mechanism for $f(.), g(.), h(.)$ is logarithmic in the number of regions in the partitions [7], it follows that the complexity of evaluation for $h(.)$ is inferior to the sequential evaluation of $f(.)$ and $g(.)$.

The existence of a compact law (MPC-RG), which inherits the qualities of the RG mechanism and the MPC performances is assured. The intermediary adjusted reference $\bar{r}_k$ and the associated governed reference model state, $\bar{z}_k$ needs no evaluation, all this process being nested in MPC-RG law.

The on-line evaluation of the control action is optimized but one may ask which is the price to be paid. As it was already known for the explicit formulations, the on-line computational gains are obtained by increasing the memory needs. It is the case for the MPC-RG which needs to store a more complex look-up table than the original MPC law. On the contrary, the RG explicit formulation stores the entire explicit solution (16) and not only the first component as it is the case for the MPC. This fact burdens in some extent the complexity of the sequential scheme. The compact MPC-RG law (Fig. 5) does not suffer from this point of view and thus memory storage disadvantages are mitigated.

The following algorithm resumes the MPC-RG design:

1. Find the explicit MPC for the model-following problem.

$$u = K_x^i x + K_z^i \bar{z} + k^i; \text{for } \begin{bmatrix} x^T & \bar{z}^T \end{bmatrix}^\mathrm{T} \in \mathrm{D_i}, \mathrm{i} = 1,..,\mathrm{d}$$

2. Determine the MPC feasible set $\tilde{X}_f = \bigcup_{i=1}^{d} D_i$.

3. Construct the explicit form of the RG:

$$\bar{z} = L_x^i x + L_z^i z + \lambda^i; \text{for } \begin{bmatrix} x^T & z^T \end{bmatrix}^\mathrm{T} \in \mathrm{R_i}, \mathrm{i} = 1,..,\mathrm{r}$$

4. Build the compact MPC-RG law:

For all $i = 1,..,r$

For all $j = 1,..,d$

Compute $DR_{ij} = \{x | \, x \in R_i \text{ and } \mathrm{f(x)} \in \mathrm{D_j}\}$

If DR_{ij} *is nondegenerate store it together with:*

$$u = K_x^i x + K_z^i L_x^j x + K_z^i L_z^j z + k^i + K_z^i \lambda^j$$

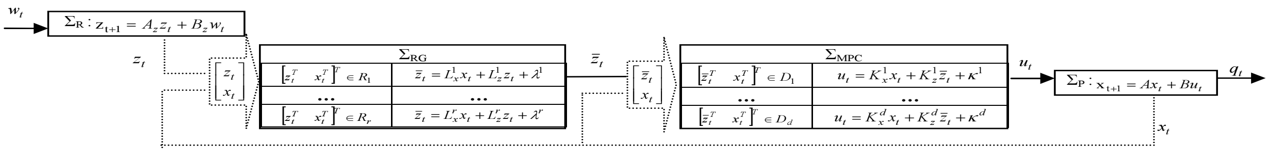

Fig. 4. Explicit description of reference governor and MPC law

Applying this algorithm one can obtain a piecewise linear control law with guarantee of feasibility. Its properties are inherited from the underlying predictive law characteristics.

Proposition The compact MPC-RG law enjoys the following properties:

i) MPC-RG is a piecewise linear and continuous function of the extended state $\tilde{x}_t = [z_t \quad x_t]^T$.

ii) If $\tilde{x}_t \in \tilde{X}_f$ then the MPC and MPC-RG are equivalent.

iii) If $r_t \to r_\infty$ then $y_t \to \hat{r}$ where $\hat{r}$ is the best feasible approximation of r_∞ with respect to the criterium in (13).

iv) If the original MPC law was designed for zero steady error for constant references r_∞, the MPC-RG law has a finite settling time to the best feasible approximation $\hat{r}$.

Proof: The property *i)* is a consequence of the fact that the composition of two piecewise affine and continuous functions inherits the same properties. For *ii)* it is sufficient to see that if the condition $\tilde{x}_t \in \tilde{X}_f$ is verified then (13) becomes an unconstrained optimization problem and thus $\bar{z}_t = z_t$ implying the equivalence of the MPC law with the MPC-RG version. In order to demonstrate *iii*) the continuity of the RG must be taken into account to generate a sequence of references leading to a steady set-point $H_z\bar{z}_t \to \hat{r}$. Further due to the stabilization properties of the MPC law this position will be regulated $y_t \to \hat{r}$. If the steady output obtained $\hat{r}$ is not corresponding to the best approximation then the optimality of the RG is denied leading to a contradiction. The point *iv*) is a special case of the former problem.

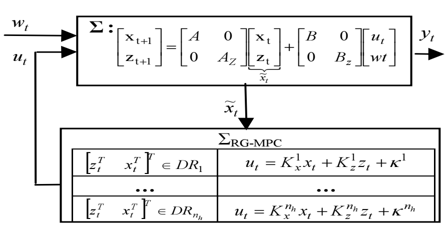

Fig. 5. Compact MPC law with guarantee of feasibility

V. EXEMPLE

Consider the discrete version of the double integrator:

$$x_{t+1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x_t + \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} u_t, \qquad (25)$$
$$y_t = \begin{bmatrix} 1 & 0 \end{bmatrix} x_t$$

For the trajectory tracking, the reference model is:

$$z_{t+1} = 0.85 z_t + 0.15 w_t; \quad r_t = z_t \qquad (26)$$

The MPC law has to consider the set of constraints:

$$-1 \leqslant u_t \leqslant 1, \quad -10 \leqslant r_t \leqslant 10,$$
$$x_t \in \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} | -10 \leqslant x_1, x_2 \leqslant 10 \right\} \qquad (27)$$

The exogenous signal will affect the evolution and using the formulations in Table I, one can obtain the explicit limitations of the reference $r_t^{min}(x_t)$, $r_t^{max}(x_t)$ (Fig. 6).

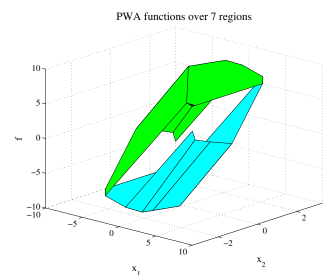

Fig. 6. The explicit description of the reference limitations. The piecewise affine function in the upper part - $r_t^{min}(x_t)$. In the lower part $r_t^{max}(x_t)$.

The prediction horizon plays a decisive role in the feasibility limitations as it can be seen in Fig. 7.

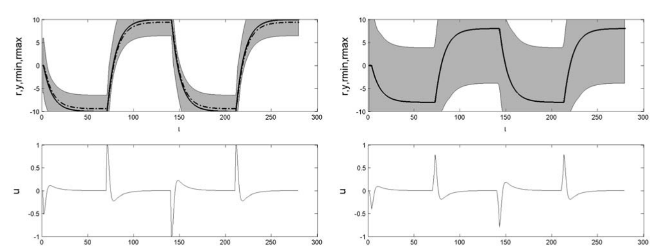

Fig. 7. Feasible interval during the trajectory tracking. Left - MPC law with $N = 1$ and a exogenous signal represented by a pulse train reference with amplitude 10. Right - $N = 4$ and a pulse train with amplitude 8.

If the feasible set available is not satisfying the feasibility requests for the tracking problem, then an avoiding redundancy mechanism can be synthesized in terms of a reference governor (RG). In order to give an idea about the complexity of its explicit solution, in Table II, one can find the number of zones (NZ_*) for prediction horizons going from 1 to 4.

The column NZ_{MPC} contains the number of domains for the predictive control explicit solution. Their union will form $\tilde{X}_f$ which further is the base for the RG scheme. The column NZ_{RG} presents the number of zones exclusively for the reference government scheme. If these two blocks function independently then the number of combinations to be explored are represented by the product $NZ_{MPC} * NZ_{RG}$

TABLE II

EXPLICIT FORMULATIONS

N	NZ_{MPC}	NZ_{RG}	Combinations to be explored	NZ_{MPC-RG}
1	5	37	185	79
2	23	154	3542	597
3	99	627	62073	3979
4	421	2373	999033	25114

which is reported in the forth column of the table. It can be observed that for large prediction horizons this indicator becomes very large and the natural question is whether all of them are representing valid combinations of the extended (reference model + system) state. The answer is given by the compact MPC law with guarantees of feasibility, constructed by the composition of the MPC and RG descriptions. The fifth column in Table II contains the number of zones NZ_{MPC-RG} for its explicit formulation.

One conclusion is imposed: the compact $MPC - RG$ scheme is avoiding the exploration of useless combinations but in the same time its number of zones is larger than the number of zones needed for the sequential implementation:

$$NZ_{MPC} + NZ_{RG} \leqslant NZ_{MPC-RG} \leqslant NZ_{MPC} * NZ_{RG}$$

From the point of view of the on-line evaluation, the second inequality is relevant. From the point of view of the memory needed to store the explicit solution is mainly the first inequality which reflects the comparison between the sequential implementation or the compact implementation.

In order to get a better insight on the "on-line evaluation vs. memory used" compromise, the search trees for the explicit solutions are constructed and their complexity compared (see Table III).

TABLE III

SEARCH TREE COMPLEXITY

	ST_{RG+MPC}			ST_{MPC-RG}		
N	Nodes	Depth	Memory(bytes)	Nodes	Depth	Memory(bytes)
1	176+6	8+4	27616+1136	228	8	25402
2	1479+71	11+6	173928 + 8408	1756	12	212963

Unfortunately, due to the huge number of zones to be explored, only the $N = 1$ and $N = 2$ are obtainable in reasonable time. However, the results are insightful, and for example for $N = 1$ one can observe that the depth of the search tree for the compact formulation is equivalent with the depth of the RG. This means that in the worst case the RG evaluation is equivalent with MPC-RG evaluation which on its turn is far less expensive that the sequential evaluation of MPC and RG (compose depth of the search tree - 12). The price to be paid as already mentioned is transparent in the memory needs. But due to the fact that the RG scheme needs the storage of the complete explicit solution, for $N = 1$ this disadvantage is mitigated and we can notice that the sequential scheme is even much memory involved that the compact scheme for some cases. For $N = 2$ the evaluation mechanism has to deal with a $12 vs. 17$ depth search tree. The memory used for the compact scheme is still comparable with the sequential case. For larger prediction horizons the differences from the memory point of view become more evident. As a conclusion the choice between the compact or the sequential scheme are dependent on the aspiration towards a small on-line evaluation time on one hand and the memory available on the other hand.

VI. CONCLUSIONS

The feasibility problem within the model predictive control framework was treated with a special attention for the tracking problems. Based on the feasibility results existing for the regulation case, we established the limitations of the feasible trajectories and classified the infeasible comportment.

In order to avoid the infeasibility, a reference adjustment mechanism was used. Based on the fact that both the MPC and the RG are in fact formulated as multiparametric optimization problems, their explicit formulation in terms of PWA functions was proposed.

The independent implementation of the RG in conjunction with MPC law was shown to be not optimal from the point of view of the evaluation mechanism. A compact predictive law with guarantee of feasibility was constructed, optimal from this on-line positioning point of view.

REFERENCES

[1] J. Maciejowski *Predictive Control with constraints* Prentice Hall 2002.

[2] E. Camacho, C. Bordons, *Model Predictive Control*, Springer, 2003.

[3] J.A. Rossiter, *Model-Based Predictive Control*, CRC Press, 2003.

[4] B. Anderson, J. Moore, *Linear Optimal Control*, Prentice Hall, 1971.

[5] A. Bemporad, M. Morari, V. Dua, E. Pistikopoulos, "The explicit linear quadratic regulator for constrained systems", *Automatica* vol.38, 2002, pp. 3-20.

[6] D.Q. Mayne, J.B. Rawlings, C.V. Rao, P.O.M. Scokaert, "Constrained model predictive control: Stability and optimality", *Automatica* vol.36, 2000, pp. 789-814.

[7] P. Tondel, T.A. Johansen, A. Bemporad, "Evaluation of piecewise affine control via binary search tree", *Automatica*, vol. 39, 2003, pp. 945 950.

[8] S. Olaru and D. Dumur, "A Parameterized Polyhedra Approach for Explicit Constrained Predictive Control", *43rd IEEE Conference on Decision and Control*, Bahamas, 2004.

[9] M.M. Seron, G.C. Goodwin, J.A. De Dona, "Characterisation of Receding Horizon Control for Constrained Linear Systems", *Asian Journal of Control*, vol. 5-2, 2003, pp. 271-286.

[10] A. Bemporad and E. Mosca, "Fulfiling hard constraints in uncertain linear systems by reference managing", *Automatica*, vol. 34, No. 4, 1998, pp. 451-461.

[11] A. Casavola, E. Mosca, M. Papini, "Control under Constraints: An application of the command governor approach to inverted pendulum", *IEEE Transactions on control system technology*, vol. 12, No. 1, 2004, pp. 193-204.

[12] E.C. Kerrigan and J.M. Maciejowski, "Invariant Sets for Constrained Nonlinear Discrete-time Systems with Application to Feasibility in Model Predictive Control", *39th IEEE Conference on Decision and Control*, Sydney, 2000.

[13] M. Kvasnica, P. Grieder, M. Baotic, M. Morari "Multi-parametric Toolbox (MPT)." url: http://control.ee.ethz.ch/ mpt/, 2004.

[14] J. Pekar, V. Havlena, "Design and analysis of model predictive control using MPT Toolbox", url: http://dsp.vscht.cz/konference_matlab/matlab04/pekar.pdf, 2004.

[15] K. Kogiso, K. Hirata, "A reference governor in a piecewise state affine function: The implementation and validation", *42nd IEEE Conference on Decision and Control*, Maui, 2003.