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Tools for analysis of Dirac Structures on Banach Spaces

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***Abstract*— Power-conserving and Dirac structures are known as an approach to mathematical modeling of physical engineering systems. In this paper connections between Dirac structures and well known tools from standard functional analysis are presented. The analysis can be seen as a possible starting framework towards the study of compositional properties of Dirac structures.**

I. Introduction

Prevailing trend in the modeling of physical systems for simulation is *port-based modeling*. The system is split into sub-systems that are interacting with each other via ports of variables called *flow* and *effort*. This way of modeling has several advantages. It represents a unified way to model physical systems from different physical domains such as mechanical, electrical, hydraulic, thermal, and so on. The knowledge about models of sub-systems (sub-models) can be stored in libraries and it is reusable for later occasions. The modeling process can be performed in an iterative manner, gradually refining model by adding the other sub-models.

An interconnection structure is a linear power-conserving part of a port based model. By analyzing an interconnection structure we can get information about the correctness of the considered model or about its dynamical behavior [5]. Also an appropriate representation of interconnection structure leads to an efficient code for numerical simulation (see for example [5]).

An interconnection structure can be considered from a geometric point of view. Namely, a subspace of admissible flows and efforts imposed by an interconnection structure represents a *Dirac structure*. Therefore, *the properties of an interconnection structure can be looked through the properties of the corresponding Dirac structures*. This approach has been initiated for electrical circuit in [13] and for rigid mechanisms in [14]. Dirac structures have been originally introduced by Courant [1] and Dorfman [2]. In [1], a generalization of Poisson and (pre)-symplectic structures has been considered. Dorfman [2] developed an algebraic theory of Dirac structures in the context of the study of completely integrable systems of partial differential equations. Dirac structures have been mainly investigated on finite dimensional vector spaces [13], [15] with few exceptions. For example in [16] the authors considered Dirac structures on vector spaces of differentiable forms and in [8], [7] Dirac structures on Hilbert spaces have been analyzed.

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The aim of this paper is to introduce and analyze Dirac structures on *reflexive Banach spaces*. Some basic properties are obtained and the connection between standard tools from functional analysis and Dirac structures is emphasized. Based on well-known mathematical tools, we provide the kernel and image representation of Dirac structures. Moreover, necessary and sufficient conditions for a subspace of a reflexive Banach space to be a Dirac structure are provided. Under some hypothesis a Dirac structure can be decomposed in the orthogonal sum of three fundamental Dirac structures: completely multivalued, completely kernel and completely skew-adjoint. The results presented in this paper can be viewed as a generalization of the results presented, for finite-dimensional spaces, in [13], [15], [17] and, for Hilbert spaces, in [6], [7].

The theory developed in this paper shows the very strong relation between the power-conserving and Dirac structures and standard tools from functional analysis. A simple academic example of ideal transmission line illustrates the analysis.

II. Dirac structures on real vector spaces

Let $\mathcal{F}$ and $\mathcal{E}$ be real vector spaces whose elements are labeled as f and e, respectively. We call $\mathcal{F}$ the space of *flows* and $\mathcal{E}$ the space of *efforts*. The space $\mathcal{B} = \mathcal{F} \times \mathcal{E}$ is called the *bond space* and an element of the space $\mathcal{B}$ is denoted by $b = (f, e)$. The spaces $\mathcal{F}$ and $\mathcal{E}$ are power conjugate. This means that there exists a map

$$\langle \cdot \mid \cdot \rangle : \mathcal{E} \times \mathcal{F} \to \mathbb{R}$$

called the *power product* which is linear in each coordinate and it is not degenerate.

Using the power product, we define a symmetric *bilinear form*

$$\ll \cdot, \cdot \gg : \mathcal{B} \times \mathcal{B} \to \mathbb{R}$$

by

$$\ll (f^1, e^1), (f^2, e^2) \gg = \langle e^1 \mid f^2 \rangle + \langle e^2 \mid f^1 \rangle,$$

for all $(f^1, e^1), (f^2, e^2) \in \mathcal{B}$. We have the following immediate relation between the power product and and the bilinear form

$$\langle e \mid f \rangle = \frac{1}{2} \ll b, b \gg,$$

for all $b = (f, e) \in \mathcal{B}$.
We recall the notion of a *Tellegen structure* (known also as *power-conserving structure*).

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Definition 2.1 (Tellegen structure): Let $\mathcal{Z}$ be a subspace of the vector space $\mathcal{B}$. We say that $\mathcal{Z}$ is a *Tellegen structure* on $\mathcal{B}$ if

$$\langle e \mid f\rangle = 0, \text{ for all } (f,e) \in \mathcal{Z}.$$

We denote by $\mathcal{Z}^{\perp}$ the orthogonal complement of $\mathcal{Z}$ with respect to the bilinear form $\ll \cdot,\cdot \gg$, namely

$$\mathcal{Z}^{\perp} := \{b \in \mathcal{B} \mid \ll b,\tilde{b} \gg = 0, \text{ for all } \tilde{b} \in \mathcal{Z}\}.$$

Remark 2.1: Let $\mathcal{Z}$ be a subspace of the vector space $\mathcal{B}$. Then $\mathcal{Z}$ is a Tellegan structure on $\mathcal{B}$ if and only if $\mathcal{Z} \subseteq \mathcal{Z}^{\perp}$. We focus on a special class of Tellegan structures namely *Dirac structures*.

Definition 2.2 (Dirac structure): Let $\mathcal{D}$ be a subset of $\mathcal{B}$. We say that $\mathcal{D}$ is a Dirac structure on $\mathcal{B}$ if

$$\mathcal{D} = \mathcal{D}^{\perp}.$$

For finite-dimensional spaces a Dirac structure is a Tellegan structure of maximal dimension. In [6], [7], Dirac structures on Hilbert spaces have been defined. For infinite-dimensional Hilbert spaces one can also approach the analysis of Dirac structures using Krein spaces which are not Pontryagin spaces. In this paper we are especially interested in the case when $\mathcal{B}$ is a reflexive Banach space.

III. Dirac structures on reflexive Banach spaces

An important tool for the analysis of Dirac structures and their properties on Hilbert spaces is the existence of the inner product. Some of the results obtained for Hilbert spaces can be carried on in the context of Banach spaces using the natural definition of the duality product.

A. Definition of Dirac structures on reflexive Banach spaces

Let $\mathcal{F}$ be a (real) Banach space and $\mathcal{E} = \mathcal{F}^*$, where $\mathcal{F}^*$ is the adjoint space of $\mathcal{F}$ (the set of all bounded semi-linear forms on $\mathcal{F}$). Then $\mathcal{E}$ is a Banach space with the norm $\|e\|$ defined by

$$\|e\| = \sup_{0\neq f\in\mathcal{F}} \frac{|e(f)|}{\|f\|}.$$

The adjoint space $\mathcal{F}^{**}$ is again a Banach space. Each $f \in \mathcal{F}$ may be regarded as an element of $\mathcal{F}^{**}$. However, this does not imply that $\mathcal{F}^{**}$ can be identified with $\mathcal{F}$ as in the case of Hilbert spaces. The Banach space $\mathcal{F}$ is said to be reflexive if $\mathcal{F}$ can be identified with $\mathcal{F}^{**}$.

Assumption 3.1: The Banach space $\mathcal{F}$ is reflexive.

We introduce the scalar product $\langle \cdot \mid \cdot\rangle : \mathcal{E}\times\mathcal{F} \to \mathbb{R}$ defined by $\langle e \mid f\rangle := e(f)$ for all $e \in \mathcal{E}$ and $f \in \mathcal{F}$. Each $f \in \mathcal{F}$ may be regarded as an element of $\mathcal{F}^{**}$. Consider also the scalar product $\langle \cdot \mid \cdot\rangle : \mathcal{F}\times\mathcal{E} \to \mathbb{R}$ which satisfies

$$\langle f \mid e\rangle_{\mathcal{F}\times\mathcal{E}} = \langle e \mid f\rangle_{\mathcal{E}\times\mathcal{F}}.$$

for all $e \in \mathcal{E}$ and $f \in \mathcal{F}$.

Remark 3.1: The scalar product on $\mathcal{E}\times\mathcal{F}$ is a power product. Indeed, since we work with real Banach spaces, the scalar product is linear in both components. From the definition of the norm on $\mathcal{E}$, we see that if $\langle e \mid f\rangle = 0$ for all $f \in \mathcal{F}$ then $\|e\| = 0$ so $e = 0$. The following equality holds in Banach spaces (see Kato [10], page 135)

$$\|f\| = \sup_{0\neq e\in\mathcal{E}} \frac{\langle e \mid f\rangle}{\|e\|}.$$

From this relation, if $\langle e \mid f\rangle = 0$ for all $e \in \mathcal{E}$ we have that $\|f\| = 0$ so $f = 0$. We have proved that the scalar product is non-degenerate.

Furthermore, the bond space $\mathcal{B} = \mathcal{F}\times\mathcal{E}$ is also a Banach space with the linear structure defined componentwise and the norm defined by

$$\|(f,e)\| = (\|f\|^2 + \|e\|^2)^{\frac{1}{2}}.$$

Other choices of the norm are also possible. We use this norm mainly because it ensures that (see Kato [10], page 164)

$$(\mathcal{F}\times\mathcal{E})^* = \mathcal{F}^*\times\mathcal{E}^* = \mathcal{E}\times\mathcal{F}.$$

The bond space $\mathcal{B}$ is reflexive. Indeed,

$$\mathcal{B}^{**} = (\mathcal{F}\times\mathcal{E})^{**} = (\mathcal{E}\times\mathcal{F})^* = \mathcal{E}^*\times\mathcal{F}^* = \mathcal{B}.$$

In the sequel we shall show that there is a very tight connection between the standard tools from the functional analysis and the above defined Tellegan and Dirac structures.

Proposition 3.1: Let $\mathcal{Z}$ be a Tellegan structure on $\mathcal{B}$. Then $\mathrm{cl}(\mathcal{Z})$ (the closure of $\mathcal{Z}$) is also a Tellegan structure on $\mathcal{B}$.

Proof: Take a sequence $(b_n)_{n\in\mathbb{N}} = (f_n, e_n)_{n\in\mathbb{N}}$ of elements in $\mathcal{Z}$ that converges to $b = (f,e)$. Using the inequality

$$|\langle e \mid f\rangle| \leq \|e\|\|f\|,$$

we have that

$$\begin{aligned} |e_n(f_n) - e(f)| &\leq |e_n(f_n - f)| + |(e_n - e)f| \\ &\leq \|e_n\|\|f_n - f\| + \|e_n - e\|\|f\|. \end{aligned}$$

Since b_n converges to b we have that $\|f_n - f\|$ and $\|e_n - e\|$ converge to zero and $\|e_n\|$ is bounded. The fact that $b_n \in \mathcal{Z}$ gives that $e_n(f_n) = 0$, and taking the limit in the above inequality we obtain that $e(f) = 0$, which means that $b = (f,e) \in \mathcal{Z}$. Therefore we conclude that $\mathrm{cl}(\mathcal{Z})$ is also a Tellegan structure on the bond space $\mathcal{B}$. ■

We consider the scalar product $\langle \cdot,\cdot\rangle_{\mathcal{B}^*\times\mathcal{B}} : \mathcal{B}^*\times\mathcal{B} \to \mathbb{R}$ defined by

$$\langle b,\tilde{b}\rangle_{\mathcal{B}^*\times\mathcal{B}} := \langle e \mid \tilde{f}\rangle + \langle \tilde{e} \mid f\rangle$$

where $b = (e,f) \in \mathcal{B}^* = \mathcal{E}\times\mathcal{F}$ and $\tilde{b} = (\tilde{f},\tilde{e}) \in \mathcal{B} = \mathcal{F}\times\mathcal{E}$. This is the scalar product which corresponds to the norm defined on $\mathcal{B}^*$ (see again Kato [10], page 164). For any subset $\mathcal{Z}$ of $\mathcal{B}$ we denote $\mathcal{Z}^c$ the orthogonal complement with respect to the scalar product $\langle \cdot,\cdot\rangle_{\mathcal{B}^*\times\mathcal{B}}$, i.e.

$$\mathcal{Z}^c := \{b \in \mathcal{B}^* \mid \langle b,\tilde{b}\rangle_{\mathcal{B}^*\times\mathcal{B}} = 0, \ \forall\tilde{b} \in \mathcal{Z}\}.$$

Remark 3.2: By definition we have that $\mathcal{Z}^c$ is a subset of $\mathcal{B}^* = \mathcal{E}\times\mathcal{F}$ and $\mathcal{Z}^{\perp}$ (the orthogonal of $\mathcal{Z}$ with respect to the bilinear form on $\mathcal{B}$ as defined in the previous section)

$$\mathcal{Z}^{\perp} := \{b \in \mathcal{B} \mid \ll b,\tilde{b} \gg = 0, \ \forall\tilde{b} \in \mathcal{Z}\}$$

is a subset of $\mathcal{B}$.
We consider R the natural embedding of $\mathcal{B}^*$ into $\mathcal{B}$. Then R is an isometric isomorphism between $\mathcal{B}^*$ and $\mathcal{B}$. More precisely R is defined by

$$R = \begin{bmatrix} 0 & r_{\mathcal{F}^{**}\mathcal{F}} \\ id_{\mathcal{E}} & 0 \end{bmatrix}$$

where $id_{\mathcal{E}}$ is the identity on $\mathcal{E}$ and $r_{\mathcal{F}^{**}\mathcal{F}}$ is the inverse of $r_{\mathcal{F}\mathcal{F}^{**}}$, the natural isometric isomorphism between $\mathcal{F}$ and $\mathcal{F}^{**}$ (see for example Dunford and Schwartz [3], vol I, page 66). The inverse of R is the isometric isomorphism $S : \mathcal{B} \rightarrow \mathcal{B}^*$ defined by

$$S = \begin{bmatrix} 0 & id_{\mathcal{E}} \\ r_{\mathcal{F}\mathcal{F}^{**}} & 0 \end{bmatrix}.$$

Remark 3.3: The bilinear form $\ll \cdot, \cdot \gg$ on $\mathcal{B}$ is related to the scalar product $\langle \cdot, \cdot \rangle_{\mathcal{B}^* \times \mathcal{B}}$ by

$$\ll b^1, b^2 \gg = \langle Sb^1, b^2 \rangle_{\mathcal{B}^* \times \mathcal{B}}.$$

for all $b^1 = (f^1, e^1)$ and $b^2 = (f^2, e^2)$ in $\mathcal{B}$. Indeed, we have (by the definition of the bilinear form) that

$$\ll b^1, b^2 \gg = \langle e^1 \mid f^2 \rangle + \langle e^2 \mid f^1 \rangle,$$

and

$$\begin{aligned} \langle Sb^1, b^2 \rangle_{\mathcal{B}^* \times \mathcal{B}} &= \langle (e^1, r_{\mathcal{F}\mathcal{F}^{**}} f^1), (f^2, e^2) \rangle_{\mathcal{B}^* \times \mathcal{B}} \\ &= \langle e^1 \mid f^2 \rangle + \langle e^2 \mid r_{\mathcal{F}\mathcal{F}^{**}} f^1 \rangle \\ &= \langle e^1 \mid f^2 \rangle + \langle e^2 \mid f^1 \rangle. \end{aligned}$$

Remark 3.4: The bilinear form $\ll \cdot, \cdot \gg$ on $\mathcal{B}$ is related to the scalar product $\langle \cdot, \cdot \rangle_{\mathcal{B}^* \times \mathcal{B}}$ by

$$\ll Rb^1, b^2 \gg = \langle b^1, b^2 \rangle_{\mathcal{B}^* \times \mathcal{B}}.$$

for all $b^1 = (e^1, f^1)$ in $\mathcal{B}^*$ and $b^2 = (f^2, e^2)$ in $\mathcal{B}$.
Using the above remarks a relation between the two orthogonal complements defined before can be very easily established.

Proposition 3.2: Let $\mathcal{Z}$ be a subspace of the bond space $\mathcal{B}$. Then the following equalities hold:

$$\mathcal{Z}^{\perp} = R\mathcal{Z}^c, \;\; S\mathcal{Z}^{\perp} = \mathcal{Z}^c.$$

Proof: This is a direct consequence of the relation between the bilinear form on $\mathcal{B}$ and the scalar product $\langle \cdot, \cdot \rangle_{\mathcal{B}^* \times \mathcal{B}}$. Let b, $\tilde{b}$ be elements of $\mathcal{Z}^{\perp}$, $\mathcal{Z}$, respectively. We have that

$$\ll b, \tilde{b} \gg = 0 \Rightarrow \langle Sb, \tilde{b} \rangle_{\mathcal{B}^* \times \mathcal{B}} \Rightarrow S\mathcal{Z}^{\perp} \subseteq \mathcal{Z}^c \Rightarrow \mathcal{Z}^{\perp} \subseteq R\mathcal{Z}^c,$$

and, for $\bar{b}$ in $\mathcal{Z}^c$, we obtain that

$$\langle \bar{b}, \tilde{b} \rangle_{\mathcal{B}^* \times \mathcal{B}} = 0 \Rightarrow \ll R\bar{b}, \tilde{b} \gg = 0 \Rightarrow R\mathcal{Z}^c \subseteq \mathcal{Z}^{\perp}.$$

Since $SR = I$ we have also that $S\mathcal{Z}^{\perp} = \mathcal{Z}^c$. ■

Remark 3.5: Let $\mathcal{Z}$ be a subspace of $\mathcal{B}$. The definition of R and the fact that $\mathcal{Z}^c$ is a closed linear subspace of $\mathcal{B}^*$ imply that $\mathcal{Z}^{\perp}$ is also a closed linear subspace of $\mathcal{B}$. Therefore, a Dirac structure will also be a closed linear subspace.
Summarizing the connections with standard functional analysis, we may state the following proposition.

Proposition 3.3: Let $\mathcal{D}$ be a vectorial subspace of $\mathcal{B}$. The following statements are equivalent:

1) $\mathcal{D}$ is a Dirac structure on $\mathcal{B}$.
2) $\mathcal{D} = R\mathcal{D}^c$.
3) $\mathcal{D}^c = S\mathcal{D}$.

Example 3.1: Let A be a skew-adjoint (unbounded in general) operator from $domA \subseteq \mathcal{F}$ to $\mathcal{E}$, that is

$$\langle Ax \mid y \rangle + \langle x \mid Ay \rangle = 0,$$

for all x, $y \in domA = domA^*$. Then the graph of A,

$$\mathcal{G}(A) = \{(x, Ax) \,:\, x \in domA\}$$

is a Dirac structure. Indeed, the definition of a skew-adjoint operator leads to

$$(\mathcal{G}(A))^{\perp} = \mathcal{G}(-A)^* = \mathcal{G}(A),$$

so that the conclusion follows.

Remark 3.6: Similarly as for $\mathcal{B}$ one may define a bilinear form on $\mathcal{B}^*$, and the orthogonal complement of a set with respect to this bilinear form. Further, Tellegan structure and Dirac structure on $\mathcal{B}^*$ may be defined. One can prove that (a closed set) $\mathcal{D}$ is a Dirac structure on $\mathcal{B}$ if and only if $\mathcal{D}^c$ is a Dirac structure on $\mathcal{B}^*$.
Before we give necessary and sufficient conditions for a Tellegan structure to be a Dirac structure, we need a technical result.

Proposition 3.4: Let $\mathcal{Z}$ be a subset of $\mathcal{B}$. The following equality holds:

$$(R\mathcal{Z}^c)^c = S\mathcal{Z}^{cc}. \tag{1}$$

Proof: First we prove that $S\mathcal{Z}^{cc} \subseteq (R\mathcal{Z}^c)^c$. Since $\mathcal{B}$ is a reflexive Banach space, we may identify $\mathcal{Z}^{cc}$ with a subset of $\mathcal{B}$ which will be denoted in the same way. Let us consider $b = (f, e) \in \mathcal{Z}^{cc}$. Then $Sb = (e, r_{\mathcal{F}\mathcal{F}^{**}}) \in S\mathcal{Z}^{cc}$. We compute the scalar product of $Sb \in \mathcal{B}^*$ with $Rz \in \mathcal{B}$, where $z = (z_e, z_f)$ is an arbitrary element in $\mathcal{Z}^c$. This is

$$\begin{aligned} \langle Sb, Rz \rangle_{\mathcal{B}^* \times \mathcal{B}} &= \langle (e, r_{\mathcal{F}\mathcal{F}^{**}} f), (r_{\mathcal{F}^{**}\mathcal{F}} z_f, z_e) \rangle_{\mathcal{B}^* \times \mathcal{B}} \\ &= \langle e \mid r_{\mathcal{F}^{**}\mathcal{F}} z_f \rangle + \langle z_e \mid r_{\mathcal{F}\mathcal{F}^{**}} f \rangle \\ &= \langle e \mid z_f \rangle + \langle z_e \mid f \rangle \\ &= \langle z, b \rangle_{\mathcal{B}^* \times \mathcal{B}}. \end{aligned}$$

Since $b \in \mathcal{Z}^{cc}$, we have

$$\langle z, b \rangle_{\mathcal{B}^* \times \mathcal{B}} = 0, \; \forall z \in \mathcal{Z}^c,$$

or equivalently

$$\langle Sb, Rz \rangle_{\mathcal{B}^* \times \mathcal{B}} = 0, \; \forall Rz \in R\mathcal{Z}^c,$$

or $Sb \in (R\mathcal{Z}^c)^c$. This means that the inclusion $S\mathcal{Z}^{cc} \subseteq (R\mathcal{Z}^c)^c$ holds. The other inclusion can be proved in a similar manner. ■

The proof of the following proposition is straightforward, well known in functional analysis, and gives necessary and sufficient conditions for a Tellegan structure to be a Dirac structure.

Theorem 3.1: Let $\mathcal{Z}$ be a closed subspace of the bond space $\mathcal{B}$. Then $\mathcal{Z}$ is a Dirac structure on $\mathcal{B}$ if and only if $\mathcal{Z}$ and $\mathcal{Z}^{\perp}$ are Tellegan structures on $\mathcal{B}$.

Proof: Suppose that $\mathcal{Z}$ is a Dirac structure on $\mathcal{B}$. Then $\mathcal{Z}$ is a Tellegan structure. Since $\mathcal{Z} = \mathcal{Z}^{\perp}$ it follows that $\mathcal{Z}^{\perp}$ is also a Tellegan structure.
Conversely, suppose that $\mathcal{Z}$ and $\mathcal{Z}^{\perp}$ are Tellegan structures. From Proposition 2.1 we have

$$\mathcal{Z} \subseteq \mathcal{Z}^{\perp} \subseteq \mathcal{Z}^{\perp\perp}. \quad (2)$$

From Proposition 3.2 we know that $\mathcal{Z}^{\perp} = R\mathcal{Z}^{c}$. Consequently,

$$\mathcal{Z}^{\perp\perp} = R(R\mathcal{Z}^{c})^{c}. \quad (3)$$

Using Proposition 3.4 the relation $(R\mathcal{Z}^{c})^{c} = S\mathcal{Z}^{cc}$ is obtained. The equality (3) becomes

$$\mathcal{Z}^{\perp\perp} = RS\mathcal{Z}^{cc} = \mathcal{Z}^{cc}.$$

Since $\mathcal{Z}$ is closed in $\mathcal{B}$ and $\mathcal{B}$ is a reflexive Banach space, then $\mathcal{Z}$ is invariant to the operation of double dual (see Kato [10], page 136), i.e. $\mathcal{Z}^{cc} = \mathcal{Z}$. We conclude that $\mathcal{Z}^{\perp\perp} = \mathcal{Z}$. From the sequence of inclusions (2) we finally obtain $\mathcal{Z} = \mathcal{Z}^{\perp}$, which means that $\mathcal{Z}$ is a Dirac structure on $\mathcal{B}$. ∎

B. Representations of Dirac structures

Many of the linear differential operators encountered in applications are closed or have a closed linear extension. Many of the important theorems which hold for continuous linear operators on Banach spaces hold also for closed linear operators. Let T be an operator acting between two Banach spaces $\mathcal{B}$ and $\mathcal{L}$. A sequence $(u_n) \subset D(T)$ is said to be *T-convergent* to $u \in \mathcal{B}$ if both (u_n) and (Tu_n) are convergent sequences and $u_n \to u$. The operator T is said to be *closed* if (u_n) is T-convergent to u implies that $u \in D(T)$ and $Tu = \lim Tu_n$. If T is a closed operator, the null space $\ker(T)$ is a closed linear subspace of $\mathcal{B}$ (see Kato [10], page 165).

Any densely defined closed operator T has a unique maximal adjoint T^* from $\mathcal{L}^*$ to $\mathcal{B}^*$ (see Goldberg [4], page 50) which is also a closed linear operator (see Goldberg [4], page 53).

In many cases we have to check if $\mathcal{D}$, a subspace of $\mathcal{B}$, expressed as the null space of a densely defined closed operator is a Dirac structure on $\mathcal{B}$. The following theorem provides necessary and sufficient conditions for $\mathcal{D} = \ker(T)$ to be a Dirac structure on $\mathcal{B}$.

Theorem 3.2: Consider a densely defined closed operator $T : \mathcal{B} \to \mathcal{L}$, where $\mathcal{B}$ is the bond space. The subspace $\mathcal{D} = \ker(T)$ is a Dirac structure on $\mathcal{B}$ if and only if $\ker(T)$ and $\mathrm{Im}(RT^*)$ are Tellegan structures on $\mathcal{B}$.

Proof: Since T is a densely defined closed operator it follows (see Goldberg [4], Theorem IV.1.2, page 95) that

$$\mathrm{cl}\,\mathrm{Im}(T^*) = (\ker(T))^{c}. \quad (4)$$

We consider the subspace $\mathcal{D} = \ker(T)$ of $\mathcal{B}$, which is closed (see the reference before). From the equality (4) and Proposition 3.2 we have that

$$\mathcal{D}^{\perp} = R\mathcal{D}^{c} = R(\ker(T))^{c} = R\,\mathrm{cl}\,\mathrm{Im}(T^*) = \mathrm{cl}\,\mathrm{Im}(RT^*).$$

Using Proposition 3.1 and Theorem 3.1 we may conclude the proof. ∎

Remark 3.7: We see from the proof of the previous theorem that if $\mathcal{D}$, the null space of the densely defined closed operator $T : \mathcal{B} \to \mathcal{L}$, is a Dirac structure on $\mathcal{B}$ then it has a image representation, namely

$$\mathcal{D} = \mathrm{cl}(\mathrm{Im}(RT^*)).$$

Using Theorem 3.2, we know how to check if the null space of a densely defined closed operator is a Dirac structure. One may ask the following question: given $\mathcal{D}$ a Dirac structure on $\mathcal{B}$, there exists a densely defined closed operator T on $\mathcal{B}$ such that $\mathcal{D}$ is the null space of T? For Hilbert spaces the answer turned out to be always positive (see [6]). In Banach spaces this is not always the case, as we will see from the following result. However, for applications, we have to verify if the null space of some operator is a Dirac structure.

We may state the following theorem, which is a direct consequence of Theorem II.1.14, page 48, Goldberg [4].

Theorem 3.3: Let $\mathcal{D}$ be a Dirac structure on the bond space $\mathcal{B}$. There exists a projection P from $\mathcal{B}$ onto $\mathcal{D}$ if and only if $\mathcal{B} = \mathcal{D} \oplus N$ for N some closed subspace of $\mathcal{B}$. Then $\mathcal{D} = \ker(I - P)$.
We make the following two remarks regarding the previous theorem.

Remark 3.8: If any of the conditions in the above theorem holds then there exists a kernel representation $\mathcal{D} = \ker(I - P)$ of a Dirac structure $\mathcal{D}$, where P is a suitable projection from $\mathcal{B}$ onto $\mathcal{D}$. Clearly, for Hilbert spaces the kernel representation always exists.

Remark 3.9: For a given closed linear manifold $\mathcal{D}$ in $\mathcal{B}$ it is not always possible to find N a closed subspace of $\mathcal{B}$ such that $\mathcal{B} = \mathcal{D} \oplus N$ (see the references in Dunford and Schwartz [3], page 553).

Three classes of Dirac structures are introduced in the sequel

1) *Completely multivalued Dirac structures* which are of the form
$$\mathcal{D}_{mul} = \{(0, e) \,:\, e \in \mathcal{E}\};$$
2) *Completely kernel Dirac structures* which are of the form
$$\mathcal{D}_{ker} = \{(f, 0) \,:\, f \in \mathcal{F}\};$$
3) *Completely skew-adjoint Dirac structures* which are determined by the graphs of injective skew-adjoint (not necessarily bounded) operators from $\mathcal{F}$ to $\mathcal{E}$.

It can be easily seen that the linear subspaces of type (1.) and type (2.) are Dirac structures, while Example 3.1 shows that the linear subspaces of type (3.) are Dirac structures as well. These particular Dirac structures are called fundamental Dirac structures. Under some conditions it can be shown that a Dirac structure can be decomposed as an orthogonal sum of the previous introduced fundamental Dirac structures. The idea of the construction of such decomposition is as follows. Define the linear subspace $\mathcal{D}_{mul} = \mathcal{D} \cap (\{0\} \times \mathcal{E})$ in $\mathcal{B}$ and the linear subspace $\mathcal{E}_{mul} = \{e \in \mathcal{E} \, : \, (0,e) \in \mathcal{D}\}$ in $\mathcal{E}$. Clearly, they are closed in $\mathcal{B}$ and in $\mathcal{E}$, respectively. Assume now that $\mathcal{E}_{mul}$ has an orthogonal complement $\mathcal{E}_1$ in $\mathcal{E}$, so that $\mathcal{E} = \mathcal{E}_{mul} \oplus \mathcal{E}_1$. Now define in $\mathcal{F}$ the linear subspace $\mathcal{F}_{mul}$ as

$$\mathcal{F}_{mul} = \{f \in \mathcal{F} \, : \, \langle e \mid f \rangle = 0, \forall \, e \in \mathcal{E}_1\},$$

and then it is easy to see that $\mathcal{F}_{mul}$ has an orthogonal complement $\mathcal{F}_1$ in $\mathcal{F}$. Therefore $\mathcal{D}_{mul}$ is a completely multivalued Dirac structure on the bond space $\mathcal{B}_{mul} := \mathcal{F}_{mul} \times \mathcal{E}_{mul}$ and there exists a Dirac structure $\mathcal{D}_1$ on the bond space $\mathcal{B}_1 := \mathcal{F}_1 \times \mathcal{E}_1$ such that

$$\mathcal{D} = \mathcal{D}_{mul} \oplus \mathcal{D}_1.$$

Furthermore, the Dirac structure $\mathcal{D}_1$ is the graph of a skew-adjoint (not necessarily bounded) operator from the Banach space $\mathcal{F}_1$ to the Banach space $\mathcal{E}_1$. Define the linear subspace $\mathcal{D}_{ker} = \mathcal{D}_1 \cap (\mathcal{F}_1 \times \{0\})$ in $\mathcal{B}_1$ and the linear subspace $\mathcal{F}_{ker} = \{f \in \mathcal{F}_1 \, : \, (f,0) \in \mathcal{D}_1\}$ in $\mathcal{F}_1$. These subspaces are closed in $\mathcal{B}_1$ and in $\mathcal{F}_1$, respectively, and assume that $\mathcal{F}_{ker}$ has an orthogonal complement $\mathcal{F}_{skew}$ in $\mathcal{F}_1$, so that $\mathcal{F}_1 = \mathcal{F}_{ker} \oplus \mathcal{F}_{skew}$. Now define in $\mathcal{E}_1$ the linear subspace $\mathcal{E}_{ker}$ as

$$\mathcal{E}_{ker} = \{e \in \mathcal{E}_1 \, : \, \langle e \mid f \rangle = 0, \forall \, f \in \mathcal{F}_{skew}\},$$

and then it follows that $\mathcal{E}_{ker}$ has an orthogonal complement $\mathcal{E}_{skew}$ in $\mathcal{E}_1$. Then $\mathcal{D}_{ker}$ is a completely kernel Dirac structure on the bond space $\mathcal{B}_{ker} := \mathcal{F}_{ker} \times \mathcal{E}_{ker}$ and there exists a Dirac structure $\mathcal{D}_{skew}$ on the bond space $\mathcal{B}_{skew} := \mathcal{F}_{skew} \times \mathcal{E}_{skew}$ such that

$$\mathcal{D}_1 = \mathcal{D}_{ker} \oplus \mathcal{D}_{skew}.$$

Clearly, the Dirac structure $\mathcal{D}_{skew}$ is the graph of an closed injective skew-adjoint (not necessarily bounded) operator from the Banach space $\mathcal{F}_{skew}$ to the Banach space $\mathcal{E}_{skew}$. Conclude that under the assumptions imposed above, a Dirac structure can be written down as an orthogonal sum of three fundamental Dirac structures on the "smaller" bond Banach spaces $\mathcal{B}_{mul}$, $\mathcal{B}_{ker}$ and $\mathcal{B}_{skew}$, respectively. Moreover, this decomposition is given by

$$\mathcal{D} = \mathcal{D}_{mul} \oplus \mathcal{D}_{ker} \oplus \mathcal{D}_{skew},$$

and is comparable to the so called "constrained input-output representation" of a Dirac structure in finite-dimensional spaces, see [17].

IV. Example

The academic example presented in this section is a straightforward adaptation, for reflexive Banach spaces, of the example of the transmission line from [6].

Consider a transmission line whose length is S. The Kirchhoff's laws describing the transmission line are given by

$$\begin{aligned} e_\phi &= -\frac{\partial e_q}{\partial z}, \\ f_q &= -\frac{\partial f_\phi}{\partial z}. \end{aligned} \tag{5}$$

Here f_q is the rate of charge density, e_q is the voltage distribution, f_ϕ is the current distribution and e_ϕ is the rate of flux density. The boundary conditions are

$$\begin{aligned} f_\phi(0) = -f_L, \, e_q(0) = e_L, \\ f_\phi(S) = f_R, \, e_q(S) = e_R. \end{aligned} \tag{6}$$

Here f_L and e_L are the current and voltage at the left boundary. Similarly, f_R and e_R are the current and voltage at the right boundary. Let p, q be two positive numbers satisfying the condition $1/p + 1/q = 1$ and let $L_p(0,S)$ and $L_q(0,S)$ be the space of p- and q-integrable functions on $[0,S]$, respectively. The space of flow variables is given by

$$\mathcal{F} = L_p(0,S) \times L_p(0,S) \times \mathbb{R}^2,$$

while the space of effort variables is given by

$$\mathcal{E} = L_q(0,S) \times L_q(0,S) \times \mathbb{R}^2.$$

An element of the space $\mathcal{F}$ is denoted by

$$f = (f_q, f_\phi, f_L, f_R),$$

and an element of the space $\mathcal{E}$ is denoted by

$$e = (e_q, e_\phi, e_L, e_R).$$

The power product is defined as

$$\begin{aligned} \langle e|f \rangle_{\mathcal{B}} &= \langle e, f \rangle_{\mathcal{F}} \\ &= \int_0^S f_q(z) e_q(z) \mathrm{d}z + \int_0^S f_\phi(z) e_\phi(z) \mathrm{d}z \\ &\quad + f_L e_L + f_R e_R. \end{aligned}$$

The first term represents the power associated to electrical domain, the second term is power associated to magnetic domain and the last two terms represents the power exchanged through the boundary. The space of admissible flows and efforts is given by

$$\mathcal{D} = \ker(T),$$

where $T : \mathcal{B} \rightarrow \mathcal{L} = L_p(0,S) \times L_p(0,S) \times \mathbb{R}^2 \times L_q(0,S) \times L_q(0,S) \times \mathbb{R}^2$ is as follows

$$T = \begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix},$$

with M and N given by

$$M = \begin{bmatrix} id_{L_p(0,S)} & \frac{\partial}{\partial z} & 0 & 0 \\ 0 & \partial_{L,p} & 1 & 0 \\ 0 & \partial_{R,p} & 0 & -1 \end{bmatrix},$$

$$N = \begin{bmatrix} \frac{\partial}{\partial z} & id_{L_q(0,S)} & 0 & 0 \\ \partial_{L,q} & 0 & -1 & 0 \\ \partial_{L,q} & 0 & 0 & -1 \end{bmatrix}.$$

Here, $\partial_{L,p} : L_p(0,S) \rightarrow \mathbb{R}$ is defined as $\partial_{L,p} x = x(0)$ and $\partial_{R,p} : L_p(0,S) \rightarrow \mathbb{R}$ is defined as $\partial_{R,p} x = x(S)$. The domain of the operator T is

$$\begin{aligned} \mathrm{Dom}(T) &= L_p(0,S) \times \mathrm{Dom}_p(\frac{\partial}{\partial z}) \times \mathbb{R} \\ &\quad \times \mathbb{R} \times \mathrm{Dom}_q(\frac{\partial}{\partial z}) \times L_q(0,S) \times \mathbb{R} \times \mathbb{R}, \end{aligned}$$

where

$$\mathrm{Dom}_p(\frac{\partial}{\partial z}) = \{x \in L_p(0,S) : x \,\mathrm{abs.\,cont.} \text{ and } \frac{\partial x}{\partial z} \in L_p(0,S)\}.$$

The subspace $\mathrm{Dom}_p(\frac{\partial}{\partial z})$ is dense on $L_p(0,S)$ (see [10], pp. 145, exercise 2.7). This means that the linear transformation T has a dense domain and thus $\ker(T)$ is a closed subspace. First we prove that $\mathcal{D}$ is a power conserving structure. Indeed, if $(f,e) \in \mathcal{D}$ then

$$\begin{aligned}\langle e|f\rangle_{\mathcal{B}} &= -\int_0^S \frac{\partial f_\phi(z)}{\partial z} e_q(z)\mathrm{d}z - \int_0^S \frac{\partial e_q(z)}{\partial z} f_\phi(z)\mathrm{d}z \\ &\quad + e_L f_L + e_R f_R \\ &= -e_q(S)f_\phi(S) + e_q(0)f_\phi(0) + e_L f_L + e_R f_R \\ &= 0.\end{aligned}$$

An element of the space $\mathcal{L}$ is denoted by

$$l = (l_\phi, l_{f_L}, l_{f_R}, l_q, l_{e_L}, l_{e_R}).$$

The linear transformation $RT^* : \mathcal{L} \to \mathcal{B}$ has the following form

$$RT^* = \begin{bmatrix} \frac{\partial}{\partial z} & 0 & 0 & 0 & 0 & 0 \\ id_{L_p(0,S)} & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\partial}{\partial z} & 0 & 0 \\ 0 & 0 & 0 & id_{L_q(0,S)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

and the domain of RT^* is given by

$$\begin{aligned}\mathrm{Dom}(RT^*) &= \mathrm{Dom}_p(\frac{\partial}{\partial z}) \times \mathbb{R}^2 \times \mathrm{Dom}_q(\frac{\partial}{\partial z}) \times \mathbb{R}^2 \\ &\quad \cap \{l \in \mathcal{L} : l_\phi(0) = l_{f_L}, l_\phi(S) \\ &= l_{f_R}, l_q(0) = -l_{e_L}, l_q(S) = l_{e_R}\}.\end{aligned}$$

Now we prove that $\mathrm{im}(RT^*)$ is a power-conserving structure. Indeed,

$$\begin{aligned}\langle e|f\rangle_{\mathcal{B}} &= -\int_0^S \frac{\partial l_\phi(z)}{\partial z} l_q(z)\mathrm{d}z - \int_0^S \frac{\partial l_q(z)}{\partial z} l_\phi(z)\mathrm{d}z \\ &\quad + l_{f_L} l_{e_L} + l_{f_R} l_{e_R} \\ &= -l_q(S)l_\phi(s) + l_q(0)l_\phi(0) + l_{f_L} l_{e_L} + l_{f_R} l_{e_R} \\ &= 0.\end{aligned}$$

Therefore the subspace $\mathcal{D}$ is a Dirac structure, cf. Theorem 3.2. This also means that equations (5) with the boundary conditions (6) represent the interconnection part of the transmission line.

V. Conclusions and further research

In this paper connections between the power-conserving and Dirac structures and standard tools from functional analysis have been identified and used to derive straightforward properties and representations for Dirac structures. The composition of two Dirac structures is not necessarily a Dirac structure (see [5]). Further research will focus on finding necessary and sufficient conditions for preserving the Dirac structure under interconnection of systems.

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