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Stability of switched seesaw systems with application to the stabilization of underactuated vehicles

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***Abstract*— This paper addresses the stabilization of a class of nonlinear systems in the presence of disturbances, using hybrid control. To this effect we introduce two new classes of switched systems and provide conditions under which they are input-to-state practically stable (ISpS). These results lead to a control design methodology called *switched seesaw control design* that allows for the development of nonlinear control laws yielding input-to-state stability, using switching. The range of applicability and the efficacy of the methodology proposed are illustrated via two non-trivial design examples. Namely, stabilization of the extended nonholonomic double integrator (ENDI) and stabilization of an underactuated autonomous underwater vehicle (AUV) in the presence of input disturbances and measurement noise.**

I. INTRODUCTION

There has been increasing interest in hybrid control in recent years, in part due to its potential to overcome the basic limitations to nonlinear system stabilization introduced by Brockett's celebrated result in the area of nonholonomic systems control [1]. Hybrid controllers that combine time-driven with event-driven dynamics have been developed by a number of authors and their design is by now firmly rooted in a solid theoretical background [2], [3].

Inspired by the progress in the area, the first part of this paper offers a new design methodology for the stabilization of nonlinear systems in the presence of external disturbances by resorting to hybrid control. To this effect, two classes of switched systems are introduced: *unstable/stable switched systems* and *switched seesaw systems*. The first have the property of alternating between an unstable and a stable mode during consecutive periods of time. The latter can be viewed as the interconnection of two unstable/stable systems such that when one is stable the other is unstable, and vice-versa. Conditions are given under which these systems are input-to-state practically stable (ISpS). The results are then used to develop a control design framework called *switched seesaw control design* that allows for the solution of robust control problems using switching.

To illustrate the scope of the new design methodology proposed, the second part of the paper solves the challenging problems of stabilizing the so-called extended nonholonomic double integrator (ENDI) and an underactuated autonomous underwater vehicle (AUV) *in the presence of input disturbances and measurement noise*. These examples were motivated by the problem of *point stabilization*, that is, the problem of steering an autonomous vehicle to a point with a desired orientation. For underactuated vehicles, i.e., systems with fewer actuators than degrees-of-freedom, point-stabilization is particularly challenging because most of the vehicles exhibit second-order (acceleration) nonholonomic constraints. As pointed out in a famous paper of Brockett [1], nonholonomic systems cannot be stabilized by continuously differentiable time invariant static state feedback control laws. To overcome this basic limitation, a variety of approaches have been proposed in the literature [2], [4]–[15].

From a practical point of view, the above problem has been the subject of much debate within the ground robotics community. However, it was only recently that the problem of point stabilization of underactuated AUVs received special consideration in the literature [16]–[19]. Point stabilization of AUVs poses considerable challenges to control system designers because the dynamics of these vehicles are complicated due to the presence of complex, uncertain hydrodynamic terms. These, in turn, require that model uncertainty be taken into account explicitly at the control design level. Furthermore, the models of the vehicles are normally second-order nonholonomic and include a drift vector field that is not in the span of the input vector fields, thus precluding the use of input transformations to bring them to driftless form. In this paper, the problem of stabilizing an underactuated AUV in the horizontal plane is solved by resorting to the seesaw control design technique referred to before. For reasons of clarity, the technique is first applied to stabilize, in the presence of input disturbances and measurement noise, the so-called extended nonholonomic double integrator (ENDI), which captures the kinematic and dynamic equations of a wheeled robot. The methodology adopted is then extended to deal with an AUV by showing that its dynamics can be cast in a form similar to (but more complex than) that of the ENDI. One of the key contributions of the paper is the fact that the solution proposed for point stabilization of an AUV addresses explicitly the existence of external disturbances and measurement errors. In a general setting this topic has only been partially addressed in the literature and in many aspects *it still remains an open problem.* Noteworthy exceptions are e.g., [9], [14], [15].

Due to space limitations, all the proofs are omitted. These can be found in [20].

Notation and definitions: $|\cdot|$ denotes the standard Euclidean norm of a vector in $\mathbb{R}^n$ and $\|u\|_I$ is the (essential) supremum norm of a signal $u : I \to \mathbb{R}^n$ on an interval $I \subset [0, \infty)$. Let $a \oplus b := \max\{a, b\}$ and denote by $M_{\mathcal{W}}$ the set of measurable, essentially bounded signals $w : [t_0, \infty) \to \mathcal{W}$, where $\mathcal{W} \subset \mathbb{R}^m$. A function $\gamma : [0, \infty) \to [0, \infty)$ is *of class* $\mathcal{K}$ ($\gamma \in \mathcal{K}$)

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if it is continuous, strictly increasing, and $\gamma(0) = 0$ and *of class* $\mathcal{K}_\infty$ if in addition it is unbounded. A function $\beta : [0,\infty)\times\mathbb{R}\to[0,\infty)$ is *of class*[1] $\mathcal{KL}$ if it is continuous, for each fixed $t\in\mathbb{R}$ the function $\beta(\cdot,t)$ is of class $\mathcal{K}$, and for each fixed $r\geq 0$ the function $\beta(r,t)$ decreases with respect to t and $\beta(r,t)\to 0$ as $t\to\infty$. We denote the identity function from $\mathbb{R}$ to $\mathbb{R}$ by $\mathbf{id}$, and the composition of two functions $\gamma_i:\mathbb{R}\to\mathbb{R};\ i=1,2$ in this order by $\gamma_2\circ\gamma_1$. The acronym w.r.t. stands for "with respect to".

II. Dwell-time switching theorems and hybrid control

This section introduces and analyzes stability related results for two classes of systems that will be henceforth called unstable/stable and seesaw switched systems.

A. Unstable/stable switched system

Consider the switched system

$$\dot{x} = f_\sigma(x,\mathrm{w}),\quad x(t_0)=x_0 \tag{1}$$

where $x\in\mathcal{X}\subset\mathbb{R}^n$ is the state, $\mathrm{w}\in M_\mathcal{W}$ is a disturbance, and $\sigma:[t_0,\infty)\to\{1,2\}$ is a piecewise constant switching signal that is continuous from the right and evolves according to

$$\sigma(t)=\begin{cases}1, & t\in[t_{k-1},t_k),\ k \text{ odd}\\ 2, & t\in[t_{k-1},t_k),\ k \text{ even}\end{cases} \tag{2}$$

In (2), $\{t_k\}:=\{t_1,t_2,t_3,\ldots\}$ is a sequence of strictly increasing infinity switching times in $[t_0,\infty)$ and t_0 is the initial time. We assume that both $f_i; i=1,2$ are locally Lipschitz w.r.t. (x,w) and that the solutions of (1) lie in $\mathcal{X}$ and are defined for all $t\geq t_0$.

Let $\omega:\mathbb{R}^n\to[t_0,\infty)$ be a continuous nonnegative real function called a *measuring function*. For a given switching signal σ, system (1) is said to be *input-to-state practically stable*[2] (ISpS) on $\mathcal{X}$ w.r.t. ω if there exist functions $\beta\in\mathcal{KL}$, $\gamma\in\mathcal{K}$, and a nonnegative constant c such that for every initial condition $x(t_0)$ and every input $\mathrm{w}\in M_\mathcal{W}$ such that the solution $x(t)$ of (1) lies entirely in $\mathcal{X}$, $x(t)$ satisfies

$$\omega(x(t))\leq\beta(\omega(x(t_0)),t-t_0)\oplus\gamma^{\mathrm{w}}(\|\mathrm{w}\|_{[t_0,t]})\oplus c. \tag{3}$$

for all $t\geq t_0$. When $\mathcal{X}=\mathbb{R}^n$, $\mathcal{W}=\mathbb{R}^m$, $\omega(x)=|x|$ and $c=0$, ISpS is equivalent to the by now classical definition of *input-to-state stability* (ISS) [21]. With respect to (1), assume the following conditions hold:

1. *Unstability* ($\sigma=1$). For $\dot{x}=f_1(x,\mathrm{w})$, there exist functions $\beta_1\in\mathcal{KL}$, $\gamma_1^{\mathrm{w}}\in\mathcal{K}$, and a nonnegative constant c_1 such that for every initial condition $x(t_0)$ and every input $\mathrm{w}\in M_\mathcal{W}$ for which the solution $x(t)$ of (1) lies entirely in $\mathcal{X}$, $x(t)$ satisfies[3]

$$\omega(x(t))\leq\beta_1\big(\omega(x(t_0))\oplus\gamma_1^{\mathrm{w}}(\|\mathrm{w}\|_{[t_0,t]})\oplus c_1,-(t-t_0)\big) \tag{4}$$

$t\geq t_0$. Notice how the term $-(t-t_0)$ in the second argument of β_1 captures the unstable characteristics of the system.

2. *Stability* ($\sigma=2$). System $\dot{x}=f_2(x,\mathrm{w})$ is ISpS on $\mathcal{X}$ w.r.t. ω, that is, for every initial condition $x(t_0)$ and every input $\mathrm{w}\in M_\mathcal{W}$ such that the solution $x(t)$ of system (1) lies entirely in $\mathcal{X}$, $x(t)$ satisfies

$$\omega(x(t))\leq\beta_2(\omega(x(t_0)),t-t_0)\oplus\gamma_2^{\mathrm{w}}(\|\mathrm{w}\|_{[t_0,t]})\oplus c_2, \tag{5}$$

$t\geq t_0$, where $\beta_2\in\mathcal{KL}$, $\gamma_2^{\mathrm{w}}\in\mathcal{K}$, $c_2\geq 0$.

If conditions 1–2 above are met, we call (1)–(2) an *unstable/stable switched system on* $\mathcal{X}$ *w.r.t.* ω. The definition of a *stable/unstable switched* is done in the obvious manner.

The following result provides conditions under which an unstable/stable switched system is ISpS.

Lemma 1: Consider an *unstable/stable switched system on* $\mathcal{X}$ *w.r.t.* ω. Let $t_i; i\in\mathbb{N}$ be a sequence of strictly increasing switching times $\{t_i\}$ such that the differences between consecutive instants of times $\Delta_i:=t_i-t_{i-1}$ satisfy

$$\beta_2\big(\beta_1(r,-\Delta_{k+1}),\Delta_{k+2}\big)\leq(\mathbf{id}-\alpha)(r),\forall r\geq r_0 \tag{6}$$

for $k=0,2,4,...$, and for some choice of class $\mathcal{K}_\infty$ function $\alpha(\cdot)$ and $r_0\geq 0$. Then, system (1)–(2) is ISpS at $t=t_k$, that is, $x(t)$ satisfies the ISpS condition (3) at $t=t_k$. Similarly, if

$$\beta_1\big(\beta_2(r,\Delta_k),-\Delta_{k+1}\big)\leq(\mathbf{id}-\alpha)(r),\quad\forall r\geq r_0 \tag{7}$$

for $k=2,4,6,\ldots$, and for some choice of class $\mathcal{K}_\infty$ function $\alpha(\cdot)$ and $r_0\geq 0$, then system (1)–(2) is ISpS at $t=t_{k+1}$. If either (6) or (7) hold and the differences between consecutive switching times Δ_i are uniformly bounded, then system (1)–(2) is ISpS. □

Remark 1: If $c_1=c_2=r_0=0$, $\omega(x)=|x|$, $\mathcal{X}=\mathbb{R}^n$, and $\mathcal{W}=\mathbb{R}^m$ and all the conditions of Lemma 1 above are met, then system (1)–(2) is ISS. □

Remark 2: If (4)–(5) hold with exponential class $\mathcal{KL}$ functions, i.e., $\beta_i(r,t)=\hat{\beta}_i r e^{-\lambda_i t}$, $i=1,2$, and α can be taken as $\alpha(r)=\hat{\alpha}r;\hat{\alpha}>0$, with $\mathbf{id}-\alpha\in\mathcal{K}$, then (6)–(7) become independent of r. In particular, (6)–(7) degenerate into

$$\Delta_{k+2}\geq\frac{\lambda_1}{\lambda_2}\Delta_{k+1}+\frac{1}{\lambda_2}\ln\frac{\hat{\beta}_1\hat{\beta}_2}{1-\hat{\alpha}};\qquad k=0,2,4,...$$
$$\Delta_{k+1}\leq\frac{\lambda_2}{\lambda_1}\Delta_k+\frac{1}{\lambda_1}\ln\frac{1-\hat{\alpha}}{\hat{\beta}_1\hat{\beta}_2};\qquad k=2,4,6,...$$

B. Switched seesaw system

This section introduces the key concept of switched seesaw system. To this effect, consider the switched system (1)–(2). Given two measuring functions ω_{su}, ω_{us} and a set $\mathcal{X}\subset\mathbb{R}^n$ we call (1) a *switched seesaw system on* $\mathcal{X}$ w.r.t. $(\omega_{su},\omega_{us})$ if it is a stable/unstable system w.r.t. ω_{su} when $\omega_{us}(x)$ and w are regarded as inputs, and an unstable/stable w.r.t. ω_{us} when $\omega_{su}(x)$ and w are regarded as inputs, see Table I. More precisely, the following conditions must hold:

C1. For $\dot{x}=f_1(x,\mathrm{w})$, that is, $\sigma=1$, there exist $\beta_{11},\beta_{12}\in\mathcal{KL}$, $\gamma_{11}^{\omega_{us}},\gamma_{12}^{\omega_{su}},\gamma_{11}^{\mathrm{w}},\gamma_{12}^{\mathrm{w}}\in\mathcal{K}$, $c_{11},c_{12}\geq 0$ such that for

[1]Our definition of $\mathcal{KL}$ functions is slightly different from the standard one because the domain of the second argument has been extended from $[0,\infty)$ to $\mathbb{R}$. This will allow us to consider the case $\beta(r,-t)$ which may grow unbounded as $t\to\infty$.

[2]On a first reading, one can consider that $\mathcal{X}=\mathbb{R}^n$. In this case, the reference to the set $\mathcal{X}$ is omitted. We will need the more general setting when we consider applications to the stabilization of underactuated vehicles.

[3]Another alternative is to consider that $x(t)$ satisfies $\omega(x(t))\leq\beta_1^x\big(\omega(x(t_0)),-(t-t_0)\big)\oplus\beta_1^{\mathrm{w}}\big(\|\mathrm{w}\|_{[t_0,t]},-(t-t_0)\big)\oplus\beta_1^c\big(c_1,-(t-t_0)\big)$ with $\beta_1^x,\beta_1^{\mathrm{w}},\beta_1^c\in\mathcal{KL}$. There is no loss of generality in considering (4), because one can always take $\beta_1(r,-t)=\beta_1^x(r,-t)\oplus\beta_1^{\mathrm{w}}(r,-t)\oplus\beta_1^c(r,-t)$ with the advantage of introducing a less complicated notation. However, this may lead to more conservative estimates.

TABLE I

TEMPORAL REPRESENTATION OF THE SWITCHED SEESAW SYSTEM

	Δ_1	Δ_2	Δ_3	Δ_4	$\cdots$
σ	1	2	1	2	$\cdots$
ω_{su}	$\searrow$	$\nearrow$	$\searrow$	$\nearrow$	$\cdots$
ω_{us}	$\nearrow$	$\searrow$	$\nearrow$	$\searrow$	$\cdots$

$\searrow$ Stable

$\nearrow$ Unstable

every solution $x(\cdot) \in \mathcal{X}$

$$\omega_{su}(x(t)) \le \beta_{11}(\omega_{su}(x(t_0)), t-t_0)) \oplus \gamma_{11}^{\omega_{us}}(\|\omega_{us}(x)\|_{[t_0,t]}) \oplus \gamma_{11}^{\mathrm{w}}(\|\mathrm{w}\|_{[t_0,t]}) \oplus c_{11}, \tag{8}$$

$$\omega_{us}(x(t)) \le \beta_{12}\big(\omega_{us}(x(t_0)) \oplus \gamma_{12}^{\omega_{su}}(\|\omega_{su}(x)\|_{[t_0,t]}) \oplus \gamma_{12}^{\mathrm{w}}(\|\mathrm{w}\|_{[t_0,t]}) \oplus c_{12}, -(t-t_0)\big). \tag{9}$$

C2. For $\dot{x} = f_2(x, \mathrm{w})$, that is, $\sigma = 2$, there exist $\beta_{21}, \beta_{22} \in \mathcal{KL}$, $\gamma_{21}^{\omega_{us}}, \gamma_{22}^{\omega_{su}}, \gamma_{21}^{\mathrm{w}}, \gamma_{22}^{\mathrm{w}} \in \mathcal{K}$, $c_{21}, c_{22} \ge 0$ such that for every solution $x(\cdot) \in \mathcal{X}$

$$\omega_{su}(x(t)) \le \beta_{21}\big(\omega_{su}(x(t_0)) \oplus \gamma_{21}^{\omega_{us}}(\|\omega_{us}(x)\|_{[t_0,t]}) \oplus \gamma_{21}^{\mathrm{w}}(\|\mathrm{w}\|_{[t_0,t]}) \oplus c_{21}, -(t-t_0)\big), \tag{10}$$

$$\omega_{us}(x(t)) \le \beta_{22}(\omega_{us}(x(t_0)), t-t_0) \oplus \gamma_{22}^{\omega_{su}}(\|\omega_{su}(x)\|_{[t_0,t]}) \oplus \gamma_{22}^{\mathrm{w}}(\|\mathrm{w}\|_{[t_0,t]}) \oplus c_{22}. \tag{11}$$

The following theorem gives conditions under which a switched seesaw system is ISpS.

Theorem 1: Let τ_1, τ_2 be two positive constants called *dwell times*, $\{t_k\}; k \in \mathbb{N}$ a sequence of strictly increasing switching times, and $\Delta_k = t_k - t_{k-1}$ a sequence of intervals satisfying

$$\Delta_1 = \Delta_3 = \Delta_5 = \cdots = \tau_1,$$
$$\Delta_2 = \Delta_4 = \Delta_6 = \cdots = \tau_2.$$

Assume there exist $\alpha_i \in \mathcal{K}_\infty; i = 1, 2$ such that

$$\beta_{21}\big(\beta_{11}(r, \tau_1), -\tau_2\big) \le (\mathbf{id} - \alpha_1)(r), \quad \forall r \ge r_0, \tag{12}$$

$$\beta_{22}\big(\beta_{12}(r, -\tau_1), \tau_2\big) \le (\mathbf{id} - \alpha_2)(r), \quad \forall r \ge r_0, \tag{13}$$

for some $r_0 \ge 0$ and

$$\bar{\gamma}_2^{\omega_{su}} \circ \bar{\gamma}_1^{\omega_{us}}(r) < r, \quad \forall r > \hat{r}_0, \tag{14}$$

$$\bar{\gamma}_1^{\omega_{us}} \circ \bar{\gamma}_2^{\omega_{su}}(r) < r, \quad \forall r > \hat{r}_0, \tag{15}$$

for some $\hat{r}_0 \ge 0$, where

$$\bar{\gamma}_1^{\omega_{us}}(r) := \alpha_1^{-1} \circ \rho_1^{-1} \circ \beta_{21}\big(\gamma_{11}^{\omega_{us}}(r) \oplus \gamma_{21}^{\omega_{us}}(r), -\tau_2\big),$$
$$\bar{\gamma}_2^{\omega_{su}}(r) := \alpha_2^{-1} \circ \rho_2^{-1} \circ \big[(\mathbf{id} - \alpha_2) \circ \gamma_{12}^{\omega_{su}}(r) \oplus \gamma_{22}^{\omega_{su}}(r)\big]$$

and $\rho_i \in \mathcal{K}_\infty; i = 1, 2$ are arbitrary functions such that $\mathbf{id} - \rho_i \in \mathcal{K}$. Then, the seesaw switched system (1) is ISpS on $\mathcal{X}$ w.r.t. to $\omega_{su} \oplus \omega_{us}$. □

Remark 3: If the $\mathcal{KL}$ functions β_{ij} are exponential, that is, if $\beta_{ij}(r,t) = \hat{\beta}_{ij} r e^{-\lambda_{ij} t}$, with $\hat{\beta}_{ij} > 1$, and α_i can be taken as $\alpha_i(r) = \hat{\alpha}_i r; \hat{\alpha}_i > 0$, with $\mathbf{id} - \alpha_i \in \mathcal{K}$, then conditions (12)–(13) imply the necessary condition

$$\frac{\lambda_{12}}{\lambda_{11}} \frac{\lambda_{21}}{\lambda_{22}} < 1.$$

The above expression sets an upper bound on the ratio of $\lambda_{12}\lambda_{21}$ (product of the rates of explosion) versus $\lambda_{11}\lambda_{22}$ (product of the rates of implosion). □

Remark 4: If $\hat{r}_0 = 0$, (14) and (15) are equivalent. If $\hat{r}_0 > 0$, the same is also true but possibly with a larger value of $\hat{r}_0$. □

C. Seesaw control systems design

Equipped with the mathematical results derived, this section proposes a new methodology for the design of stabilizing feedback control laws for nonlinear systems of the form

$$\dot{x} = f(x, u, \mathrm{w}), \tag{16}$$

where $x \in \mathcal{X} \subset \mathbb{R}^n$ is the state, $u \in \mathcal{U} \subset \mathbb{R}^m$ is the control input, and $\mathrm{w} \in M_{\mathcal{W}}$, $\mathcal{W} \subset \mathbb{R}^{n_w}$ is a disturbance signal.

The first step consist of finding two measuring functions $\omega_{su}(x), \omega_{us}(x)$ such that (16) is *measure-to-state practically stable* (MSpS) w.r.t. $\omega_{su}(x) \oplus \omega_{us}(x)$, that is, there exist a class $\mathcal{KL}$ function β, a class $\mathcal{K}$ function γ, and a nonnegative constant c such that

$$|x(t)| \le \beta(|x(t_0)|, t-t_0) \oplus \gamma(\|\omega_{su}(x) \oplus \omega_{us}(x)\|_{[t_0,t]} \oplus \|\mathrm{w}\|_{[t_0,t]}) \oplus c. \tag{17}$$

The choice of $\omega_{su}(x)$, $\omega_{us}(x)$ is strongly motivated by the physics of the problem at hand, as the examples in Section 3 reveal. In fact, given the original state x, $\omega_{su}(x)$ and $\omega_{us}(x)$ will in general be functions of disjoint, yet complementary partitions of x.

The next step involves the design of two feedback laws $\alpha_1(x)$, $\alpha_2(x)$, such that (16) together with the switching controller

$$u = \alpha_\sigma(x), \quad \sigma \in \{1, 2\}$$

is a switched seesaw system w.r.t. $\omega_{su} \oplus \omega_{us}$. It is now easy to show that if $\sigma(t)$ is chosen such that the conditions of Theorem 1 hold and if the MSpS condition applies, then the closed-loop system

$$\dot{x} = f(x, \alpha_\sigma(x), \mathrm{w}),$$

is ISpS w.r.t. $\omega(x) = |x|$.

III. STABILIZATION OF UNDERACTUATED VEHICLES

A. The Extended nonholonomic double integrator

In [1], the equations of the *nonholonomic integrator* system were introduced. The nonholonomic integrator displays all basic properties of nonholonomic systems and is often quoted in the literature as a benchmark for control system design. Under suitable state and control transformations, it captures the kinematics of a wheeled robot. However, to tackle the realistic case where both the kinematics and dynamics of a wheeled robot must be taken into account, the nonholonomic integrator model must be extended. In [11], it is shown that the dynamic equations of motion of a mobile robot of the unicycle type can be transformed into the system

$$\ddot{x}_1 = u_1, \quad \ddot{x}_2 = u_2, \quad \dot{x}_3 = x_1\dot{x}_2 - x_2\dot{x}_1, \tag{18}$$

where $x := (x_1, x_2, x_3, \dot{x}_1, \dot{x}_2)' \in \mathbb{R}^5$ is the state vector and $u := (u_1, u_2)' \in \mathbb{R}^2$ is a two-dimensional control vector. System (18) will be referred to as *the extended nonholonomic double integrator* (ENDI). The ENDI falls into the class of control affine nonlinear systems with drift and cannot be stabilizable via a time-invariant continuously differentiable feedback law (cf., e.g., [22]).

1) Seesaw control design: We now solve the problem of practical stabilization of the ENDI system (18) subject to input disturbances $v \in M_{\mathcal{V}}$, $\mathcal{V} := \{v \in \mathbb{R}^2 : \|v\|_\infty \le \bar{v}\}$ and measurement noise $n \in M_{\mathcal{N}}$, $\mathcal{N} := \{n \in \mathbb{R}^5 : \|n\| \le \epsilon\}$, where $\bar{v}$ and ϵ are finite but otherwise arbitrary. To this effect, the dynamics of (18) are first extended to

$$\ddot{x}_1 = u_1 + v_1, \quad \ddot{x}_2 = u_2 + v_2, \quad \dot{x}_3 = x_1\dot{x}_2 - x_2\dot{x}_1, \tag{19}$$

$$y = x + n \tag{20}$$

where $y \in \mathbb{R}^5$ is the vector of state measurements corrupted by noise n. Following the procedure described in Section II-C we first introduce the measuring functions

$$\omega_{su} := z^2, \quad z := \dot{x}_3 + \lambda_1 x_3,\ \lambda_1 > 0 \tag{21}$$

$$\omega_{us} := x_1^2 + \dot{x}_1^2 + x_2^2 + \dot{x}_2^2. \tag{22}$$

and the feedback laws

$$\alpha_1(x) := \begin{bmatrix} -k_2\dot{x}_1 \\ -k_2\dot{x}_2 - \frac{k_3}{x_1}z \end{bmatrix},\ \alpha_2(x) := \begin{bmatrix} -k_2\dot{x}_1 - k_1(x_1-\kappa) \\ -k_2\dot{x}_2 - k_1x_2 \end{bmatrix} \tag{23}$$

where $\kappa, k_1, k_2, k_3 > 0$. To provide some insight into the choice of the relevant functions note that ω_{su} and ω_{us} can be viewed as positive semi-definite Lyapunov functions of z and $(x_1, \dot{x}_1, x_2, \dot{x}_2)'$, respectively, the time-derivatives of which are given by

$$\dot{\omega}_{su} = 2z\big[x_1(u_2 + v_2 + \lambda_1\dot{x}_2) - x_2(u_1 + v_1 + \lambda_1\dot{x}_1)\big] \tag{24}$$

$$\dot{\omega}_{us} = 2\dot{x}_1(x_1 + u_1 + v_1) + 2\dot{x}_2(x_2 + u_2 + v_2) \tag{25}$$

In the absence of input disturbances and measurement noise, it is straightforward to conclude that with the control law $u = \alpha_1(x)$, the measuring function ω_{su} satisfies $\dot{\omega}_{su} = -2k_3\omega_{su}$ as long as $x_1 \neq 0$. This in turn implies that ω_{su} converges exponentially fast to zero during the intervals of time in which $u = \alpha_1(x)$ is applied. In a similar vein, consider the evolution of ω_{us} under the influence of the control law $u = \alpha_2(x)$. Simple computations show that

$$\ddot{x}_1 = -k_2\dot{x}_1 - k_1(x_1 - \kappa), \quad \ddot{x}_2 = -k_2\dot{x}_2 - k_1x_2$$

and therefore ω_{us} converges exponentially fast to κ^2 during the intervals of time in which $u = \alpha_2(x)$ is applied.

We now proceed with the seesaw control design as explained in Section II-C

Proposition 1: The ENDI system together with the measuring functions $\omega_{su}(x)$ and $\omega_{us}(x)$ is *measure-to-state stable*[4] (MSS) w.r.t. $\omega_{su}(x) \oplus \omega_{us}(x)$. □

Proposition 2: The ENDI system (19)–(20) in closed-loop with

$$u = \alpha_\sigma(y) \tag{26}$$

defined in (23) verifies conditions C1 and C2 of a switched seesaw system w.r.t. $\omega_{su} \oplus \omega_{us}$ on $\mathcal{X} \subset \{x \in \mathbb{R}^5 : |x_1| \ge \delta\}$ for some $\delta > 0$.

Proof: [Outline] We start by showing that C1 is observed when $\sigma = 1$. In the presence of input disturbances and measurement noise, the control input $u = \alpha_1(x+n)$ is given by $u_1 = -k_2(\dot{x}_1 + n_4)$, $u_2 = -k_2(\dot{x}_2 + n_5) - \frac{k_3}{x_1+n_1}(z + n_z)$, $n_z := x_1n_5 + n_1\dot{x}_2 + n_1n_5 - x_2n_4 - n_2\dot{x}_1 - n_2n_4 + k_2n_3$ which, from the fact that $\|n\|_\infty \le \epsilon$, satisfies $|n_z| \le 4\epsilon\sqrt{\omega_{us}} + 2\epsilon^2 + \lambda\epsilon$. For $|x_1| > \delta > \epsilon$, a bound for ω_{su} is determined as

$$\dot{\omega}_{su} \le -\lambda_{11}\omega_{su} + \frac{\lambda_{11}}{3}\hat{\gamma}_{11}^{\omega_{us}}\omega_{us} + \frac{\lambda_{11}}{3}c_{11}, \quad \theta_1, \theta_2 > 0$$

where $\lambda_{11} = 2\Big[\frac{k_3}{1+\frac{\epsilon}{\delta}} - \frac{k_3}{1-\frac{\epsilon}{\delta}}\frac{\theta_1}{2} - \epsilon k_2 - \theta_2\bar{v}\Big]$, $\frac{\lambda_{11}}{3}\hat{\gamma}_{11}^{\omega_{us}} = 2\frac{k_3}{1-\frac{\epsilon}{\delta}}\frac{16}{\theta_1}\epsilon^2 + \epsilon k_2 + \frac{1}{\theta_2}\bar{v}$, and $\frac{\lambda_{11}}{3}c_{11} = 2\frac{k_3}{1-\frac{\epsilon}{\delta}}\frac{1}{\theta_1}\epsilon^2(2\epsilon + \lambda)^2$. Therefore,

$$\omega_{su}(t) \le 3\omega_{su}(t_0)e^{-\lambda_{11}(t-t_0)} \oplus \hat{\gamma}_{11}^{\omega_{us}}\|\omega_{us}\|_{[t_0,t]} \oplus c_{11} \tag{27}$$

Notice the absence of the term γ_{11}^{w} due to the fact that the disturbances and noise are assumed to be bounded and their bounds are known in advance[5]. We now establish a bound for ω_{us}. Computing its time-derivative yields

$$\dot{\omega}_{us} \le \lambda_{12}\omega_{us} + \lambda_{12}\hat{\gamma}_{12}^{\omega_{su}}\omega_{su} + \lambda_{12}\gamma_{12}^{v}(|v|) + \lambda_{12}c_{12},$$

where $\lambda_{12} = 2 + \frac{k_3}{\delta-\epsilon} + 4\epsilon + \frac{\theta_3}{2}$, $\theta_3 > 0$, $\lambda_{12}\hat{\gamma}_{12}^{\omega_{su}} = \frac{k_3}{\delta-\epsilon}$, $\lambda_{12}\gamma_{12}^{v}(r) = 2r^2$, and $\lambda_{12}c_{12} = 4k_2^2\epsilon^2 + \frac{k_3\epsilon^2}{\delta-\epsilon}\frac{(2\epsilon+k_2)^2}{2\theta_3}$. Therefore, ω_{us} satisfies

$$\begin{aligned}\omega_{us}(t) \le 4\big(&\omega_{us}(t_0) \oplus \hat{\gamma}_{12}^{\omega_{su}}\|\omega_{su}\|_{[t_0,t]} \\ &\oplus \gamma_{12}^{v}(\|v\|_{[t_0,t]}) \oplus c_{12}\big)e^{\lambda_{12}(t-t_0)}.\end{aligned} \tag{28}$$

Similarly, we check that condition C2 is satisfied when $\sigma = 2$. In this case, the control input $u = \alpha_2(x+n)$ is given by $u_1 = -k_2(\dot{x}_1 + n_4) - k_1(x_1 + n_1 - \kappa)$, $u_2 = -k_2(\dot{x}_2 + n_5) - k_1(x_2 + n_2)$. Substituting the above equations into (24) yields

$$\dot{\omega}_{su} \le \lambda_{21}\omega_{su} + \lambda_{21}\hat{\gamma}_{21}^{\omega_{us}}\omega_{us}, \quad \theta_4, \theta_5 > 0$$

where $\lambda_{21} = \frac{2(k_2+k_1)+k_1\kappa}{\theta_4} + 2\frac{\bar{v}}{\theta_5}$ and $\lambda_{21}\hat{\gamma}_{21}^{\omega_{us}} = \theta_4[(k_2 + k_1)\epsilon + k_1\kappa] + \theta_5\bar{v}$. Therefore,

$$\omega_{su} \le 2\big(\omega_{su}(t_0) \oplus \hat{\gamma}_{21}^{\omega_{us}}\|\omega_{us}\|_{[t_0,t]}\big)e^{\lambda_{21}(t-t_0)}. \tag{29}$$

To compute a bound for $\omega_{us}(t)$, we first observe that $\omega_{us} = |\chi_1|^2 + |\chi_2|^2$, where $\chi_1 := (x_1, \dot{x}_1)'$, $\chi_2 := (x_2, \dot{x}_2)'$, and χ_1, χ_2 satisfy

$$\dot{\chi}_i = A\chi_i + Bd_i, \quad i = 1, 2$$

with $A := \begin{bmatrix} 0 & 1 \\ -k_1 & -k_2 \end{bmatrix}$, $B = [0, 1]'$, $d_1 = -k_2n_4 - k_1n_1 + \kappa + v_1$, and $d_2 = -k_2n_5 - k_1n_2 + v_2$. Let $\lambda > 0$ be an arbitrary constant such that $(A + \frac{\lambda}{2}I)$ is Hurwitz. Further, let $P > 0$ satisfy $\big(A + \frac{\lambda}{2}I\big)P + P\big(A + \frac{\lambda}{2}I\big)' + BB' \le 0$. Define $V_i := \chi_i'P^{-1}\chi_i$ and compute $\dot{V}_i$ to obtain

$$\begin{aligned}\dot{V}_i &= \chi_i'(P^{-1}A + A'P^{-1})\chi_i + 2\chi'P^{-1}Bd_i \\ &\le -(\lambda - \theta_6)V_i, \quad V_i \ge \frac{|d_i|^2}{\theta_6}, \quad \theta_6 \in (0, \lambda)\end{aligned}$$

From the above, it follows that $V_i(t) \le V_i(t_0)e^{-(\lambda-\theta_6)(t-t_0)} \oplus \frac{|d_i|^2}{\theta_6}$, and therefore

$$\omega_{us} \le \hat{\beta}_{22}\omega_{us}(t_0)e^{-\lambda_{22}(t-t_0)} \oplus \gamma_{22}^{v}(\|v\|_{[t_0,t]}) \oplus c_{22},$$

where $\hat{\beta}_{22} = 3\frac{\lambda_{max}(P)}{\lambda_{min}(P)}$, $\lambda_{22} = (\lambda - \theta_6)$, $\gamma_{22}^{v}(r) = 6r^2$, and $c_{22} = 6\big[((k_2 + k_1)\epsilon + k_1\kappa)^2 + (k_2 + k_1)^2\epsilon^2\big]$. ■

[4]A system is MSS w.r.t. $\omega_{su}(x) \oplus \omega_{us}(x)$ if it is MSpS w.r.t. $\omega_{su}(x) \oplus \omega_{us}(x)$ and c = 0 (see (17)).

[5]It is possible to avoid this at the cost of introducing the $\mathcal{K}$ function $\gamma_{11}^{v}(r)$ and making $\gamma_{12}^{\omega_{su}}(r)$ a quadratic function.

It is now easy to conclude that if the switched seesaw controller (26) is applied to the ENDI system and a suitable selection of the dwell times τ_1, τ_2 is made such that conditions (12)–(15) hold, then the resulting closed-loop system is ISpS as long as $|x_1(t)| \geq \delta$. It remains to state conditions under which $|x_1|$ is indeed bounded away from 0.

Proposition 3: Consider the closed-loop system (19)–(20), (23) and (26). Let $S_0 := \{x \in \mathbb{R}^5 : |(x_1(t_0) - \kappa, \dot{x}_1(t_0))| \leq \mu\}$, for some $\mu > 0$. Then, under a suitable choice of the controller gains, for every initial condition $x(t_0) \in S_0$, the resulting solution $x(\cdot)$ lies in $\mathcal{X} \subset \{x \in \mathbb{R}^5 : |x_1| \geq \delta; \delta > 0\}$. □

From Propositions 1–3 and Theorem 1 we finally conclude

Theorem 2: Consider the ENDI system (19)–(20) subject to input disturbances and measurement noise, together with the switching control law (23), (26). Assume the conditions of Theorem 1 hold and let the initial conditions of the closed-loop system be in S_0, defined in Proposition 3. Then, the switching controller stabilizes the state around a neighborhood of the origin, that is, it achieves ISpS of the closed-loop system on $\mathcal{X}$ w.r.t. $\omega(x) = |x|$. □

Remark 5: It is always possible to make sure that x starts in S_0 because $(x_1, \dot{x}_1)$ can initially be brought as close as required to $(\kappa, 0)$ by applying $u = \alpha_2(x)$ during a finite amount of time before the normal switching takes over. In fact, from (23) it is clear that with $u = \alpha_2(x)$, $(x_1, \dot{x}_1)$ converges to $(\kappa, 0)$. □

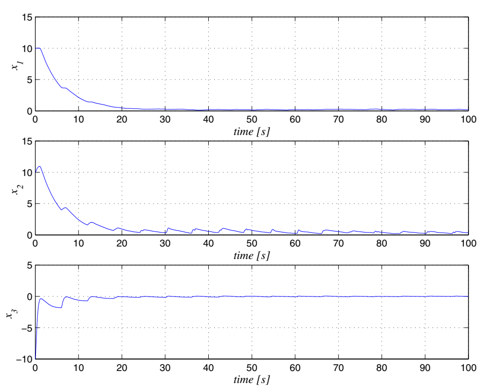

Fig. 1. Time evolution of state variables $x_1(t)$, $x_2(t)$, and $x_3(t)$.

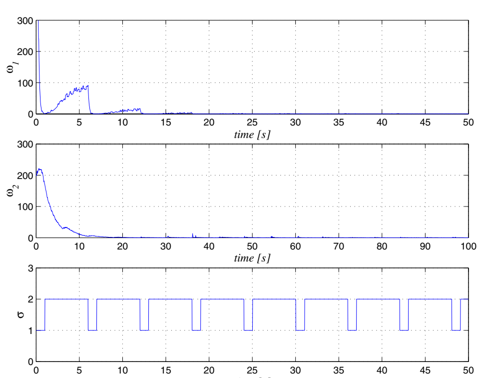

Fig. 2. Time evolution of measuring functions $\omega_{su}(t)$, $\omega_{us}(t)$, and the switching signal $\sigma(t)$.

2) Simulation results: Figures 1–2 show simulations results. The measurement noise is a zero mean uniform random noise with amplitude 0.1, and the input disturbances are $v_1 = 0.1\sin(t)$ and $v_2 = 0.1\sin(t + \pi/2)$.With the dwell-time constants set to $\tau_1 = 1.0\,s$ and $\tau_2 = 5.0\,s$, the assumptions of Theorem 2 were verified to hold. Notice how the state variables converge to a small neighborhood of the origin. Fig. 2 shows clearly, during the first switching intervals, how the behavior of ω_{su} and ω_{us} capture the successive "stable/unstable" and "unstable/stable" cycles, respectively.

B. The underactuated autonomous underwater vehicle

Based on the results derived, this section addresses the problem of stabilizing an underactuated AUV in the horizontal plane to a point, with a desired orientation. The AUV has no side thruster, and its control inputs are the thruster surge force τ_u and the thruster yaw torque τ_r. The AUV model is a second-order nonholonomic system, falls into the class of control affine nonlinear systems with drift, and there is no time-invariant continuously differentiable feedback law that asymptotically stabilizes the closed-loop system to an equilibrium point [22].

1) Vehicle Modeling: In the horizontal plane, the kinematic equations of motion of the vehicle can be written as

$$\dot{\mathbf{x}} = \mathbf{u}\cos\psi - \mathbf{v}\sin\psi,$$
$$\dot{\mathbf{y}} = \mathbf{u}\sin\psi + \mathbf{v}\cos\psi,$$
$$\dot{\psi} = \mathbf{r},$$

where, following standard notation, $\mathbf{u}$ (surge speed) and $\mathbf{v}$ (sway speed) are the body fixed frame components of the vehicle's velocity, $\mathbf{x}$ and $\mathbf{y}$ are the cartesian coordinates of its center of mass, ψ defines its orientation, and $\mathbf{r}$ is the vehicle's angular speed. Neglecting the motions in heave, roll, and pitch the simplified dynamic equations of motion for surge, sway and heading yield [22]

$$m_u\dot{\mathbf{u}} - m_v\mathbf{v}\mathbf{r} + d_u\mathbf{u} = \tau_u, \tag{30}$$
$$m_v\dot{\mathbf{v}} + m_u\mathbf{u}\mathbf{r} + d_v\mathbf{v} = 0, \tag{31}$$
$$m_r\dot{\mathbf{r}} - m_{uv}\mathbf{u}\mathbf{v} + d_r\mathbf{r} = \tau_r, \tag{32}$$

where the positive constants $m_u = m - X_{\dot{u}}$, $m_v = m - Y_{\dot{v}}$, $m_r = I_z - N_{\dot{r}}$, and $m_{uv} = m_u - m_v$ capture the effect of mass and hydrodynamic added mass terms, and $d_u = -X_u - X_{|u|u}|u|$, $d_v = -Y_v - Y_{|v|v}|v|$, and $d_r = -N_r - N_{|r|r}|r|$ capture hydrodynamic damping effects.

2) Coordinate Transformation: Consider the global diffeomorphism given by the state and control coordinate transformation [22]

$$x_1 = \psi$$
$$x_2 = \mathbf{x}\cos\psi + \mathbf{y}\sin\psi$$
$$x_3 = -2(\mathbf{x}\sin\psi - \mathbf{y}\cos\psi) + \psi(\mathbf{x}\cos\psi + \mathbf{y}\sin\psi)$$
$$u_1 = \frac{1}{m_r}\tau_r + \frac{m_{uv}}{m_r}\mathbf{u}\mathbf{v} - \frac{d_r}{m_r}\mathbf{r}$$
$$u_2 = \frac{m_v}{m_u}\mathbf{v}\mathbf{r} - \frac{d_u}{m_u}\mathbf{u} + \frac{1}{m_u}\tau_u - u_1\frac{x_1x_2 - x_3}{2} + \mathbf{v}\mathbf{r} - \mathbf{r}^2 z_2$$

that yields

$$\ddot{x}_1 = u_1, \quad \ddot{x}_2 = u_2, \quad \dot{x}_3 = x_1\dot{x}_2 - x_2\dot{x}_1 + 2\mathbf{v}, \tag{33}$$

$$m_v\dot{\mathbf{v}} + m_u\left(\dot{x}_2 + \dot{x}_1\frac{x_1x_2 - x_3}{2}\right)\dot{x}_1 + d_v\mathbf{v} = 0. \tag{34}$$

Throughout the paper, $q := \mathrm{col}(x, \mathbf{v})$, $x := (x_1, x_2, x_3, \dot{x}_1, \dot{x}_2)'$ and $u = (u_1, u_2)'$ denote the state vector and the input vector of (33)–(34), respectively.

3) Seesaw control design: We now design a switching feedback control law for system (33)–(34) so as to stabilize (in an ISpS sense) the state q around a small neighborhood of the origin. A comparison of (33)–(34) with the ENDI system (18) shows the presence of an extra state variable $\mathbf{v}$ that is not in the span of the input vector field but enters as an input perturbation in the x_3 dynamics. We also note that since $\frac{d_v}{m_v} > 0$, (34) is ISS when x is regarded as input. Motivated by these observations, we select for measuring functions $\omega_{su}(\cdot)$, $\omega_{us}(\cdot)$ the ones given in (21)–(22). Using Proposition 1 and the fact that $\mathbf{v}$ satisfies

$$|\mathbf{v}(t)| \leq \hat{\beta}_v |\mathbf{v}(t_0)| e^{-\lambda_v (t-t_0)} \oplus \gamma_v(\|\omega_{su}\|_{[t_0,t]} \oplus \|\omega_{us}\|_{[t_0,t]})$$

for some $\hat{\beta}_v, \lambda_v > 0$, and $\gamma_v(r) \in \mathcal{K}$ we conclude that system (33)–(34) is MSS w.r.t. $\omega_{su} \oplus \omega_{us}$.

Before we define the feedback laws $\alpha_1(\cdot)$, $\alpha_2(\cdot)$ we compute the time-derivatives of ω_{su} and ω_{us} to obtain

$$\dot{\omega}_{su} = 2z\left[x_1(u_2 + v_2 + k_2\dot{x}_2) - x_2(u_1 + v_1 + k_2\dot{x}_1 + 2\nu)\right]$$
$$\dot{\omega}_{us} = 2\dot{x}_1(x_1 + u_1 + v_1) + 2\dot{x}_2(x_2 + u_2 + v_2),$$

where $\nu := \dot{\mathbf{v}} + k_2\mathbf{v}$ satisfies the linear bound $|\nu| \leq \hat{\gamma}_v|\mathbf{v}| + \hat{\gamma}_{\omega_{us}}|\omega_{us}| + \hat{\gamma}_z|z|$ provided that $\|(x_1 - \kappa, \dot{x}_1)\|_\infty \leq \mu$, for a given $\mu > 0$. Comparing $\dot{\omega}_{su}$, $\dot{\omega}_{us}$ with (24)–(25) and using the bound on $|\nu|$ together with the previous results for the ENDI case, it is straightforward to conclude that if $\alpha_1(\cdot)$, $\alpha_2(\cdot)$ are selected as in (23), then the following result holds:

Theorem 3: For every initial condition in $S_0 := \{q \in \mathbb{R}^6 : |(x_1, \dot{x}_1)| \leq \mu_0\}$, for some $\mu_0 > 0$, and selecting σ such that the assumptions of Theorem 1 hold, system (33)–(34) subject to input disturbances and measurement noise in closed-loop with the seesaw controller $u = \alpha(q + n)$ is ISpS on $\mathcal{X}$ w.r.t. $\omega(q) = |q|$. □

4) Simulation results: Simulations were done using a dynamic model of the *Sirene* AUV [22]. Figure 3 shows the simulation results for a sample initial condition given by $(\mathbf{x}, \mathbf{y}, \psi, \mathbf{u}, \mathbf{v}, \mathbf{r}) = (-4\,m, -4\,m, \pi/4, 0, 0, 0)$. The amplitudes of the noise signals were set to $(0.5\,m, 0.5\,m, 5\pi/180, 0.1, 0.1, 0.1)$. There is also a small input disturbance: $v_1 = 10\sin(t)$, $v_2 = 10\sin(t + \pi/2)$. The dwell-time constants were set to $\tau_1 = 15\,s$ and $\tau_2 = 20\,s$.

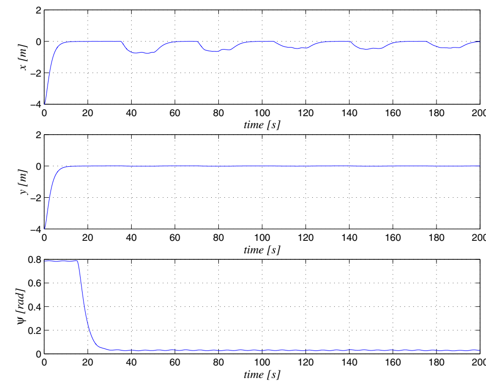

Fig. 3. Time evolution of the position $\mathbf{x}$, $\mathbf{y}$ and orientation ψ.

IV. Conclusions

A new class of switched systems was introduced and mathematical tools were developed to analyze their stability and disturbance/noise attenuation properties. A so-called seesaw control design methodology was also proposed. To illustrate the potential of this control design methodology, applications were made to the stabilization of the ENDI and to the dynamic model of an underactuated AUV in the presence of input disturbances and measurement noise. Future research will address the generalization of the seesaw control design methodology to tackle a wider range of control problems.

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