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Robust root-clustering of a matrix in intersections or unions of regions: Addendum

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Abstract— **This paper considers robust stability analysis for a matrix affected by LFT-based complex uncertainty (LFT for linear fractional transformation). A method is proposed to compute a bound on the amount of uncertainty ensuring robust root-clustering in a combination (intersection and/or union) of several possibly nonsymmetric half planes, discs, and exteriors of discs. In some cases to be detailed, this bound is not conservative. The conditions are expressed in terms of (linear matrix inequalities) LMIs and derived through Lyapunov's second method. As a distinctive feature of the approach, the Lyapunov matrices proving robust root-clustering (one per subregion) are not necessarily positive definite, but have prescribed inertias depending on the number of roots in the corresponding subregions. As a special case, when root-clustering in a single half plane, disc or exterior of a disc is concerned, the whole clustering region reduces to only one convex subregion and the corresponding unique Lyapunov matrix has to be positive definite as usual. The extension to polytopic LFT-based uncertainty is also addressed.**

I. Introduction

Robust stability has been raising much interest in the last three decades. Indeed, in a linear state-space context, it matters to attest whether an uncertain state matrix has its eigenvalues in the open left half plane (OLHP) for continuous-time analysis or in the open unit disc (OUD) for discrete-time analysis. More precisely, assuming nominal stability, it can be useful to estimate the maximal size of the uncertainty domain for which stability is preserved.

The way to estimate this size obviously depends on the form of the uncertainty. The structured (parametric) case should be distinguished from the unstructured (nonparametric) one as pointed out in one of the first contributions due to Patel and Toda [24]. The present contribution is restricted to a rather unstructured uncertainty, namely the so-called LFT-based uncertainty (LFT for linear fractional transformation). In this context, the maximal acceptable size of uncertainty was clearly defined, in continuous-time, as the complex stability radius [16]. Such a stability radius was shown to equal the reciprocal of the $\mathcal{H}_\infty$-norm of a proper transfer in [20] and, thus, also appears to be the reciprocal of the maximal structured singular value μ [11] along frequency. The discrete-time counterpart is described in [23]. In these references, the computation of the stability radius could be carried out with iterative solving of Riccati equations [8], [9]. Another technique consists in computing μ while sweeping frequencies. With the emergence of convex optimization over linear matrix inequalities (LMIs), the stability radius can be computed owing to the LMI version of the bounded real lemma [1].

It is also important to differentiate between the complex stability radius and the real one. The former concerns a complex uncertainty and can be computed with LMI software as just mentioned. The latter takes the realness of the uncertainty into account (what is more discerning to analyze practical plants in automatic control) and is a bit more difficult to obtain [26], [15]. The present contribution is restricted to the complex case.

When further performances are required, such as transient ones, it might be shrewd to consider a more sophisticated region for the state matrix root-clustering, different from the OLHP or the OUD. Based on the notions of Ω-regions and generalized Lyapunov equations (GLEs) due to Gutman and Jury [13], Yedavalli has proposed significant robustness bounds [34] (later improved by other authors) but the results are still conservative. The reader is also invited to see [32]. Moreover, the considered regions are usually connected, which might not be suitable for plants with separate dynamics or with specified robust damping ratios. One of the first attempts to consider unions of regions is provided in [2]. The concept of $\mathcal{D}_U$-stability (root-clustering in a region $\mathcal{D}_U$ whose form encompasses many unions of possible disjoint and nonsymmetric subregions) enables more general results [4]. However, these results remain quite conservative.

This paper is an attempt to consider sophisticated clustering regions by extending the notion of complex stability radius to some combinations (unions and/or intersections) of half planes, discs, and exteriors of discs. Besides, the conservatism of the previously proposed methods is reduced. In some typical cases to be further detailed, the exact value of the complex radius is reached.

This paper must be considered both as reminder and a sequel of journal paper [3]. It is organized as follows: section 2 states the problem, presenting the clustering regions and extending the concept of complex $\mathcal{D}$-stability radius to the case where $\mathcal{D}$ is some combination of regions. Section 3 introduces the notion of $\partial\mathcal{D}$-regularity of a nominal matrix, which is the nonmembership of the matrix eigenvalues to a geometric curve $\partial\mathcal{D}$. Such a property can

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often be checked through the derivation of a Lyapunov matrix which is not necessarily positive definite but is just nonsingular with constant inertia. When $\mathcal{D}$-stability is concerned, the Lyapunov matrix is required to be strictly positive or negative definite. In section 4, the considered matrix is affected by an LFT-based complex uncertainty. Based upon the notion of $\partial\mathcal{D}$-regularity, a method to reach the complex $\mathcal{D}$-stability radius is proposed. In some cases to be detailed, the exact value is obtained. In section 5, the polytopic LFT uncertainty is considered. A numerical example is provided in section 6 before the conclusion. Some proofs are omitted for the sake of conciseness. The reader is invited to refer to [3] for missing technical details and further references.

Notations

M' denotes the transpose conjugate of matrix M. Hence, s' is the conjugate of complex number s. M^H is the Hermitian matrix $M+M'$. The 2-norm of M induced by the Euclidean vector norm (maximal singular value) is denoted by $||M||_2$. $\mathbb{I}_n$ is the identity matrix of order n and 0 is a null matrix of appropriate dimensions. Matrix inequalities are considered in the sense of Löwner *i.e.* > 0 (resp. < 0) means positive (resp. negative) definite. Symbol $\mathbf{i}$ denotes the imaginary unit and $\lambda(A)$ denotes the spectrum of square matrix A. At last, the vector $\text{In}(M) = [n_+ \ n_- \ n_0]$ is the inertia of a square matrix M if n_+, n_-, n_0 are the numbers of eigenvalues of M with positive, negative and zero real part, respectively.

II. Problem statement

First, the form of the uncertain matrix to be analyzed is given. Then, the clustering region is introduced. At last, the problem to be solved is stated.

A. The uncertain matrix

The considered matrix reads:

$$A_c = A + B\bar{\Delta}C \ \in \mathbb{C}^{n\times n}, \quad \text{with} \quad \bar{\Delta} = \Delta(I - D\Delta)^{-1}. \tag{1}$$

In the above expression, uncertainty Δ is constant, unknown and assumed to belong to $\mathcal{B}(\rho)$, the ball of all matrices $\Delta \in \mathbb{C}^{q\times r}$ satisfying $||\Delta||_2 \leq \rho$. Matrices $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{n\times q}$, $C \in \mathbb{C}^{r\times n}$, and $D \in \mathbb{C}^{n\times q}$ are known. Such a description is referred to as LFT uncertainty.

Remark 2.1: In this work, all the matrices are assumed to be complex. However, in practice, state matrix A_c is real. In that case, Δ is then real and $\mathcal{B}(\rho)$ must be restricted to real matrices. Actually, this restriction is more difficult to take into account and this case is not investigated in the paper although some interesting results exist [26], [15].

B. Clustering region $\mathcal{D}$

Consider the following geometric curves $\forall k \in \{1, ..., m\}$:

$$\partial\mathcal{D}_k = \left\{ s \in \mathbb{C} | f_k(s) = r_{k_{00}} + (r_{k_{10}}s)^H + r_{k_{11}}s's = 0 \right\}, \\ \{r_{k_{00}}, r_{k_{10}}, r_{k_{11}}\} \in \mathbb{R}\times\mathbb{C}\times\mathbb{R} \tag{2}$$

Relevant curves are lines or circles. Each curve $\partial\mathcal{D}_k$ enables us to define an associated region:

$$\mathcal{D}_k = \{ s \in \mathbb{C} \,|\, f_k(s) < 0 \} \quad \forall k \in \{1, ..., m\}. \tag{3}$$

Clearly, $\mathcal{D}_k$ denotes either one side or the other side of the boundary $\partial\mathcal{D}_k$. It can then be a half plane, a disc or the exterior of a disc. It is an open region (*i.e.* not including $\partial\mathcal{D}_k$) in order to encompass the concept of asymptotic stability for linear time invariant systems. $\mathcal{D}_k$ can actually correspond to the scalar case of regions defined in [25] or to a special case of second order Ω-regions [13]. Also define the region $\mathcal{D}$ as a combination, *i.e.* any union and/or intersection of the various subregions $\mathcal{D}_k$. Such a formulation of $\mathcal{D}$ clearly enables a very large choice of clustering regions.

C. Problem Statement

This contribution aims at computing the complex $\mathcal{D}$-stability radius. More precisely, assume that A is $\mathcal{D}$-stable *i.e* $\lambda(A) \subset \mathcal{D}$. Define $r_{\mathcal{D}}$ as the largest value of ρ, the radius of $\mathcal{B}(\rho)$, such that A_c defined in (1) remains $\mathcal{D}$-stable for any $\Delta \in \mathcal{B}(\rho)$. Such a value is the so-called complex $\mathcal{D}$-stability radius. A lower bound $\rho^\star$ of $r_{\mathcal{D}}$, as tight as possible, is to be computed. For this purpose, the concept of $\partial\mathcal{D}$-regularity is introduced in the next section.

III. $\partial\mathcal{D}$-regularity

In this section, only nominal matrices are considered. The concepts of matrix $\partial\mathcal{D}$-regularity and matrix $\partial\mathcal{D}$-singularity are introduced. A necessary and sufficient condition for matrix $\partial\mathcal{D}$-regularity to be satisfied when $\partial\mathcal{D}$ is defined as in (2) is expressed in terms of an LMI. After preliminary notions and assumptions in subsection 1, subsection 2 presents this condition through a first theorem. In subsection 3, the distribution of the matrix eigenvalues with respect to the boundary $\partial\mathcal{D}$ is connected to the inertia of the solution to the LMI, owing to a second theorem. Subsection 4 is devoted to a discussion of these theorems.

A. Preliminaries

Definition 3.1: Let $\partial\mathcal{D}$ be any curve in the complex plane, then matrix $A \in \mathbb{C}^{n\times n}$ is called:

- $\partial\mathcal{D}$-singular when $\lambda(A) \cap \partial\mathcal{D} \neq \emptyset$.
- $\partial\mathcal{D}$-regular when $\lambda(A) \cap \partial\mathcal{D} = \emptyset$.

Remark 3.1: Assume that $\partial\mathcal{D}$ is a boundary separating two open regions $\mathcal{D}$ and $\bar{\mathcal{D}}^C$ (then $\mathbb{C} = \mathcal{D} \cup \partial\mathcal{D} \cup \bar{\mathcal{D}}^C$). Matrix A is $\mathcal{D}$-stable if and only if it is $\partial\mathcal{D}$-regular and the whole of its spectrum lies in $\mathcal{D}$. Otherwise, it is $\mathcal{D}$-unstable.

First assume that $\partial\mathcal{D}$ reduces to one curve *i.e.* $m = 1$:

$$\partial\mathcal{D} = \{s \in \mathbb{C} \,|\, f(s) = r_{00} + (r_{10}s)^H + r_{11}s's = 0\}. \quad (4)$$

In parallel with the work of Hill [14], we state two theorems in the next two parts. The result in [14] is based on Ostrowski and Schneider's theorem [22] and on Frobenius's theorem. Our contribution is more part of Lyapunov's framework [21] and its extensions to root-clustering [13].

B. LMI condition for $\partial\mathcal{D}$-regularity

Theorem 3.1: Let A and $\partial\mathcal{D}$ be, respectively, a complex square matrix of dimension n and a curve as defined in (4). Matrix A is $\partial\mathcal{D}$-regular if and only if there exists a matrix $P = P' \in \mathbb{C}^{n\times n}$ such that

$$F(A,P) = r_{00}P + (r_{10}PA)^H + r_{11}A'PA < 0. \quad (5)$$

Proof: See [3]. ■

C. Root-distribution and the inertia of P

Theorem 3.2: Let A and $\partial\mathcal{D}$ be, respectively, a complex square matrix of dimension n and a curve as defined in (4). Matrix A is $\partial\mathcal{D}$-regular with n_+ eigenvalues in $\mathcal{D}$ and n_- eigenvalues outside $\mathcal{D}$ if and only if any solution $P = P'$ to LMI (5) has inertia $\mathrm{In}(P) = [n_+,\ n_-,\ 0]$.

Proof: see [3]. ■

As mentioned in subsection III-A, our result is an alternative to results by Hill [14]. However, our result consists of one single theorem valid for any line or any circle. It does not require preliminary lemmas except Sylvester's well-known theorem ($\mathrm{In}(H) = \mathrm{In}(MHM')$ for any nonsingular M). The proofs basically require simple algebraic manipulations that might help extension to other curves. In that sense, it is closer to the result of [17]. Nevertheless, some differences are pointed out in the forthcoming discussion.

Corollary 3.1: Let A and $\partial\mathcal{D}$ be, respectively, a complex square matrix of dimension n and a curve as defined in (4). Also let $\mathcal{D}$ and $\bar{\mathcal{D}}^C$ be the regions defined by $f(s) < 0$ and $f(s) > 0$ respectively. Matrix A is $\mathcal{D}$-stable (resp. $\bar{\mathcal{D}}^C$-stable) if and only if there exists a positive (resp. negative) definite matrix $P = P' \in \mathbb{C}^{n\times n}$ such that (5) holds.

Proof: The proof follows directly from theorem 3.2. ■

As special cases, $\mathcal{D}$ can be the OLHP or the OUD. Lyapunov and Stein's theorems [21], [29] are then recovered.

Remark 3.2: Since any solution to (5) is nonsingular then there is no need to specify it as a constraint, and (5) is a simple LMI in P. Note that the nonstrict LMI cannot be considered because it would allow $\partial\mathcal{D}$-singularity.

D. Discussion

With appropriate changes, the above reasoning might be adapted to any second order Ω-transformable region. In that sense, this could be seen as a special case of [17]. However, although [17, Theorem 1] seems suitable to prove the first statement in theorem 3.1, we do not agree with the proof of [17, Theorem 2] related to the inertia of P. Indeed after having proven that some solution P to an LMI exists and has expression [17, equation (6)], it is claimed that for any choice of the negative definite right handside member of the associated equality, the solution to this equality keeps the same expression. We do not agree with that point. Perhaps the same doubt led Jury to achieve a special proof for the nonsingularity of the solution of a GLE [19, Theorem 3.16]. The notion of Ω-transformability was required to prove this nonsingularity (see also [13, Theorem 12]). It could at first sight directly be derived from [17, Theorem 2] but we also think that this special proof was necessary.

Going on with Ω-transformability, it is interesting to see that although transformability seemed to be required to prove the nonsingularity of the solution to a GLE [13], we show here that, owing to the notion of $\partial\mathcal{D}$-regularity (rather than just $\mathcal{D}$-stability), $\mathcal{D}$ and $\bar{\mathcal{D}}^C$ are considered altogether (the reader is here reminded of the fact that $\mathcal{D}$-stability and $\bar{\mathcal{D}}^C$-stability are only special cases of $\partial\mathcal{D}$-regularity as stated in corollary 3.1). As a consequence, the exterior of a disc is a non Ω-transformable region for which it is impossible to find a singular solution to a corresponding GLE. Otherwise, it would be in contradiction with Remark 3.2. In other words, any solution to a GLE attesting matrix root-clustering is necessarily nonsingular (existence and uniqueness of the solution to a GLE is another problem; see [13]). The exterior of the disc is then a region for which Ω-transformability is not required to guarantee the nonsingularity of the solution to a corresponding GLE. It is what was illustrated by an example proposed in [31].

Apart from our doubt about the proof of [17, Theorem 2], we would like to add that this contribution seems to have been overlooked. Actually, [17, Theorem 1] is nothing but an LMI test for matrix root-clustering in an Ω-region. In 1971, such a test was not tractable from a computational point of view (at about the same time, Willems was just beginning to warn the control community about the great interest in handling LMIs [33]). For this reason, it mattered to "convert" this LMI test into a GLE [13]. Now that LMIs have become classical tools, although some significant contributions enabled to test matrix root-clustering via LMI conditions [10], many authors should remember the pioneer work [17].

Furthermore, if solving an LMI is not an obstacle, results in [17] can be used to consider problem 85 formulated by Wang in the electronic book proposed by Blondel and Megretski [6]. In problem 85, the analysis of root-clustering in Ω-transformable regions of order greater than 2 is concerned and expressed in terms of GLEs. Of course, it is an interesting challenge from a mathematical point of view. For low-dimensional problems, the LMI approach of [17] might enable us to be free of the transformability assumption and of GLEs but for high-dimensional problems, the GLEs are still fundamental. So problem 85 is discerning. See [3] for further discussion.

IV. COMPLEX $\mathcal{D}$-STABILITY RADIUS

In this section, the uncertain case is studied. First, a condition for the uncertain A_c defined in (1) to be $\partial\mathcal{D}$-regular when $\partial\mathcal{D}$ complies to (4) is given. This condition is then used to compute the $\mathcal{D}$-stability radius when $\mathcal{D}$ is some combination of regions as defined in subsection II-B.

Theorem 4.1: Let A_c and $\partial\mathcal{D}$ be, respectively, an uncertain matrix as defined in (1) and a geometric curve as defined in (4). Assume that matrix A is $\partial\mathcal{D}$-regular. A_c is robustly $\partial\mathcal{D}$-regular against $\mathcal{B}(\rho)$ if and only if there exists $P = P'$ with inertia $\text{In}(P) = [n_+,\ n_-,\ 0]$ such that

$$\mathcal{Q}(P,\gamma) = \begin{bmatrix} C'C & C'D \\ D'C & D'D - \gamma\mathbb{I}_q \end{bmatrix} + \begin{bmatrix} r_{00P} + (r_{10}PA)^H + r_{11}A'PA & r_{10}PB + r_{11}A'PB \\ r_{10}'B'P + r_{11}B'PA & r_{11}B'PB \end{bmatrix} < 0. \tag{6}$$

with $\gamma = \rho^{-2}$. In this event, A_c keeps n_+ eigenvalues inside $\mathcal{D}$ and n_- outside $\mathcal{D}$.

Proof: The proof is based on the Kalman-Yakubovich-Popov (KYP) lemma [27]. See [3]. ∎

The value of ρ obtained when minimizing γ under (6) is the largest acceptable value of ρ. It is the complex $\partial\mathcal{D}$-regularity radius, denoted here by $\varrho_{\partial\mathcal{D}}$.

Now come back to region $\mathcal{D}$ defined as a combination of several regions $\mathcal{D}_k$ (see subsection II-B). Referring to previous works on stability radii [16], [26], the complex $\mathcal{D}$-stability radius can also be defined as follows:

$$r_{\mathcal{D}} = \inf\{||\Delta||_2 \,|\, \Delta \in \mathbb{C}^{q\times r} :\ A + B\Delta C \ \text{ is } \mathcal{D}\text{-unstable}\}. \tag{7}$$

The complex $\partial\mathcal{D}$-regularity of a complex matrix A is

$$\varrho_{\partial\mathcal{D}} = \inf\{||\Delta||_2 \,|\, \Delta \in \mathbb{C}^{q\times r} :\ \lambda(A + B\Delta C) \cap \partial\mathcal{D} \neq \emptyset\} \tag{8}$$

where $\partial\mathcal{D}$ is the boundary of $\mathcal{D}$. The formulation of $r_{\mathcal{D}}$ implicitly assumes that A is $\mathcal{D}$-stable. That is the basic difference from the formulation of $\varrho_{\partial\mathcal{D}}$. It follows that

$$A \text{ is } \mathcal{D}\text{-stable} \ \Rightarrow\ \varrho_{\partial\mathcal{D}} = r_{\mathcal{D}}. \tag{9}$$

Also define the set $\partial\mathcal{U}$ by

$$\partial\mathcal{U} = \bigcup_{k=1}^{m} \partial\mathcal{D}_k. \tag{10}$$

It is clear that $\partial\mathcal{D} \subset \partial\mathcal{U}$. Hence, A_c has no eigenvalue on $\partial\mathcal{D}$ if it has no eigenvalue on $\partial\mathcal{U}$, so we have

$$\varrho_{\partial\mathcal{U}} = \min_{k\in\{1,\dots,m\}} \varrho_{\partial\mathcal{D}_k} \ \leq\ \varrho_{\partial\mathcal{D}}. \tag{11}$$

Moreover, if $\partial\mathcal{U} = \partial\mathcal{D}$, the above inequality becomes an equality. Besides, if A is assumed to be $\mathcal{D}$-stable and if $\partial\mathcal{U} \cap \mathcal{D} = \emptyset$, it can be assessed that $\varrho_{\partial\mathcal{U}} = r_{\mathcal{D}}$.

From the previous reasoning, one deduces:

Theorem 4.2: Let A_c, $\mathcal{D}$, $\partial\mathcal{D}$ be, respectively, an uncertain matrix as defined in (1), a clustering region as defined in subsection II-B and its boundary. Then A_c is robustly $\partial\mathcal{D}$-regular against $\mathcal{B}(\rho)$ if there exist m Hermitian matrices P_k, $k = 1, \dots, m$, such that

$$\mathcal{Q}_k(P_k,\gamma) < 0 \quad \forall k \in \{1,\dots,m\}, \tag{12}$$

where $Q_k(P_k,\gamma)$ complies with the same description as $Q(P,\gamma)$ (but with $r_{k_{ij}}$ and P_k instead of r_{ij} and P) and where and $\gamma = \rho^{-2}$. In this event, A_c keeps the same number of eigenvalues inside $\mathcal{D}$ for any $\Delta \in \mathcal{B}(\rho)$.

Moreover, if $\partial\mathcal{D} = \partial\mathcal{U}$, then LMIs (12) are also necessary.

Proof: See [3]. ∎

In the light of this theorem, the following statements, which can be seen as corollaries, can be formulated:

- γ can be minimized under LMI constraints (12) down to $\gamma^\star$ and then $\rho^\star = (\gamma^\star)^{-1/2}$ equals $\varrho_{\partial\mathcal{U}}$. If A is $\mathcal{D}$-stable, then $\rho^\star$ is a robust $\mathcal{D}$-stability bound.
- If $\partial\mathcal{U} = \partial\mathcal{D}$, then $\rho^\star$ equals $\varrho_{\partial\mathcal{D}}$.
- If A is $\mathcal{D}$-stable and if $\partial\mathcal{U} \cap \mathcal{D} = \emptyset$, then $\rho^\star$ equals both $\varrho_{\partial\mathcal{D}}$ and $r_{\mathcal{D}}$, the complex $\mathcal{D}$-stability radius.

When $\partial\mathcal{U} \cap \mathcal{D} \neq \emptyset$, $\rho^\star$ might not equal $\varrho_{\partial\mathcal{D}}$ or $r_{\mathcal{D}}$ and the condition (12) in Theorem 4.2 might even fail as in the case illustrated by Figure 1. The two eigenvalues of the matrix are here symbolized by the big dots which belongs to $\partial\mathcal{D}_1$ and $\partial\mathcal{D}_2$ respectively but lie inside $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2$. Therefore, the matrix is here $\mathcal{D}$-stable but is both $\partial\mathcal{D}_1$-singular and $\partial\mathcal{D}_2$-singular. As a consequence, condition (12) cannot be checked although $\mathcal{D}$-stability holds.

eigenvalues of A

$\partial\mathcal{D}_1$ $\partial\mathcal{D}_2$

$\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2$

Fig. 1. $\varrho_{\partial\mathcal{U}}$ cannot be computed

V. EXTENSION TO POLYTOPIC LFT UNCERTAINTY

It is now assumed matrices A, B, C and D belong to a polytope of matrices so that one can define the following convex combination:

$$M(\theta) = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \sum_{i=1}^{N} \theta_i \begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix}, \tag{13}$$

where the vertices A_i, B_i, C_i, and D_i are known and the coordinates θ_i are unknown but belong to the set $\boldsymbol{\Theta}$ defined by $\theta_i \geq 0\,\forall i$ and $\sum_{i=1}^{N} \theta_1 = 1$. This section does not aim at computing a "polytopic" complex stability radius but simply aims at computing a good lower bound $\rho^\star$ through LMI machinery.

Theorem 5.1: Let A_c, $\mathcal{D}$, $\partial\mathcal{D}$ be, respectively, an uncertain matrix as defined in (1) with (13), a clustering region as defined in subsection II-B and its boundary. Then A_c is robustly $\partial\mathcal{D}$-regular against $\mathcal{B}(\rho)$ and $\mathbf{\Theta}$ if there exist Nm matrices $P_{k_i} = P'_{k_i}$, $k = 1, ..., m$, $i = 1, ..., N$, and m matrices G_k, $k = 1, ..., m$, such that

$$\mathbf{W}_{k_i} = \left(G_k \begin{bmatrix} A_i & -I_n & B_i & 0 \end{bmatrix}\right)^H +,$$

$$\begin{bmatrix} r_{k_{00}}P_{k_i} & r_{k_{10}}P_{k_i} & 0 & C'_i \\ r'_{k_{10}}P_{k_i} & r_{k_{11}}P_{k_i} & 0 & 0 \\ 0 & 0 & -\mathbb{I}_q & D'_i \\ C_i & 0 & D_i & -\gamma\mathbb{I}_r \end{bmatrix} < 0 \ \forall\{k,i\} \quad (14)$$

with $\gamma = \rho^{-2}$. In this event, A_c keeps the same number of eigenvalues inside $\mathcal{D}$ for any $\Delta \in \mathcal{B}(\rho)$ and for any $\theta \in \mathbf{\Theta}$.

Proof: This is just an outline. The convex combination

$$\sum_{i=1}^{N} \theta_i \mathbf{W}_{k_i} \quad (15)$$

is necessarily negative definite. Using [28, Theorem 2.3.12] and Schur's complement, one equivalently gets the same condition as (12) but for any $\theta \in \mathbf{\Theta}$ *i.e.* with $M(\theta)$ and

$$P_k(\theta) = \sum_{i=1}^{N} \theta_i P_{k_i} = P'_k(\theta), \quad \theta \in \mathbf{\Theta}. \quad (16)$$

Hence, by virtue of Theorem 4.2 applied for each θ, A_c is robustly $\partial\mathcal{D}$-regular against $\mathcal{B}(\rho)$ and $\mathbf{\Theta}$. The statement on root-distribution relies on the fact that inertias cannot change, by virtue of Theorem 3.2. ■

Once again, it is possible to minimize γ in order to reach a better value pf $\rho^\star$ which is however conservative because (14) is only sufficient for (15).

Remark 5.1: The derivation of the implicit parameter-dependent "Lyapunov"-matrices given in (16) follows the same idea as in [12], later used in [25]. Besides, it can be proven that the matrix G_k can comply with the structure

$$G_k = \begin{bmatrix} \bar{G}_k \\ 0 \end{bmatrix} \quad \text{where} \quad \bar{G}_k \in \mathbb{C}^{2n\times n}, \quad (17)$$

with no loss of generality, following arguments given in [7].

VI. Numerical illustration

In this section, we propose a simple illustration of the application of Theorem 5.1. This model is inspired from [2]. The lateral dynamic of an aircraft is modeled by uncertain state and input matrices:

$$A_0 = \begin{bmatrix} -0.3400 & 0.0517 & 0.0010 & -0.9970 & 0.0000 \\ 0.0000 & 0.0000 & 1.0000 & 0.0000 & 0.0000 \\ -2.6900 & 0.0000 & -1.1500 & 0.7380 & 0.0000 \\ 5.9100 & 0.0000 & 0.1380 & -0.5060 & 0.0000 \\ -0.3400 & 0.0517 & 0.0010 & 0.0031 & 0.0000 \end{bmatrix} \pm$$

$$\delta_1 \begin{bmatrix} 1 & 0.1 & 0.01 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0.1 & 0.01 & 0 \end{bmatrix},$$

$$B_0 = \begin{bmatrix} 0.0755 & 0.0000 & 0.0246 \\ 0.0000 & 0.0000 & 0.0000 \\ 4.4800 & 5.2200 & -0.7420 \\ -5.0300 & 0.0998 & 0.9848 \\ 0.0755 & 0.0000 & 0.0246 \end{bmatrix} \pm \delta_2 \begin{bmatrix} 1 & 0 & 0.5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0.5 \end{bmatrix},$$

where the additional terms correspond to an uncertain parametric uncertainty with two deflecting parameters δ_1 and δ_2 such that $|\delta| \leq 0.1$ and $|\delta_2| \leq 0.02$. A static state feedback control law associated with matrix

$$K = \begin{bmatrix} -3.9063 & -0.2869 & 0.0006 & -1.5109 & -1.8135 \\ 0.8077 & -2.4178 & -0.9356 & -0.2877 & -0.0296 \\ -21.5282 & -1.2212 & -0.0424 & -10.1518 & -11.1581 \end{bmatrix}$$

is applied to obtain nominal spectrum:

$$\{-0.5; -2 \pm 2\mathbf{i}; -3 \pm 2\mathbf{i}\}. \quad (18)$$

It is assumed that K is also subject to an additive uncertainty $\Delta \in \mathcal{B}(\rho)$ so analyzing both robustness against uncertain parameters and nonfragility of the control corresponds to the robustness analysis of (1) where $A = A_0 + B_0K$, $B = B_0$, $C = I_5$, and $D = 0$. The polytope has four vertices. To analyze the robustness of the pole location in the presence of the uncertainty, $\mathcal{D}$ is chosen as the union of 5 discs $\mathcal{D}_k$, $k = 1, ..., 5$, centered around the nominal eigenvalues given by (18) and all of radius 0.5. Since $\mathcal{D}_2$ and $\mathcal{D}_3$ (respectively $\mathcal{D}_4$ and $\mathcal{D}_5$) are symmetric to each other with respect to the real axis, it is only required to consider three regions. Theorem 5.1 leads to:

$$\rho^\star = \min\{0.1944; 0.0601; 0.0618\} = 0.0601.$$

$\rho^\star$ appears here as a nonfragility index with respect to some desired transient performances.

Many spectra are plotted for various values of $\Delta \in \mathcal{B}(\rho^\star)$ and θ_i, $i = 1, ..., N$; see Figure 2. It can be seen that the bound is not very conservative although this example is not trivial. The pole migration is not very far from $\partial\mathcal{D}$. Since the nominal part of A is $\mathcal{D}$-stable and since the five open subregions are disjoint (which implies that $\partial\mathcal{U} \cap \mathcal{D} = \emptyset$), the only source of conservatism is the polytopic uncertainty. The LFT-based uncertainty (in this case, only norm-bounded one) is taken into account with no conservatism.

Computing $\rho^\star$ leads to derive 20 Lyapunov matrices P_{k_i}. Each matrix P_{k_i} is such that $\mathrm{In}(P_{k_i}) = [1,\ 4,\ 0]$, as forecast by Theorem 3.2 since A_c has one root inside and four roots outside each $\mathcal{D}_k$.

VII. Conclusion

In this paper, the concept of matrix $\partial\mathcal{D}$-regularity was used to compute, through a Lyapunov approach, a robust $\mathcal{D}$-stability bound. An original point in the paper is the wide class of allowed clustering regions since $\mathcal{D}$ can be a combination (*i.e.* the union and/or intersection) of several possibly non symmetric half planes, discs and exteriors of discs. Such an originality in the choice of the region is made possible by using Lyapunov matrices which are not necessarily positive definite, but that preserve inertia over

Fig. 2. Pole migration with polytopic norm-bounded complex uncertainty

the uncertainty domain. When the boundaries of various subregions do not intersect $\mathcal{D}$, this bound turns to be the exact complex $\mathcal{D}$-stability radius, that is the largest bound on a complex LFT-based uncertainty preserving $\mathcal{D}$-stability. The bound is deduced from the solution of an LMI problem. It is also extended to polytopic LFT-based uncertainty with weak conservatism. This extension to the polytopic case might be seen a step further compared to [3] where only norm-bounded uncertainty is considered.

As extensions of our work, the case of real matrices could be addressed and the associated real $\mathcal{D}$-stability radius could be seeked, maybe with the use of arguments of [26], [15]. But the most interesting perspective would be a way to circumvent the limit induced by the possible intersection between $\partial\mathcal{U}$ and $\mathcal{D}$ (see Fig. 1). We conjecture that the generalized KYP lemma [18], with appropriate modifications, could solve this problem.

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