On feedback stabilization of nonlinear systems under quantization

Claudio De Persis

Abstract— **The aim of this note is to show how the results in D. Liberzon, "Hybrid feedback stabilization of systems with quantized signals"**, *Automatica*, **39, 1543-1554, 2003, concerning asymptotic stabilization using quantized feedback, still hold under the assumption of asymptotic stabilizability only. As a consequence, we are able to examine as special interesting cases nonlinear systems which are e.g. globally asymptotically and locally exponentially stabilizable and stabilizable by dynamic observer-based feedback. The results are also discussed for discrete-time nonlinear systems.**

I. INTRODUCTION

As the use of communication networks is spreading in control applications, researches are turning their interest to study – among other phenomena – the effect of converting or coding feedback signals into digital quantities. Most of the efforts (to cite a few, [18], [23], [1], [4], [20], [8], [19], [5], [13], [12], [17], [11], [24], [14], [10], [6], [21], [7], [15], [3], [2]) have relied upon the availability of models of the dynamic system which generates the feedback signal, as opposed to more information-theoretic approaches which view the source of information as a purely statistical one. While some of the works above have focused their attention on *static* encoding, some others have investigated what can be achieved when *dynamic* encoding is allowed. Among the latter, the seminal paper [1] has introduced the so-called zooming-in/zooming-out technique to achieve *asymptotic* stabilization using dynamic quantization. The technique has then been investigated in [12], focusing on nonlinear systems which can be made input-to-state stable with respect to quantization errors. In this note, we point out that the same results of [12] can be achieved even for those nonlinear systems which can be globally asymptotically stabilized by feedback with no encoding, and that is proven by simple modifications of the arguments in [12]. An analogous idea was pursued in [3] to rephrase the results in [14] under the stabilizability assumption. However, the dynamic encoder in [14] is quite different from the zooming-in strategy of [1] and [12], and so are the proofs. As special interesting cases, we examine nonlinear systems which are globally asymptotically and locally exponentially stabilizable and stabilizable by dynamic quantized-output-feedback. For their interest in practical implementation of this approach to quantized feedback, discrete-time nonlinear systems are (succinctly) investigated as well.

In Section II we consider the case in which the quantization affects the state, whereas the problem under input and output quantization is studied in Section III. Discrete-time systems are studied in Section IV. Section V draws the conclusion.

II. STATE QUANTIZATION

We consider systems of the form

$$\dot{x} = f(x, u) \,, \tag{1}$$

$x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, in which the measured state undergoes quantization. We recall the notion of quantization as proposed in [12]. More specifically, the quantization function $q(\cdot) : \mathbb{R}^n \to \mathcal{Q}$, where $\mathcal{Q}$ is a finite subset of $\mathbb{R}^n$, is a function with the properties that [1], for all $|z| \leq M$

$$|q(z) - z| \leq \Delta \,, \tag{2}$$

and, additionally,

$$|q(z)| \leq M \,,$$

where M, Δ are suitable positive constant to be specified later. In order to complete the definition of q, we should specify the values taken by the function outside the ball of radius M. However, assuming without loss of generality that the ball of radius M includes the set of initial conditions, we will see that the state can not leave the ball. In other words, we will be only interested in *semi-global* stabilizability results, and therefore, there is no need in this paper to define the function $q(\cdot)$ for $|z| > M$. However, were we interested in *global* results, the results can be immediately extended to cover also this case by using the zooming-out arguments of [1], [12].

Rather than input-to-state stabilizability with respect to measurement errors as in [12], [14], we shall assume here asymptotic stabilizability as in [3]. In particular, we consider the nonlinear system under quantized feedback

$$\dot{x} = f(x, k(q_\mu(x))) \,, \tag{3}$$

where [12]

$$q_\mu(x) := \mu q(x/\mu) \,,$$

μ is a positive constant, and $k(\cdot)$ is the map for which system

$$\dot{x} = f(x, k(x))$$

is globally asymptotically stable. To be more precise, we assume the following:

Dipartimento di Informatica e Sistemistica "A. Ruberti", Università di Roma "La Sapienza", Via Eudossiana 18, 00184 Roma, Italy, `depersis@dis.uniroma1.it`

[1]We refer the reader to [12] for more details.

Assumption 1: There exist smooth functions $V(\cdot) : \mathbb{R}^n \to \mathbb{R}_+$ and $k(\cdot) : \mathbb{R}^n \to \mathbb{R}^m$ for which

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|)$$
$$\frac{\partial V}{\partial x} f(x, k(x)) \leq -\alpha_3(|x|) , \tag{4}$$

for all $x \in \mathbb{R}^n$, with $\alpha_1(\cdot)$, $\alpha_2(\cdot)$ and $\alpha_3(\cdot)$ suitable class-$\mathcal{K}_\infty$ functions.

The signal available to the controller is the quantized feedback $q_\mu(x)$, and hence there will be a discrepancy between the actual closed-loop system and the system with the desired asymptotic properties. This discrepancy can be described by the following expression (see e.g. [3])

$$f(x, q_\mu(x)) = f(x, k(x)) + g(x, q_\mu(x))\mu(q(x/\mu) - x/\mu) , \tag{5}$$

with $g(\cdot, \cdot)$ a suitable smooth function, such as e.g.

$$g(x, \bar{x}) = -\int_0^1 \left[\frac{\partial f(x, y)}{\partial y}\right]_{y = \alpha \bar{u} + (1-\alpha)u} d\alpha \cdot$$
$$\cdot \int_0^1 \left[\frac{\partial k(y)}{\partial y}\right]_{y = \alpha \bar{x} + (1-\alpha)x} d\alpha ,$$

where $u = k(x)$ and

$$\bar{u} = k(q_\mu(x)) .$$

In what follows, the function below will be useful

$$\pi(r) := \max_{|x| \leq r} \left|\frac{\partial V(x)}{\partial x}\right| \max_{|x|,|y| \leq r} |g(x, y)| . \tag{6}$$

As in [12], the stabilization results descend from a Lemma which characterizes the convergence in finite time of the state of the system from an outer level set to an inner level set. In particular, the lemma below is a version of Lemma 2 in [12] when the assumption of input-to-state stabilizability with respect to measurements (quantization) errors is replaced by an asymptotic stabilizability assumption. Namely, we have:

Lemma 1: Let Assumption 1 hold. For any $\bar{\mu} > 0$, any $M > 0$, any $\bar{t} \in \mathbb{R}$, if

$$V(x(\bar{t})) \leq \alpha_1(\mu M) ,$$

and

$$\alpha_1(\lambda \mu M) \geq \alpha_2 \circ \alpha_3^{-1}(2\pi(\mu M) \cdot \mu \Delta) , \tag{7}$$

for some $0 < \lambda < 1$ and for some $\mu \in (0, \bar{\mu}]$, then

$$V(x(\bar{t} + t)) \leq \alpha_1(\mu M) \tag{8}$$

for all $0 \leq t < T(\mu M, \mu \Delta)$, and

$$V(x(\bar{t} + t)) \leq \alpha_2 \circ \alpha_3^{-1}(2\pi(\mu M) \cdot \mu \Delta) \tag{9}$$

for all $t \geq T(\mu M, \mu \Delta)$, with

$$T(\mu M, \mu \Delta) := \frac{\alpha_1(\mu M) - \alpha_2 \circ \alpha_3^{-1}(2\pi(\mu M)\mu \Delta)}{\pi(\mu M)\mu \Delta} . \tag{10}$$

Proof: The result is proven as Lemma 2 in [12], replacing $\rho(\Delta \mu)$ in [12] with $\alpha_3^{-1}(2\pi(\mu M) \cdot \mu \Delta)$. For

convenience of the reader, the proof is reported below. Note that $V(x) \leq \alpha_1(\mu M)$ implies $|x| \leq \mu M$ and hence

$$|q(x/\mu) - x/\mu| \leq \Delta$$
$$|q(x/\mu)| \leq M .$$

Therefore,

$$\dot{V}(x) := \frac{\partial V}{\partial x} f(x, q_\mu(x)) \leq$$
$$\leq -\alpha_3(|x|) + \pi(\mu M)\mu \Delta$$
$$\leq -\alpha_3 \circ \alpha_2^{-1}(V(x)) + \pi(\mu M)\mu \Delta .$$

Consider now the set

$$\mathcal{S} := \{x \in \mathbb{R}^n :$$
$$\alpha_2 \circ \alpha_3^{-1}(2\pi(\mu M)\mu \Delta) \leq V(x) \leq \alpha_1(\mu M)\} .$$

Note that condition (7) guarantees the set $\mathcal{S}$ not to be void. Then, for all $x \in \mathcal{S}$,

$$\dot{V}(x) \leq -\frac{1}{2}\alpha_3 \circ \alpha_2^{-1}(V(x)) \leq -\pi(\mu M)\mu \Delta , \tag{11}$$

the latter inequality being true as

$$x \in \mathcal{S} \Rightarrow V(x) \geq \alpha_2 \circ \alpha_3^{-1}(2\pi(\mu M)\mu \Delta)$$
$$\Downarrow$$
$$\frac{1}{2}\alpha_3 \circ \alpha_2^{-1}(V(x)) \geq \pi(\mu M)\mu \Delta .$$

Bearing in mind that $V(x(\bar{t})) \leq \alpha_1(\mu M)$, if $x(\bar{t}) \notin \mathcal{S}$, then the inequality (9) holds for all $t \geq 0$, for, otherwise, inequality (11) would be contradicted. Inequality (9) also implies inequality (8). On the other hand, if $x(\bar{t}) \in \mathcal{S}$, for all the times $t \geq 0$ for which

$$V(x(\bar{t} + t)) \geq \alpha_2 \circ \alpha_3^{-1}(2\pi(\mu M)\mu \Delta) ,$$

it is also true that (see the second inequality in (11))

$$\dot{V}(x(\bar{t} + t)) \leq -\pi(\mu M)\mu \Delta ,$$

or, integrating,

$$V(x(\bar{t} + t)) \leq \alpha_1(\mu M) - \pi(\mu M)\mu \Delta t .$$

Note now that $T(\mu M, \mu \Delta) > 0$ by condition (7). It is then straightforward to verify that, if $t \geq T(\mu M, \mu \Delta)$, then

$$V(x(\bar{t} + T(\mu M, \mu \Delta))) \leq \alpha_2 \circ \alpha_3^{-1}(2\pi(\mu M)\mu \Delta) ,$$

from which both (8) and (9) follow immediately. ∎

A result analogous to Theorem 2 in [12] can be given:

Proposition 1: Let Assumption 1 hold. For any $X > 0$, let $\bar{\mu} > 0$, $M > 0$ be such that $\bar{\mu} M \geq \alpha_1^{-1} \circ \alpha_2(X)$. If

$$\alpha_1(\lambda \mu M) \geq \alpha_2 \circ \alpha_3^{-1}(2\pi(\mu M) \cdot \mu \Delta) , \quad \forall \mu \in [0, \bar{\mu}] , \tag{12}$$

for some $0 < \lambda < 1$, then there exists a hybrid quantized feedback control policy that makes system (3) locally asymptotically stable and, moreover,

$$|x(0)| \leq X \quad \Rightarrow \quad \lim_{t \to \infty} |x(t)| = 0 .$$

Proof: See [12]. ∎

Remark. We briefly recall the hybrid control strategy proposed in [12], as it will be useful later. Note first that $|x(0)| \leq$

X and $\bar{\mu} M \geq \alpha_1^{-1} \circ \alpha_2(X)$ imply $V(x(0)) \leq \alpha_1(\bar{\mu} M)$. Define the sequence of positive real numbers

$$
\begin{aligned}
\mu_0 &= \bar{\mu} \\
\mu_{j+1} &= \frac{1}{M} \alpha_1^{-1} \circ \alpha_2 \circ \alpha_3^{-1}(2\pi(\mu_j M)\mu_j \Delta) ,
\end{aligned} \quad (13)
$$

and any sequence of times $\{t_j\}_{j \in \mathbb{N}}$ satisfying

$$
\begin{aligned}
t_0 &= 0 \\
t_{j+1} &\geq t_j + T(\mu_j M, \mu_j \Delta) .
\end{aligned} \quad (14)
$$

Define also the control law

$$
u(t) = k(q_{\mu_j}(x(t))) , \quad t \in [t_j, t_{j+1}) , \ j \in \mathbb{N} .
$$

Then, repeated application of Lemma 1 shows that

$$
V(x(t)) \leq \alpha_1(\mu_j M) , \quad \forall t \geq t_j , \quad \forall j \in \mathbb{N} .
$$

This in particular implies that $|x(t)| \leq \mu_j M$ for all $t \geq t_j$, for each $j \in \mathbb{N}$, with $\mu_j \to 0$ as $j \to +\infty$. In fact, by definition (13),

$$
\alpha_1(M\mu_{j+1}) = \alpha_2 \circ \alpha_3^{-1}(2\pi(\mu_j M)\mu_j \Delta) .
$$

For $k = 0$, $\mu_0 = \bar{\mu}$, and by (12),

$$
\alpha_1(M\mu_1) = \alpha_2 \circ \alpha_3^{-1}(2\pi(\mu_0 M)\mu_0 \Delta) \leq \alpha_1(\lambda \mu_0 M) ,
$$

that is $\mu_1 \leq \lambda \mu_0 < \bar{\mu}$. Let $\underline{j} \in \mathbb{N}$ be such that $\mu_{j+1} \leq \lambda \mu_j$, $j = 0, 1, \ldots, \underline{j}$. By (13)

$$
\alpha_1(M\mu_{\underline{j}+2}) = \alpha_2 \circ \alpha_3^{-1}(2\pi(\mu_{\underline{j}+1} M)\mu_{\underline{j}+1} \Delta) .
$$

As $\mu_{\underline{j}+1} \leq \lambda^{\underline{j}+1}\bar{\mu} < \bar{\mu}$, (12) yields $\mu_{\underline{j}+2} \leq \lambda \mu_{\underline{j}+1} < \bar{\mu}$. That is, $\mu_{j+1} = \lambda \mu_j$, for each $j \in \mathbb{N}$. ◁

As in [12], the main obstacle in the application of this result descends from condition (12), which – depending on the expression of the comparison functions involved – may or may not be satisfied for all $\mu \in [0, \bar{\mu}]$. In order to avoid sluggish response as the state is approaching the origin, we also would like to have $T(\mu M, \mu \Delta) < +\infty$ as $\mu \to 0^+$. (The eventuality for $T(\mu M, \mu \Delta)$ to go to zero does not raise any problem, for the state is guaranteed to belong to an inner level set for *all* the times *after* $t_j + T(\mu M, \mu \Delta)$.) Of course, as far as the first issue is concerned, the same considerations given in [12] after Theorem 2, apply also to this case. For instance, bearing in mind that the results stated above hold if the term $2\pi(\mu M)$ in (10) and (12) is replaced by the constant

$$
\bar{\pi} := 2\pi(\bar{\mu} M) ,
$$

which is independent of μ, then, as in [12], we can state that condition (12) is fulfilled for all $\mu \in [0, \bar{\mu}]$ provided that e.g. (cf. (30) in [12])

$$
(\alpha_1^{-1} \circ \alpha_2 \circ \alpha_3^{-1})'(0) < \infty .
$$

However, this condition is not necessary, and there are important classes of nonlinear systems which do not satisfy this condition but for which condition (12) is fulfilled. In what follows, we point out one of such classes: The class of nonlinear systems which are globally asymptotically and locally *exponentially* stabilizable. That is, in addition to Assumption 1, we require:

Assumption 2: There exist real numbers $a_i > 0$, $i = 1, \ldots, 4$ and $\chi > 0$ for which $\alpha_i(r) = a_i \bar{r}^2$ for all $r \in [0, \chi]$. Furthermore,

$$
\left| \frac{\partial V}{\partial x} \right| \leq a_4 |x|
$$

for all $|x| \in [0, \chi]$. ◁

Remark. Important classes of systems are globally asymptotically and locally exponentially stable. For instance, systems of the form

$$
\begin{aligned}
\dot{x}_1 &= A_1 x_1 + p_1(x_1, x_2, \ldots, x_n) + b_1(x_1, x_2, \ldots, x_n)u \\
\dot{x}_2 &= A_2 x_2 + p_2(x_2, x_3, \ldots, x_n) + b_2(x_2, x_3, \ldots, x_n)u \\
&\vdots \\
\dot{x}_{n-1} &= A_{n-1} x_{n-1} + p_{n-1}(x_{n-1}, x_n) + b_{n-1}(x_{n-1}, x_n)u \\
\dot{x}_n &= \varphi(x_n) + \theta(x_n)u ,
\end{aligned}
$$

under suitable technical conditions (e.g. all the matrices A_i are stable in the sense of Lyapunov, functions p_i, b_i satisfy appropriate growth conditions, the equilibrium $x_n = 0$ of the system $\dot{x}_n = \varphi(x_n)$ is globally asymptotically and locally exponentially stable, etc.), can be made globally asymptotically and locally exponentially stable by a smooth control law (see e.g. [9]).

We have the following:

Corollary 1: Let Assumptions 1 and 2 hold. For any $X > 0$, let $\bar{\mu}$ and M be such that $\bar{\mu} M \geq \alpha_1^{-1} \circ \alpha_2(X)$. Then there exists Δ for which

$$
\alpha_1(\lambda \mu M) \geq \alpha_2 \circ \alpha_3^{-1}(2\pi(\mu M) \cdot \mu \Delta) , \quad \forall \mu \in [0, \bar{\mu}] , \quad (15)
$$

for some $0 < \lambda < 1$, and there exists a hybrid quantized feedback control policy that makes the system (3) locally asymptotically stable and, moreover,

$$
|x(0)| \leq X \quad \Rightarrow \quad \lim_{t \to \infty} |x(t)| = 0 .
$$

Proof: It is enough to verify that a number Δ for which (15) is fulfilled always exists. As mentioned in the remark after Proposition 1, $|x(t)| \leq \lambda^j \bar{\mu} M$, for all $t \geq t_j$, with t_j defined as in (14). Therefore, there exists a non-negative integer j_* for which $|x(t)| \leq \chi$, for all $t \geq t_{j_*}$. Notice that there always exists a number Δ_1 for which (15) is fulfilled for all $\mu \in [\mu_{j_*-1}, \bar{\mu}]$, as μ is ranging over a compact interval and is bounded away from zero. From time t_{j_*} on, we can still repeatedly apply Lemma 1, provided that condition (15) is met for all $\mu \in [0, \mu_{j_*-1})$. But this is actually the case, because the comparison functions can be replaced by their quadratic expressions, and the function $\pi(\mu M)$ can be replaced by the linear function $a_5(\chi)\mu M$, where

$$
a_5(\chi) = a_4 \max_{|x|,|y| \leq \chi} |g(x,y)| .
$$

Condition (15) then becomes

$$
a_1 \lambda^2 M \geq 2\frac{a_2}{a_3} a_5(\chi)\Delta , \quad (16)
$$

which, being independent of μ, can always be satisfied by an appropriate choice – say Δ_2 – of Δ. Then the thesis holds setting $\Delta = \min\{\Delta_1, \Delta_2\}$. ∎

Remark. We also observe that in the present case there is no possibility to have a sluggish response as $\mu \to 0^+$. In fact, with the same arguments as before, it can be seen that, from time t_{j^*} on, the state is guaranteed to enter a smaller level set after a *constant* period of time whose length is given by the expression:

$$\bar{T}(M, \Delta) := \frac{a_1 M}{a_5(\chi)\Delta} - 2\frac{a_2}{a_3} > 0 . \tag{17}$$

III. INPUT AND OUTPUT QUANTIZATION

In the following subsections we see how the result stated in the case of state quantization still hold in the case the input and, respectively, the output are quantized. The two cases are treated similarly and succinctly, with most of the details omitted.

A. Input quantization

As in [12], the results can be modified to deal with the case in which quantization affects the control input rather than the state. In this case, the closed-loop control system takes the form

$$\dot{x} = f(x, q_\mu(k(x))) .$$

As in the previous section, it is convenient to rewrite the right-hand side as:

$$f(x, q_\mu(k(x)) = f(x, k(x)) + \\ + \ell(x, q_\mu(k(x)))(q_\mu(k(x)) - k(x)) ,$$

with

$$\ell(x, q_\mu(k(x))) = -\int_0^1 \left[\frac{\partial f(x, y)}{\partial y}\right]_{y=\alpha q_\mu(k(x))+(1-\alpha)k(x)} d\alpha .$$

Also in the present case, we set, with a slight abuse of notation,

$$\pi(r) := \max_{|x|\leq r}\left|\frac{\partial V(x)}{\partial x}\right| \max_{|x|\leq r}|\ell(x, q_\mu(k(x)))| ,$$

and, as in [12], we let $\kappa(\cdot)$ be a class-$\mathcal{K}_\infty$ function such that

$$|k(x)| \leq \kappa(|x|) , \quad \forall x .$$

Then, Lemma 4 and Theorem 4 in [12] hold under stabilizability assumption only:

Lemma 2: Let Assumption 1 hold. For any $\bar{\mu} > 0$, for any $M > 0$, any $\bar{t} \in \mathbb{R}$, if

$$V(x(\bar{t})) \leq \alpha_1 \circ \kappa^{-1}(\mu M) ,$$

and

$$\alpha_1 \circ \kappa^{-1}(\lambda\mu M) \geq \alpha_2 \circ \alpha_3^{-1}(2\pi(\mu M) \cdot \mu\Delta) ,$$

for some $0 < \lambda < 1$ and for some $\mu \in (0, \bar{\mu}]$, then

$$V(x(\bar{t} + t)) \leq \alpha_1 \circ \kappa^{-1}(\mu M)$$

for all $0 \leq t < T(\mu M, \mu\Delta)$, and

$$V(x(\bar{t} + t)) \leq \alpha_2 \circ \alpha_3^{-1}(2\pi(\mu M) \cdot \mu\Delta)$$

for all $t \geq T(\mu M, \mu\Delta)$, with

$$T(\mu M, \mu\Delta) := \frac{\alpha_1 \circ \kappa^{-1}(\mu M) - \alpha_2 \circ \alpha_3^{-1}(2\pi(\mu M)\mu\Delta)}{\pi(\mu M)\mu\Delta} .$$

Proof: It is straightforward from Lemma 1 above and Lemma 4 in [12]. ∎

Proposition 2: Let Assumption 1 hold. For any $X > 0$, let $\bar{\mu} > 0$, $M > 0$ be such that $\bar{\mu}M \geq \kappa \circ \alpha_1^{-1} \circ \alpha_2(X)$. If

$$\alpha_1 \circ \kappa^{-1}(\lambda\mu M) \geq \alpha_2 \circ \alpha_3^{-1}(2\pi(\mu M) \cdot \mu\Delta) , \quad \forall \mu \in [0, \bar{\mu}] ,$$

for some $0 < \lambda < 1$, then there exists a hybrid quantized feedback control policy that makes system (3) locally asymptotically stable and, moreover,

$$|x(0)| \leq X \quad \Rightarrow \quad \lim_{t\to\infty} |x(t)| = 0 .$$

Proof: The result descends from the previous Lemma and the same arguments of Theorem 1 in [12]. ∎

Remark. It is also possible to give the analogous of Corollary 1 in the case of input quantization. This would require the additional condition for the function $\kappa(\cdot)$ to be $\mathcal{O}(r)$ near $r = 0$. ◁

B. Output quantization

The approach chosen in Section II allows us also to address rather straightforwardly the case when only output measurements are available, i.e. the case when the readout map is different from the identity:

$$\begin{aligned} \dot{x} &= f(x, u) \\ y &= h(x) \end{aligned} \tag{18}$$

We assume that:

Assumption 3: An observer-based dynamic controller of the form

$$\begin{aligned} \dot{\hat{x}} &= f(\hat{x}, u) + g(\hat{x}, y, u)(y - h(\hat{x})) \\ u &= k(\hat{x}) , \end{aligned} \tag{19}$$

exists and globally asymptotically stabilizes system (18).

Remark. There are several classes of nonlinear systems for which this assumption is satisfied. For a recent and quite general characterization of nonlinear systems which, under the assumptions of asymptotic stabilizability and uniform detectability, can be globally asymptotically stabilized by dynamic observer-based controllers, we refer the reader to the paper [22]. ◁

The measurement y being available only in quantized form, to actually implement the dynamic controller above, y must be replaced by the quantity

$$q_{\bar{h}\mu}(y) = \bar{h}\mu q\left(\frac{y}{\bar{h}\mu}\right) ,$$

with $\bar{h}$ a positive constant to be specified later. Borrowing the notation in [12], the closed-loop system takes the form

$$\dot{x}^e = f^e(x^e) + \begin{bmatrix} 0 \\ g(\hat{x}, q_{\bar{h}\mu}(y), k(\hat{x})) \end{bmatrix} (q_{\bar{h}\mu}(y) - h(x)) ,$$

where $k(\cdot)$ is as in Assumption 1, and the unforced system $\dot{x}^e = f^e(x^e)$ satisfies Assumption 1, where x, V, and $f(x, k(x))$ are replaced respectively by x^e, V^e, and $f^e(x^e)$. Analogously to the previous sections, we introduce the function:

$$\pi(r) := \max_{|x^e| \leq r} \left| \frac{\partial V^e(x^e)}{\partial \hat{x}} \right| \max_{|x| \leq r, |y| \leq \bar{h}r} |g(x, y, k(x))| \ .$$

Then the following Lemma can be immediately stated:

Lemma 3: Let Assumptions 1 and 3 hold. For any $\bar{\mu} > 0$, any $M > 0$, any $\bar{t} \in \mathbb{R}$, if $V^e(x^e(\bar{t})) \leq \alpha_1(\mu M)$, and

$$\alpha_1(\lambda \mu M) \geq \alpha_2 \circ \alpha_3^{-1}(2\pi(\mu M) \cdot \bar{h}\mu\Delta) \ ,$$

for some $0 < \lambda < 1$ and some $\mu \in (0, \bar{\mu}]$, and where $\bar{h}$ is the Lipschitz constant for $h(x)$ when $|x| \leq \bar{\mu}M$, then

$$V^e(x^e(\bar{t} + t)) \leq \alpha_1(\mu M)$$

for all $0 \leq t < T(\mu M, \mu\Delta)$, and

$$V^e(x^e(\bar{t} + t)) \leq \alpha_2 \circ \alpha_3^{-1}(2\pi(\mu M) \cdot \bar{h}\mu\Delta)$$

for all $t \geq T(\mu M, \mu\Delta)$, with

$$T(\mu M, \mu\Delta) := \frac{\alpha_1(\mu M) - \alpha_2 \circ \alpha_3^{-1}(2\pi(\mu M)\bar{h}\mu\Delta)}{\pi(\mu M)\bar{h}\mu\Delta} \ .$$

Proof: The proof is immediately obtained from Lemma 1 by replacing $\mu\Delta$ there with $\bar{h}\mu\Delta$. ∎

Also the analogous of Proposition 1 can be stated quite straightforwardly:

Proposition 3: Let Assumptions 1 and 3 hold. For any $X > 0$, let $\bar{\mu} > 0$, $M > 0$ be such that $\bar{\mu}M \geq \alpha_1^{-1} \circ \alpha_2(X)$. If

$$\alpha_1(\lambda \mu M) \geq \alpha_2 \circ \alpha_3^{-1}(2\pi(\mu M) \cdot \bar{h}\mu\Delta) \ , \quad \forall \mu \in [0, \bar{\mu}] \ ,$$

for some $0 < \lambda < 1$, then there exists a hybrid quantized feedback control policy that makes system (3) locally asymptotically stable and, moreover,

$$|x(0)| \leq X \quad \Rightarrow \quad \lim_{t \to \infty} |x(t)| = 0 \ .$$

Proof: See [12]. ∎

Remark. It is immediate to see that also an analogous of Corollary 1 holds in the case in which output quantization is being used. Classes of systems which admit dynamic observer-based feedback able to globally asymptotically and locally exponentially stabilize the system include for instance those considered in [16]. ◁

IV. QUANTIZED STABILIZATION OF NONLINEAR DISCRETE-TIME SYSTEMS

For practical implementation of the schemes examined above, it may be useful to consider how the previous results can be translated for nonlinear discrete-time systems. Consider the system

$$x(t+1) = f(x(t), u(t)) \ , \quad t \in \mathbb{Z}$$

and assume the following discrete-time counterpart of Assumption 1:

Assumption 4: There exist smooth functions $V(\cdot) : \mathbb{R}^n \to \mathbb{R}^+$ and $k(\cdot) : \mathbb{R}^n \to \mathbb{R}^m$ for which

$$\begin{aligned} \alpha_1(|x|) &\leq V(x) \leq \alpha_2(|x|) \\ V(f(x, k(x))) - V(x) &\leq -\alpha_3(|x|) \ , \end{aligned} \tag{20}$$

for all $x \in \mathbb{R}^n$, with $\alpha_1(\cdot)$, $\alpha_2(\cdot)$ and $\alpha_3(\cdot)$ suitable class-$\mathcal{K}_\infty$ functions.

Set now:

$$v(x, q_\mu(x)) := \int_0^1 \left[\frac{\partial V(f(x, y))}{\partial y} \right]_{y = \alpha k(q_\mu(x)) + (1-\alpha)k(x)} d\alpha \cdot$$
$$\cdot \int_0^1 \left[\frac{\partial k(y)}{\partial y} \right]_{y = \alpha q_\mu(x) + (1-\alpha)x} d\alpha$$

and

$$\pi(r) := \max_{|x|, |y| \leq r} |v(x, y)| \ .$$

Then the following holds:

Lemma 4: Let Assumption 4 hold. For any $\bar{\mu} > 0$, any $M > 0$, any $\bar{t} \in \mathbb{Z}_+$, if

$$V(x(\bar{t})) \leq \alpha_1(\mu M) \ ,$$

and

$$\alpha_1(\lambda \mu M) \geq \alpha_2 \circ \alpha_3^{-1}(2\pi(\mu M) \cdot \mu\Delta) \ , \tag{21}$$

for some $0 < \lambda < 1$ and for some $\mu \in (0, \bar{\mu}]$, then, for some $\bar{t} \leq t_* \leq K(\mu M, \mu\Delta)$,

$$V(x(t)) \leq \alpha_1(\mu M)$$

for all $\bar{t} \leq t < t_*$, and

$$V(x(t_*)) \leq \alpha_2 \circ \alpha_3^{-1}(2\pi(\mu M) \cdot \mu\Delta)$$

where $K(\mu M, \mu\Delta)$ is the minimal integer not smaller than

$$\frac{\alpha_1(\mu M) - \alpha_2 \circ \alpha_3^{-1}(2\pi(\mu M)\mu\Delta)}{\pi(\mu M)\mu\Delta} + \bar{t} \ . \tag{22}$$

Moreover, if (21) is replaced by

$$\alpha_1(\lambda \mu M) \geq \alpha_2 \circ \alpha_3^{-1}(2\pi(\mu M) \cdot \mu\Delta) + \pi(\mu M) \cdot \mu\Delta \ , \tag{23}$$

then

$$V(x(t)) \leq \alpha_2 \circ \alpha_3^{-1}(2\pi(\mu M) \cdot \mu\Delta) + \pi(\mu M) \cdot \mu\Delta$$

for all t greater than or equal to (22).

Proof: (Sketch) As far as $V(x) \leq \alpha_1(\mu M)$, the Lyapunov function satisfies

$$V(f(x, k(q_\mu(x)))) - V(x) \leq -\alpha_3(|x|) + \pi(\mu M)\mu\Delta \ .$$

As in the proof of Lemma 1, one proves that, along the solution of the closed-loop system

$$x(t+1) = f(x(t), k(q_\mu(x(t)))) \ ,$$

as far as $V(x(t)) \geq \alpha_2 \circ \alpha_3^{-1}(2\pi(\mu M) \cdot \mu\Delta)$, the Lyapunov function satisfies

$$V(x(t+1)) - V(x(t)) \leq -\pi(\mu M)\mu\Delta \ .$$

From this, it is immediate to conclude the first part of the statement. On the other hand, if

$$V(x(t)) < \alpha_2 \circ \alpha_3^{-1}(2\pi(\mu M) \cdot \mu\Delta) \ ,$$

then

$$V(x(t+1)) \leq \alpha_2 \circ \alpha_3^{-1}(2\pi(\mu M) \cdot \mu\Delta) + \pi(\mu M)\mu\Delta \ ,$$

that is, by (23), $V(x(t+1)) < \alpha_1(\mu M)$ and this proves the thesis. ∎

To the purpose of achieving asymptotic convergence, this result can be iteratively used in two different ways. One is illustrated in [1] and requires that, as soon as $V(x) \leq \alpha_2 \circ \alpha_3^{-1}(2\pi(\mu M) \cdot \mu\Delta)$, the parameter μ is updated according to the law proposed in [12] and recalled in Section 2. To this purpose only condition (21) must be satisfied. The other way of using the result requires (23) to be fulfilled, and the parameter μ to be updated as

$$
\begin{aligned}
\mu_0 &= \bar{\mu} \\
\mu_{j+1} &= \frac{\alpha_1^{-1}(\alpha_2 \circ \alpha_3^{-1}(2\pi(\mu_j M)\mu_j\Delta) + \pi(\mu_j M)\mu_j\Delta)}{M} \ ,
\end{aligned}
\tag{24}
$$

at the discrete times $t_j \in \mathbb{Z}_+$ satisfying

$$
\begin{aligned}
t_0 &= 0 \\
t_{j+1} &\geq t_j + \frac{\alpha_1(\mu_j M) - \alpha_2 \circ \alpha_3^{-1}(2\pi(\mu_j M)\mu_j\Delta)}{\pi(\mu_j M)\mu_j\Delta} \ .
\end{aligned}
$$

Note that the lemma guarantees that, if $V(x(t_j)) \leq \alpha_1(\mu_j M)$, then $V(x(t_{j+1})) \leq \alpha_1(\mu_{j+1} M)$, and by (24) and (23), $V(x(t_{j+1})) \leq \alpha_1(\lambda\mu_j M)$, which shows that the magnitude of the state is actually decreased by a factor $\lambda < 1$ at each sampling time. Results analogous to the previous lemma, which may allow to conclude asymptotic convergence, can be drawn in the case of input and output quantization, in the same way as in the previous section.

V. CONCLUSION

We have shown how the so-called zooming-in technique introduced in the papers [1], [12] can also be used to deal with nonlinear systems which are asymptotically stabilizable. This allowed us to investigate specific cases which were not considered before. In particular, several good features of the zooming-in technique pointed out for the case of linear systems also hold for nonlinear systems which exhibit *locally* an exponentially stable behavior under appropriate control. In the case in which only output measurements are available, we have seen that the zooming-in technique can be applied to observer-based dynamic feedback controllers. In view of a possible practical implementation of the technique, we have also studied the same problem for nonlinear discrete-time systems. Along the way, we have pointed out examples of classes of systems to which the results of the paper can be applied. Discussion on chattering and the conservativeness of the approach goes beyond the scope of the paper.

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