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Asymptotic Tracking for Nonlinear Systems using Fictitious Inputs

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***Abstract*— In this contribution the recently introduced concept of fictitious inputs (see [1]) for the design of feedforward controllers is investigated for the case of SISO systems, when only a single fictitious input needs to be introduced. It is shown that the internal dynamics and the input-ouput linearizing controller can be derived from the differential parameterization. Thus, in the case of stable internal dynamics, a desired trajectory can be stabilized based on the differential parameterization resulting from the introduction of the fictitious input. The results of the paper are illustrated for the Van de Vusse type continuous stirred tank reactor (CSTR).**

I. Introduction

The flatness based approach to the analysis and control of nonlinear systems is an important design strategy for nonlinear control systems. This approach has been introduced e.g. in [2] and [3]. For an affine input nth order SISO system

$$\dot{x} = f(x) + g(x)u \quad (1)$$
$$y = h(x) \quad (2)$$

the flatness property of (1) implies the existence of an (eventually fictitious) flat output $y_f \in \mathbb{R}$, such that

$$y_f = \Phi(x) \quad (3)$$
$$x = \psi_x(y_f, \dot{y}_f, \ldots, y_f^{(n-1)}) \quad (4)$$
$$u = \psi_u(y_f, \dot{y}_f, \ldots, y_f^{(n)}) \quad (5)$$

The feedforward controller is then obtained by inserting the arbitrary but sufficiently smooth reference trajectory for y_f into (5). If system (1) is not flat, a flat system can always be constructed by the introduction of fictitious inputs (see [1]). Setting the fictitious inputs of the resulting differential paramterization to zero yields a differential parameterization for the original system (1). Yet, the components of the parameterizing output for system (1) are differentially dependent in contrast to the situation of a real flat system. This contribution clarifies the structure of the differential parameterization for the case of SISO systems, when a single fictitious input u_f is introduced. This is done by comparison with the derivation of the Byrnes-Isidori normal form for system (1)–(2). Section II recalls some facts about input-output linearization using the Byrnes-Isidori normal form and feedforward controller design using fictitious inputs. Section III shows that the Brunovský states of the fictitious system are naturally related to the coordinates of a Byrnes-Isidori normal form for system (1)–(2). In Section IV the input-output linearizing controller is derived from the differential parameterization. Section V shows that the involved theoretic investigation of the differential parameterization in Sections III–IV results in a very simple and systematic step-by-step procedure to derive the input-output linearizing controller which can be performed without knowledge of the previously acquired theoretic background. Section VI then extends the method to the design of tracking controllers for output tracking for minimum phase systems. In Section VII the approach is applied to a Van de Vusse type CSTR. This example shows the advantage of this approach compared to the well established transformation to Byrnes-Isidori normal form when dealing with tracking control for nonminimum phase systems.

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II. Problem Formulation

If the SISO system (1)–(2) has relative degree r locally about x_0 (see e.g. [4]), then for x in a neighbourhood $U(x_0)$ of x_0

$$L_g h(x) = L_g L_f h(x) = \ldots = L_g L_f^{r-2} h(x) = 0 \quad (6)$$

and

$$L_g L_f^{r-1} h(x_0) \neq 0 \quad (7)$$

If (6)–(7) holds, the time derivatives of y can be written as

$$y^{(i)} = L_f^i h(x), \qquad i = 1(1)r-1 \quad (8)$$

In this case a coordinates transformation

$$(\xi, \eta) = \Phi(x) \quad (9)$$

which is given by

$$\xi_i = L_f^{i-1} h(x), \qquad i = 1(1)r \quad (10)$$
$$\eta_j = \varphi_j(x), \qquad j = 1(1)n-r \quad (11)$$

exists, where the φ_j can always be chosen such that (9) is nonsingular about x_0. In the new coordinates (ξ, η) system (1)–(2) is represented by the Byrnes-Isidori normal form

$$\begin{aligned} \dot{\xi}_i &= \xi_{i+1}, \qquad i = 1(1)r-1 \\ \dot{\xi}_r &= b(\xi,\eta) + a(\xi,\eta)u \quad (12) \\ \dot{\eta} &= p(\xi,\eta) + q(\xi,\eta)u \\ y &= \xi_1 \quad (13) \end{aligned}$$

where

$$a(\xi,\eta) = L_g L_f^{r-1} h(x)\big|_{x=\Phi^{-1}(\xi,\eta)} \quad (14)$$
$$b(\xi,\eta) = L_f^r h(x)\big|_{x=\Phi^{-1}(\xi,\eta)} \quad (15)$$

It is well known that based on the Byrnes-Isidori normal form (12)–(13) an asymptotic tracking controller for output (2) can be designed (see e.g. [4]). In the following it is shown that the tracking problem can also be solved on the basis of a differential parameterization for system (1) even if system (1) is not flat. To this end, a fictitious scalar input u_f is introduced to system (1)

$$\dot{x} = f(x) + g(x)u + g_f(x)u_f \quad (16)$$

such that

$$\text{rank}[g(x)\ g_f(x)] = 2 \tag{17}$$

holds locally. If (17) is satisfied, u_f is independent from u and qualifies as a new input. As a consequence of the fact that every flat output satisfies $\dim y_f = \dim u$ (see [2]), a possible flat output for system (16) has $\dim y_f = \dim[u\ \ u_f]^T = 2$. It is assumed that it is possible to find a flat output for (16) of the kind

$$y_f = [y_{f1}\ \ y_{f2}]^T = [y\ \ y_{f2}]^T = [h(x)\ \ h_f(x)]^T \tag{18}$$

where the first component y_{f1} is the original output (2) of system (1)–(2). If additionally system (16) is static feedback linearizable, there exists a differential parameterization of the inputs

$$u = \psi_u(y_{f1}, \dot{y}_{f1}, \ldots, y_{f1}^{(r_1)}, y_{f2}, \dot{y}_{f2}, \ldots, y_{f2}^{(r_2)}) \tag{19}$$

$$u_f = \psi_{u_f}(y_{f1}, \dot{y}_{f1}, \ldots, y_{f1}^{(r_1)}, y_{f2}, \dot{y}_{f2}, \ldots, y_{f2}^{(r_2)}) \tag{20}$$

and of the states

$$x = \psi_x(y_{f1}, \dot{y}_{f1}, \ldots, y_{f1}^{(r_1-1)}, y_{f2}, \dot{y}_{f2}, \ldots, y_{f2}^{(r_2-1)}) \tag{21}$$

such that the controllability indices of (16) satisfy $r_1+r_2 = n$ (see [3]). In [1] it has been shown that it is always possible to determine a differential parameterization (19)–(21) for general nonlinear systems, although not necessarily with the introduction of only one fictitious input u_f. However, the application of this approach to various examples shows that the assumptions made above are not too restrictive for the case of SISO systems.

In contrast to the case of a flat output for system (1), the components of y_f in (18) cannot be assigned freely but have to respect $u_f \equiv 0$ to be trajectories of the original system (1). In view of (20) this yields

$$0 \equiv \psi_{u_f}(y_{f1}, \dot{y}_{f1}, \ldots, y_{f1}^{(r_1)}, y_{f2}, \dot{y}_{f2}, \ldots, y_{f2}^{(r_2)}) \tag{22}$$

Thus, the components of y_f are obviously differentially dependent. If the ouput y is supposed to track a given trajectory y^* i.e. y_{f1}^*, y_{f2}^* can be determined as the solution of the differential equation

$$\psi_{u_f}(y_{f1}^*, \dot{y}_{f1}^*, \ldots, y_{f1}^{(r_1)*}, y_{f2}, \dot{y}_{f2}, \ldots, y_{f2}^{(r_2)}) \equiv 0 \tag{23}$$

The feedforward controller is then obtained by inserting y_{f1}^* and y_{f2}^* into (19). In this paper it will be shown how a tracking controller can be derived from the differential parameterization (19)–(21). This extents the results in [1] to the design of tracking controllers.

III. Natural coordinates based on the differential parameterization

In this section it will be investigated how the Brunovský states (see [5]) of system (16)

$$\begin{aligned} \zeta = (\zeta^1, \zeta^2) &= (\zeta_1^1, \ldots, \zeta_{r_1}^1, \zeta_1^2, \ldots, \zeta_{r_2}^2) \\ &= (y_{f1}, \ldots, y_{f1}^{(r_1-1)}, y_{f2}, \ldots, y_{f2}^{(r_2-1)}) \end{aligned} \tag{24}$$

are related to the coordinates of a Byrnes-Isidori normal form for system (1)–(2). The coordinates transformation

$$\zeta = \psi_x^{-1}(x) = \Phi_f(x) \tag{25}$$

transforms system (16) into nonlinear controller form (see [4])

$$\begin{aligned} \dot{\zeta}_i^1 &= \zeta_{i+1}^1, & i &= 1(1)r_1 - 1 \\ \dot{\zeta}_{r_1}^1 &= b_1(\zeta) + a_{11}(\zeta)u + a_{12}(\zeta)u_f \\ \dot{\zeta}_j^2 &= \zeta_{j+1}^2, & j &= 1(1)r_2 - 1 \\ \dot{\zeta}_{r_2}^2 &= b_2(\zeta) + a_{21}(\zeta)u + a_{22}(\zeta)u_f \end{aligned} \tag{26}$$

where the controllability indices r_1 and r_2 follow from (19)–(21). The decoupling matrix $A(\zeta)$ (see [4]) of system (26) is given by

$$\begin{aligned} A(\zeta) &= \left. \begin{bmatrix} L_g L_f^{r_1-1} h(x) & L_{g_f} L_f^{r_1-1} h(x) \\ L_g L_f^{r_2-1} h_f(x) & L_{g_f} L_f^{r_2-1} h_f(x) \end{bmatrix} \right|_{x=\Phi_f^{-1}(\zeta)} \\ &= [a_{ij}(\zeta)] \end{aligned} \tag{27}$$

As static feedback linearizability is assumed, it follows that

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21} \neq 0 \tag{28}$$

in a neighbourhood of $\zeta_0 = \Phi_f(x_0)$. The normal form (26) can be related to the original system (1) by setting $u_f \equiv 0$

$$\begin{aligned} \dot{\zeta}_i^1 &= \zeta_{i+1}^1, & i &= 1(1)r_1 - 1 \\ \dot{\zeta}_{r_1}^1 &= b_1(\zeta) + a_{11}(\zeta)u \\ \dot{\zeta}_j^2 &= \zeta_{j+1}^2, & j &= 1(1)r_2 - 1 \\ \dot{\zeta}_{r_2}^2 &= b_2(\zeta) + a_{21}(\zeta)u \end{aligned} \tag{29}$$

Output (2) reads in the ζ-coordinates

$$y = y_{f1} = \zeta_1^1 = h(x)\big|_{x=\Phi_f^{-1}(\zeta)} \tag{30}$$

From (29) and (8) it can be deduced that for the coordinates transformation Φ_f in (25)

$$\Phi_f(x) = [h(x)\ \ldots\ L_f^{r_1-1}h(x)\ h_f(x)\ \ldots\ L_f^{r_2-2}h_f(x)]^T \tag{31}$$

holds. Furthermore a_{11} and b_1 in (29) are given by

$$a_{11}(\zeta) = L_g L_f^{r_1-1} h(x)\big|_{x=\Phi_f^{-1}(\zeta)} \tag{32}$$

$$b_1(\zeta) = L_f^{r_1} h(x)\big|_{x=\Phi_f^{-1}(\zeta)} \tag{33}$$

Comparing (32)–(33) with (14)–(15) it follows, that for $r_1 = r$ system (29) with output (30) is already in Byrnes-Isidori normal form with $\xi = \zeta^1$ and $\eta = \zeta^2$. So, the coordinates transformation into Byrnes-Isidori normal form (29)–(30) is given by (25) which is a function of x only and thus independent from setting $u_f = 0$. Note that Φ_f is simply the inverse of the differential parameterization (21) of the states x.

If $r_1 < r$ it follows from (6) and (32) that $a_{11} = 0$. Consequently, $a_{12} \neq 0$ in view of (26) and (28). Thus, the relative degree of (2) as output of the fictitious system (16) has been reduced by the introduction of u_f. As a consequence, in this case an additional coordinates transformation into Byrnes-Isidori normal form has to be determined. The drift term $\tilde{f}$ and input vector $\tilde{g}$ of system (29) for this situation are

$$\tilde{f}(\zeta) = \begin{bmatrix} \zeta_2^1 \\ \ldots \\ \zeta_{r_1}^1 \\ b_1(\zeta) \\ \zeta_2^2 \\ \ldots \\ \zeta_{r_2}^2 \\ b_2(\zeta) \end{bmatrix}, \qquad \tilde{g}(\zeta) = \begin{bmatrix} 0 \\ \ldots \\ 0 \\ a_{21}(\zeta) \end{bmatrix} \tag{34}$$

where $a_{21} \neq 0$ because of $a_{11} = 0$ and (28). In view of (30)

$$L_{\tilde{g}} L_{\tilde{f}}^{k} \zeta_1^1 = 0, \qquad k = 0(1)r-2 \quad (35)$$

$$L_{\tilde{g}} L_{\tilde{f}}^{r-1} \zeta_1^1 \Big|_{\zeta_0 = \Phi_f(x_0)} \neq 0 \quad (36)$$

holds for the original system (29) in the new coordinates since the relative degree r is independent from the choice of local coordinates. Thus, the ξ-coordinates for system (1)–(2) can be introduced as

$$\xi_i = L_{\tilde{f}}^{i-1} \zeta_1^1 = \zeta_i^1 \ \ (= y^{(i-1)}), \qquad i = 1(1)r \quad (37)$$

in view of (10), (30) and (34). In the following it will be shown that introducing the η-coordinates as

$$\eta_j = \zeta_j^2, \qquad j = 1(1)n-r \quad (38)$$

yields a nonsingular coordinates transformation $(\xi, \eta) = \bar{\Phi}_f(\zeta)$ which transforms system (1)–(2) into Byrnes-Isidori normal form (12)–(13). To this end, the structure of $b_1(\zeta)$ in (34) is investigated. From the structure of $\tilde{f}$ in (34) it can be seen that (35) is fulfilled independently from $b_1(\zeta)$ for $k = 0(1)r_1 - 1$. At $k = r_1$ one has

$$\begin{aligned} L_{\tilde{g}} L_{\tilde{f}}^{r_1} \zeta_1^1 &= L_{\tilde{g}} b_1(\zeta) \\ &= \tfrac{\partial b_1(\zeta)}{\partial \zeta} \tilde{g} = [\tfrac{\partial b_1(\zeta)}{\partial \zeta_1^1} \cdots \tfrac{\partial b_1(\zeta)}{\partial \zeta_{r_2}^2}] \tilde{g} = 0 \end{aligned} \quad (39)$$

This yields $\frac{\partial b_1(\zeta)}{\partial \zeta_{r_2}^2} = 0$ in view of $a_{21} \neq 0$. The succeeding conditions in (35) can be expressed as

$$\begin{aligned} L_{\tilde{g}} L_{\tilde{f}}^{r_1+i} \zeta_1^1 = L_{\tilde{g}} L_{\tilde{f}}^{i} b_1 = (\tfrac{\partial}{\partial \zeta} L_{\tilde{f}}^{i} b_1) \tilde{g} \overset{!}{=} 0, \\ i = 1(1)r - r_1 - 2 \end{aligned} \quad (40)$$

(see (33)). With the structure of $\tilde{f}$ and similar arguments as before this leads to

$$\begin{aligned} \frac{d^i}{dt^i} b_1 &= L_{\tilde{f}}^{i} b_1, \qquad i = 0(1)r - r_1 - 1 \quad (41) \\ &= b_1^{(i)}(\zeta^1, \zeta_1^2, \ldots, \zeta_{r_2-(r-r_1)+1+i}^2) \end{aligned}$$

where especially

$$b_1 = b_1(\zeta^1, \zeta_1^2, \ldots, \zeta_{r_2-(r-r_1)+1}^2) \quad (42)$$

Finally, (36) can be expressed as

$$\begin{aligned} L_{\tilde{g}} L_{\tilde{f}}^{r-1} \zeta_1^1 \Big|_{\zeta_0 = \Phi_f(x_0)} &= L_{\tilde{g}} L_{\tilde{f}}^{r-r_1-1} b_1(\zeta) \Big|_{\zeta_0 = \Phi_f(x_0)} \\ &= (\frac{\partial}{\partial \zeta} L_{\tilde{f}}^{r-r_1-1} b_1) \tilde{g} \Big|_{\zeta_0 = \Phi_f(x_0)} \overset{!}{\neq} 0 \end{aligned} \quad (43)$$

and, as before, the structure of $\tilde{f}$ and $\tilde{g}$ yields

$$\frac{\partial b_1^{(i)}}{\partial \zeta_{r_2-(r-r_1)+1+i}^2} \Big|_{\zeta_0 = \Phi_f(x_0)} \neq 0, \quad i = 0(1)r - r_1 - 1 \quad (44)$$

Using the above results the transformation $(\xi, \eta) = \bar{\Phi}_f(\zeta)$ into Byrnes-Isidori normal form has the following structure

$$\begin{aligned} \xi_i &= \zeta_i^1, & i &= 1(1)r_1 \\ \xi_{r_1+j} &= \frac{d^{j-1}}{dt^{j-1}} b_1 = L_{\tilde{f}}^{j-1} b_1, & j &= 1(1)r - r_1 \\ &= \xi_{r_1+j}(\zeta^1, \zeta_1^2, \ldots, \zeta_{r_2-(r-r_1)+j}^2) & & \quad (45) \\ \eta_l &= \zeta_l^2, & l &= 1(1) \underbrace{r_2 - (r - r_1)}_{= r_1 + r_2 - r = n - r} \end{aligned}$$

In view of (44) and (45) the Jacobian of $\bar{\Phi}_f(\zeta)$ is nonsingular about ζ_0 and consequently $\bar{\Phi}_f(\zeta)$ qualifies as a coordinates transformation. Thus, the Byrnes-Isidori normal form for $r < r_1$ of the original system (1) is given by

$$\begin{aligned} \dot{\xi}_i &= \xi_{i+1}, & i &= 1(1)r-1 \\ \dot{\xi}_r &= \bar{b}(\xi, \eta) + \bar{a}(\xi, \eta) u & & \quad (46) \\ \dot{\eta}_j &= \eta_{j+1}, & j &= 1(1)n-r-1 \\ \dot{\eta}_{n-r} &= \zeta_{r_2-(r-r_1)+1}^2 \circ \bar{\Phi}_f^{-1}(\xi, \eta) \\ &= q(\xi_1, \ldots, \xi_{r_1+1}, \eta) \end{aligned}$$

with output

$$y = \xi_1 \quad (47)$$

In case of $r_1 = r$ the transformation $\bar{\Phi}_f$ becomes identity and

$$\bar{b} = b_1 \qquad \bar{a} = a_{11} \quad (48)$$

(see (29)). So, the transformation of system (1) into the coordinates of the corresponding Byrnes-Isidori normal form is given by

$$(\xi, \eta) = \bar{\Phi}_f \circ \Phi_f(x) \quad (49)$$

in either case. This will be used for a unified notation.

IV. Input-output linearizing controller

This section clarifies the relation between the input-output linearizing controller and the differential parameterization as introduced in Section II. A major result of this section is that equation (22) is related to the internal dynamics of system (1)–(2). Furthermore, it is shown that the input-output linearizing feedback for the Byrnes-Isidori normal form (46)–(47) can be derived from the differential parameterization (19)–(20) of the inputs. The analysis of the differential parameterization (19)–(20) is done mainly in the ζ-coordinates where the fictitious system (16) is given in nonlinear controller form (26). The exact linearizing feedback law which transforms system (26) into Brunovský normal form with the new inputs v_1 and v_2 is given by

$$\begin{bmatrix} u \\ u_f \end{bmatrix} = A^{-1}(\zeta) \begin{bmatrix} v_1 - b_1(\zeta) \\ v_2 - b_2(\zeta) \end{bmatrix} \quad (50)$$

in view of (28). The same feedback controller is obtained by setting

$$v_1 = \dot{\zeta}_{r_1}^1 = y_{f1}^{(r_1)}, \qquad v_2 = \dot{\zeta}_{r_2}^2 = y_{f2}^{(r_2)} \quad (51)$$

in (19)–(20), i.e.

$$\begin{aligned} \begin{bmatrix} u \\ u_f \end{bmatrix} &= \begin{bmatrix} \psi_u(\zeta^1, v_1, \zeta^2, v_2) \\ \psi_{u_f}(\zeta^1, v_1, \zeta^2, v_2) \end{bmatrix} = A^{-1}(\zeta) \begin{bmatrix} v_1 - b_1 \\ v_2 - b_2 \end{bmatrix} \\ &= \tfrac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix} a_{22}(v_1 - b_1) - a_{12}(v_2 - b_2) \\ -a_{21}(v_1 - b_1) + a_{11}(v_2 - b_2) \end{bmatrix} \end{aligned} \quad (52)$$

which follows from the properties of the flat system (16) (see [3]). In the following it is important to recall that the differential parameterization for (1) is obtained from (19)–(20) by setting $u_f \equiv 0$. For the further discussions two different cases have to be distinguished:

A. $r_1 = r$

In this case the dynamics of system (1) in the ζ-coordinates are given by the controller normal form (26) where a_{11} and b_1 are given by (32)–(33) for $r_1 = r$ with $a_{11}(\zeta_0) \neq 0$ in view of (7). Setting $u_f = 0$ in (52) gives

$$0 = \frac{1}{a_{11}a_{22} - a_{12}a_{21}}(-a_{21}(v_1 - b_1) + a_{11}(v_2 - b_2)) \quad (53)$$

As $\det A \neq 0$ (see (28)) (53) can be solved for v_2

$$v_2 = \psi_{u_f}^{-1}(\zeta^1, \zeta^2, v_1) = \frac{a_{21}}{a_{11}}(v_1 - b_1) + b_2 = \frac{a_{21}}{\bar{a}}(v_1 - \bar{b}) + b_2 \quad (54)$$

in view of (48). Substituting v_2 according to (54) in $\psi_u(\zeta^1, v_1, \zeta^2, v_2))$ (see (52)) yields

$$\begin{aligned} u &= \frac{1}{a_{11}a_{22} - a_{12}a_{21}}(a_{22}(v_1 - b_1) - a_{12}[\frac{a_{21}}{a_{11}}(v_1 - b_1)]) \\ &= \frac{1}{a_{11}}(v_1 - b_1) = \frac{1}{\bar{a}}(v_1 - \bar{b}) \end{aligned} \quad (55)$$

where again (48) was used. Thus, (55) is the input-output linearizing feedback law for the Byrnes-Isidori normal form (46)–(47). Application of (55) to (46)–(47) yields the following system representation

$$\begin{aligned} \dot{\xi}_i &= \xi_{i+1}, & i &= 1(1)r-1 \\ \dot{\xi}_r &= v_1 \\ \dot{\eta}_j &= \eta_{j+1}, & j &= 1(1)n-r-1 \\ \dot{\eta}_{n-r} &= b_2 + \frac{a_{21}}{\bar{a}}(v_1 - \bar{b}) = q(\xi, \eta, v_1) \end{aligned} \quad (56)$$

(56) with output (47) is in Byrnes-Isidori normal form. The η-subsystem of (56), which represents the internal dynamics, is a state space representation of the implicit differential equation (53) (i.e. 22). This is due to the fact that $q(\xi, \eta, v_1) = \psi_{u_f}^{-1}(\zeta^1, \zeta^2, v_1)\big|_{\zeta = \bar{\Phi}_f^{-1}(\xi, \eta)}$ in view of (54).

B. $r_1 < r$

In this case $a_{11} = 0$ holds in (26), as derived in Section III. Thus, with $\psi_{u_f} = 0$ (52) simplifies to

$$u = \frac{1}{a_{21}}(v_2 - b_2) - \frac{a_{22}}{a_{12}a_{21}}(v_1 - b_1) \quad (57)$$

$$0 = \frac{1}{a_{12}}(v_1 - b_1) \quad (58)$$

For this case $a_{12} \neq 0$ (as $\det A \neq 0$) and thus (58) holds if and only if

$$0 = v_1 - b_1(\zeta^1, \zeta_1^2, \ldots, \zeta_{r_2-(r-r_1)+1}^2) = \bar{\psi}_{u_f} \quad (59)$$

where also (42) was used. Inserting (59) into (57) yields

$$u = \frac{1}{a_{21}}(v_2 - b_2) \quad (60)$$

However, in contrast to the situation in Section IV-A additional constraints are needed to determine the unknown v_2 in (60). These can be derived from the fact that admissible trajectories for the original system (1) which respect $u_f \equiv 0$ obviously also provide for the time derivatives of (59) to vanish. Together with (41) this yields for the first $r - r_1 - 2$ time derivatives

$$\frac{d^i}{dt^i}(59) \Leftrightarrow \begin{aligned} 0 &= v_1^{(i)} - L_{\tilde{f}}^i b_1 \\ &= v_1^{(i)} - b_1^{(i)}(\zeta^1, \zeta_1^2, \ldots, \zeta_{r_2-(r-r_1)+i+1}^2), \\ & \qquad i = 1(1)r - r_1 - 2 \end{aligned} \quad (61)$$

For the $(r - r_1 - 1)$th time derivative one has

$$\begin{aligned} 0 &= v_1^{(r-r_1-1)} - b_1^{(r-r_1-1)}(\zeta^1, \zeta_1^2, \ldots, \zeta_{r_2}^2) \\ &= v_1^{(r-r_1-1)} - L_{\tilde{f}}^{r-r_1-1} b_1 \end{aligned} \quad (62)$$

so that the next time derivative can be formulated as follows

$$\begin{aligned} 0 &= v_1^{(r-r_1)} \\ &\quad - \frac{\partial(b_1^{(r-r_1-1)})}{\partial(\zeta^1, \zeta_1^2, \ldots, \zeta_{r_2-1}^2)} \frac{\partial(\zeta^1, \zeta_1^2, \ldots, \zeta_{r_2-1}^2)}{\partial t} - \frac{\partial b_1^{(r-r_1-1)}}{\partial \zeta_{r_2}^2} \dot{\zeta}_{r_2}^2 \\ &= \tilde{\psi}_{u_f}(\zeta^1, v_1^{(r-r_1)}, \zeta^2, \dot{\zeta}_{r_2}^2) \end{aligned} \quad (63)$$

By $\frac{\partial}{\partial \zeta_{r_2}^2} b_1^{(r-r_1-1)} \neq 0$ (see (44)) it is possible to solve (63) for $\dot{\zeta}_{r_2}^2$

$$\dot{\zeta}_{r_2}^2 = \tilde{\psi}_{u_f}^{-1}(v_1^{(r-r_1)}, \zeta) \quad (64)$$

In view of (51) this yields the desired input v_2

$$v_2 = \tilde{\psi}_{u_f}^{-1}(v_1^{(r-r_1)}, \zeta) \quad (65)$$

Thus, $v_1^{(r-r_1)}$ can be chosen freely, as condition (63) can be fulfilled for any $v_1^{(r-r_1)}$ by a suitable input v_2. So, $v_1^{(r-r_1)}$ can be seen as an input, whereas $v_1^{(i)}$, $i = 0(1)r - r_1 - 1$, are state variables. This becomes obvious in view of the coordinates transformation $\bar{\Phi}_f$. Comparing (61)–(62) with (45) yields

$$v_1^{(i)} = \xi_{r_1+i+1} \; (= y_{f1}^{(r_1+i)}), \qquad i = 0(1)r - r_1 - 1 \quad (66)$$

Finally, substituting (65) in (60) yields the control law

$$u = \frac{1}{a_{21}}(\tilde{\psi}_{u_f}^{-1}(v_1^{(r-r_1)}, \zeta) - b_2) \quad (67)$$

It is essential to realize that the next time derivative of (62) can also be written as

$$\begin{aligned} 0 &= v_1^{(r-r_1)} - b_1^{(r-r_1)} \\ &= v_1^{(r-r_1)} - \underbrace{L_{\tilde{f}}^{r-r_1} b_1}_{\bar{b}} - \underbrace{L_{\tilde{g}} L_{\tilde{f}}^{r-r_1-1} b_1}_{\bar{a}} u \end{aligned} \quad (68)$$

where in view of (39)

$$\bar{a} = L_{\tilde{g}} L_{\tilde{f}}^{r-r_1-1} b_1 = L_{\tilde{g}} L_{\tilde{f}}^{r-1} \zeta_1^1, \quad \bar{a}(\zeta_0) \neq 0 \quad (69)$$

$$\bar{b} = L_{\tilde{f}}^{r-r_1} b_1 = L_{\tilde{f}}^r \zeta_1^1 \quad (70)$$

Thus, $\bar{a}, \bar{b}$ are exactly the terms appearing in (46). Solving (68) for u yields

$$u = \frac{1}{\bar{a}}(v_1^{(r-r_1)} + \bar{b}) \quad (71)$$

This is the input-output linearizing feedback law for the Byrnes-Isidori normal form (46)–(47) with the new input $v_1^{(r-r_1)}$. So, the application of (67) is equivalent to (71). As a consequence, the application of (67) results in the following system dynamics in the (ξ, η)-coordinates

$$\begin{aligned} \dot{\xi}_i &= \xi_{i+1}, & i &= 1(1)r_1 - 1 \\ \dot{\xi}_{r_1+j} &= \xi_{r_1+j}, & j &= 0(1)r - r_1 - 1 \\ \dot{\xi}_r &= v_1^{(r-r_1)} \\ \dot{\eta}_l &= \eta_{l+1}, & l &= 1(1)n - r - 1 \\ \dot{\eta}_{n-r} &= \zeta_{r_2-(r-r_1)+1}^2 = \bar{\psi}_{u_f}^{-1}(\zeta^1, v_1, \zeta_1^2, \ldots, \zeta_{r_2-(r-r_1)}^2) \\ &= q(\xi_1, \ldots, \xi_{r_1+1}, \eta) \end{aligned} \quad (72)$$

with output (47). This is an input-normalized Byrnes-Isidori normal form with new input

$$v_1^{(r-r_1)} = y_{f1}^{(r)} \quad (73)$$

In view of (45) and (72). The right hand side of $\dot{\eta}_{n-r}$ stems from the fact that $\bar{\psi}_{u_f}$ in (59) can be solved for $\zeta^2_{r_2-(r-r_1)+1}$ in view of $\frac{\partial b_1}{\partial \zeta^2_{r_2-(r-r_1)+1}} \neq 0$ (see (44)). Thus, the η-subsystem, which represents the internal dynamics, is a state space representation of the implicit differential equation (59) which is equivalent to $\psi_{u_f} = 0$.

V. Input-ouput linearization using the differential parameterization

The previous results can now be used to derive a systematic procedure for determining the input-output linearizing controller on the basis of (19)–(20). The parameters r_1 and r_2 can directly be derived from the differential parameterization (19)–(20). If additionally (22) can be solved for $v_2 = y_{f2}^{(r_2)}$, then the relative degree r of y is equal to r_1 (see (54)). The input-output linearizing controller with new input $v_1 = y_{f1}^{(r_1)}$ is then given by inserting $v_2 = y_{f2}^{(r_2)} = \psi_{u_f}^{-1}$ into (19)

$$u = \psi_u(y_{f1}, \ldots, y_{f1}^{(r_1)}, y_{f2}, \ldots, y_{f2}^{(r_2-1)}, \psi_{u_f}^{-1}(\ldots y_{f_1}^{(r_1)})) \tag{74}$$

in view of (55). If in contrast $y_{f2}^{(r_2)}$ cannot be obtained directly from (22), then (22) should be normalized such that the coefficient of $v_1 = y_{f1}^{(r_1)}$ is equal to one (see (59)). This yields

$$0 = \bar{\psi}_{u_f}(y_{f1}, \ldots, y_{f1}^{(r_1)}, y_{f2}, \ldots, y_{f2}^{(\kappa)}), \qquad \kappa < r_2 \tag{75}$$

The relative degree r of y can then be determined as

$$r = r_1 + r_2 - \kappa = n - \kappa \tag{76}$$

in view of (59) and $\zeta^2_{r_2-(r-r_1)+1} = y_{f2}^{(r_2-(r-r_1))} = y_{f_2}^{(\kappa)}$. In this case $y_{f2}^{(r_2)}$ has to be determined as

$$\begin{aligned} y_{f_2}^{(r_2)} &= (\bar{\psi}_{u_f}^{(r-r_1)})^{-1}(y_{f1}, \dot{y}_{f1}, \ldots, y_{f1}^{(r_1)}, y_{f1}^{(r)}, \\ &\qquad y_{f2}, \dot{y}_{f2}, \ldots, y_{f2}^{(r_2-1)}) \\ &= \tilde{\psi}_{u_f}^{-1}(y_{f1}, \dot{y}_{f1}, \ldots, y_{f1}^{(r_1)}, y_{f1}^{(r)}, y_{f2}, \dot{y}_{f2}, \ldots, y_{f2}^{(r_2-1)}) \end{aligned} \tag{77}$$

in view of (24), (65) and (73). The input-output linearizing controller is then given by

$$\begin{aligned} u = \psi_u(&y_{f1}, \dot{y}_{f1}, \ldots, y_{f1}^{(r_1)}, \\ &y_{f2}, \dot{y}_{f2}, \ldots, y_{f2}^{(r_2-1)}, \tilde{\psi}_{u_f}^{-1}(\ldots, y_{f1}^{(r)})) \end{aligned} \tag{78}$$

where the new input is $v_1^{(r-r_1)} = y_{f1}^{(r)}$.

VI. Tracking controller design

In the following a tracking controller is derived on the basis of the differential parameterization to stabilize the tracking of the reference trajectory y^*. To this end, the tracking error e for the control output is defined as (see 18)

$$e = y - y^* \quad (= y_{f_1} - y_{f_1}^*) \tag{79}$$

For the case $r_1 = r$ the states of the tracking error system can be introduced as (see (37), which also holds for $r_1 = r$)

$$e_i = \xi_i - \xi_i^*, \qquad i = 1(1)r_1 \tag{80}$$

In these coordinates the tracking error system is given by

$$\begin{aligned} \dot{e}_i &= e_{i+1}, \qquad i = 1(1)r_1 - 1 \\ \dot{e}_{r_1} &= \dot{\xi}_{r_1} - \dot{\xi}_{r_1}^* \end{aligned} \tag{81}$$

Setting v_1 in (55) i.e. $y_{f1}^{(r_1)}$ in (74) equal to

$$v_1 = \dot{\xi}_{r_1}^* - \sum_{i=1}^{r_1} \lambda_i e_i \tag{82}$$

yields

$$\dot{\xi}_{r_1} = v_1 = \dot{\xi}_{r_1}^* - \sum_{i=1}^{r_1} \lambda_i e_i \tag{83}$$

in view of (56). Comparing with (81) the tracking error then respects

$$\dot{\xi}_{r_1} - \dot{\xi}_{r_1}^* + \sum_{i=1}^{r_1} \lambda_i e_i = e_1^{(r)} + \sum_{i=1}^{r} \lambda_i e_1^{(i-1)} = 0 \tag{84}$$

where $r = r_1$ was used. The λ_i can now be chosen such that the tracking error dynamics are stable.

In the case $r_1 < r$ additional states have to be introduced. According to (37) and (45) one has

$$\begin{aligned} e_{r_1+i} &= \xi_{r_1+i} - \xi_{r_1+i}^*, \qquad i = 1(1)r - r_1 \\ &= b_1^{(i-1)} - \xi_{r_1+i}^* \end{aligned} \tag{85}$$

The tracking error system is then given by

$$\begin{aligned} \dot{e}_i &= e_{i+1}, \qquad i = 1(1)r - 1 \\ \dot{e}_r &= \dot{\xi}_r - \dot{\xi}_r^* = b_1^{(r-r_1)} - \dot{\xi}_r^* \end{aligned} \tag{86}$$

Then, setting $v_1^{(r-r_1)}$ in (71) i.e. $y_{f1}^{(r)}$ in (78) equal to

$$v_1^{(r-r_1)} = \dot{\xi}_r^* - \sum_{i=1}^{r} \lambda_i e_i \tag{87}$$

together with (72) and the last row of (86) yields

$$0 = \dot{e}_r + \sum_{i=1}^{r} \lambda_i e_i = e_1^{(r)} + \sum_{i=1}^{r} \lambda_i e_1^{(i-1)} = 0 \tag{88}$$

If the λ_i are chosen adequately, the tracking error system is stable. However, it is a well known fact that despite of the tracking error system beeing stable the internal dynamics can still be unstable and cannot be influenced using an input-output linearizing controller. In this situation the proposed control scheme allows easy switching of the control variables as illustrated in the following section.

VII. Example

In process control applications the necessity of operation point changes occurs quite often. In [6] the Van de Vusse type CSTR is investigated in detail as a benchmark example. The system equations of this CSTR reactor are given by

$$\begin{aligned} \dot{x}_1 &= -k_1 x_1 - k_3 x_1^2 + (C_{A0} - x_1)u \qquad (89) \\ \dot{x}_2 &= k_1 x_1 - k_2 x_2 + (-x_2)u \\ y &= x_2 \qquad (90) \end{aligned}$$

The control output is the state x_2 which is the product concentration in the output stream, the state x_1 is the reactant concentration in the reactor and the input u is the dilution rate. A transition between the two operation points A: $(2.1534\,\frac{\text{mol}}{\text{liter}}, 0.9\,\frac{\text{mol}}{\text{liter}})$ and B: $(2.9175\,\frac{\text{mol}}{\text{liter}}, 1.1\,\frac{\text{mol}}{\text{liter}})$ is considered. These operating points and the corresponding parameters are taken from [6]. If a fictitious input u_f is introduced with the input vector g_f

$$g_f = [1\ \ 0]^T \tag{91}$$

$\mathrm{rank}[g\ \ g_f] = 2$ holds in a neighbourhood of the operation points. As flat output $y_f = [x_2\ \ x_1]^T$ (see (18)) is chosen, where (21) is obvious. After a few algebraic manipulations one arrives at the differential parameterization of the inputs

$$u = -\frac{1}{y_{f1}}(\dot{y}_{f1} - k_1 y_{f2} + k_2 y_{f1}) \tag{92}$$

$$\begin{aligned} u_f &= \dot{y}_{f2} - (-k_1 y_{f2} - k_3 y_{f2}^2) \\ &\quad + \frac{C_{A0} - y_{f2}}{y_{f1}}(\dot{y}_{f1} - k_1 y_{f2} + k_2 y_{f1}) \end{aligned} \tag{93}$$

From (92)–(93) it can be deduced that $r_1 = r_2 = 1$ in view of (19)–(20). The equation resulting from setting $u_f = 0$ in (93) can be solved for $y_{f2}^{(r_2)}$

$$\begin{aligned} \dot{y}_{f2} &= (-k_1 y_{f2} - k_3 y_{f2}^2) \\ &\quad - \frac{C_{A0} - y_{f2}}{y_{f1}}(\dot{y}_{f1} - k_1 y_{f2} + k_2 y_{f1}) \end{aligned} \tag{94}$$

This yields $r = r_1$. (92) does not depend on $y_{f2}^{(r_2)} = \dot{y}_{f2}$ and can easily be verified to be the input-output linearizing feedback for (89)–(90). (94) are the internal dynamics for the output (90). For both operation points A and B these are unstable. According to [6] a trajectory y_f^* for the transition has been planned using backwards integration. In Figure 1 the resulting trajectory is shown. The output reference trajectory y_{f1}^* starts at $t = 2\,\mathrm{min}$. Due to the instability of the internal dynamics the trajectory for y_{f2}^* has a noncausal part (see [6]). However, even for very small deviations from the planned trajectory, the internal dynamics converge to another operation point, when the input output linearizing controller is used. The proposed control scheme allows easy switching of the control variables, i.e. the trajectory is stabilized by stabilization of the tracking error for y_{f2}. If (93) with $u_f = 0$ is solved for y_{f1}, one gets

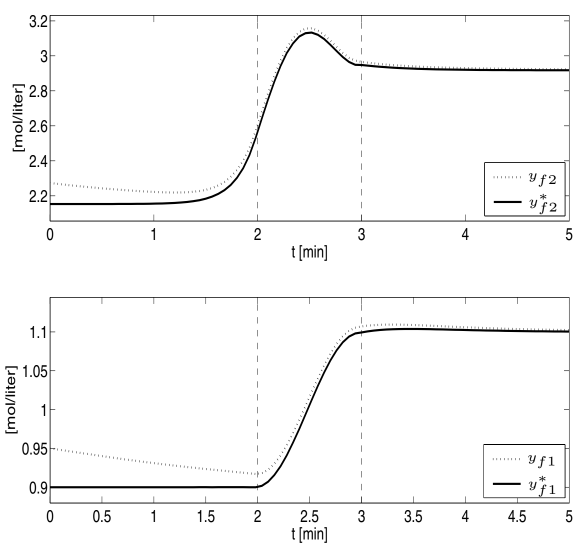

Fig. 1. y_{f1} and y_{f2} trajectories

$$\dot{y}_{f1} = \frac{y_{f1}}{C_{A0} - y_{f2}}(-\dot{y}_{f2} - k_1 y_{f2} - k_3 y_{f2}) + k_1 y_{f2} + k_2 y_{f1} \tag{95}$$

It can be verified that these internal dynamics (for output $y_{f2} = x_1$) are stable for both operation points. Consequently, asymptotic tracking can be achieved using the input-ouput linearizing feedback law for y_{f2}, which results from inserting (95) into (92).

$$u = \frac{1}{C_{A0} - y_{f2}}(\dot{y}_{f2} + k_1 y_{f2} + k_3 y_{f2}^2) \tag{96}$$

The stabilization of the trajectory is then achieved by replacing $\dot{y}_{f2}$ in (96) with

$$\dot{y}_{f2} = \dot{y}_{f2}^* - \lambda_1(y_{f2} - y_{f2}^*) \tag{97}$$

corresponding to (82), where y_{f2}^* and $\dot{y}_{f2}^*$ stem from the original reference trajectory. The result for $\lambda_1 = 35\,\frac{1}{\mathrm{h}}$ can be seen in Figure 1, for an initial error $(0.05\,\frac{\mathrm{mol}}{\mathrm{liter}}, 0.12\,\frac{\mathrm{mol}}{\mathrm{liter}})$. It can be verified that the desired trajectory for y_{f1} is approached asymptotically. It has to be mentioned that system (89) is flat with flat output $y_f = \frac{x_2}{C_{A0} - x_1}$. A tracking controller based on flat feedback thus would obviously require x_1 and x_2 to be measured, whereas for the implementation of the tracking controller (96)–(97) only $y_{f2} = x_1$ has to be measured. Furthermore in [1] a differential parameterization of a non-flat helicopter model could be derived by the introduction of a single fictitious input. This shows that the proposed approach can also be used for non-flat systems.

VIII. Conclusions and Future Work

This contribution clarified the structure of the differential parameterization obtained from the introduction of a fictitious input. The input-output linearizing feedback has been determined from the differential parameterization and an asymptotic tracking controller was derived. Additionally, as shown in the example, the proposed controller design allows amazing flexibility to achieve the tracking. Future work includes the investigation of the case with several fictitious inputs and the extension to MIMO systems.

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