Proceedings of the
44th IEEE Conference on Decision and Control, and
the European Control Conference 2005
Seville, Spain, December 12-15, 2005

Input-to-Output Stabilization of Nonlinear Systems via Backstepping

Denis V. Efimov, *Member, IEEE*, Alexander L. Fradkov, *Fellow, IEEE*

***Abstract*— An extension of backstepping design method to stabilization of nonlinear systems with respect to a set is presented. Robust version of controller providing the system with input-to-output stability (IOS) is proposed. The performance of obtained solution is demonstrated by the pendulum with actuator example.**

I. Introduction

During last two decades the problem of stabilizing control design for nonlinear dynamical systems was intensively studied see, e.g. survey [10]. As a result the number of approaches were developed. Method of Control Lyapunov function (CLF) [4], [5], [22] gives necessary and sufficient conditions for affine in control nonlinear systems stabilizable by continuous control. Feedback linearization approach [8] provides elegant geometric design tool for a class of nonlinear systems transforming them to linear ones and allowing to apply a wide spectrum of solutions available for linear systems. The control design problem for cascades systems plays an important role among fundamental control design problems. There are several methods like backstepping, nested saturation design or forwarding [10], [12], [11], [16], [29] which allow to design a stabilizing control for a class of nonlinear dynamical systems fitting some structural conditions (e.g. well defined relative degree, minimum phase property, low-triangular model of the plant). Passification design method is focused on stabilization of a class of nonlinear systems possessing weak minimum phase property [17], [7]. There exist robust versions of the above approaches mainly based on input-to-state stability (ISS) theory [23], [25], [26].

Another promising topic deals with the problem of nonlinear systems stabilization with respect to set [7], [15], [18]–[21], [30] (part of variables or output). Such problem arises in oscillation or synchronization control, energy level stabilization in mechanical systems, maneuvering problem or in robotic applications. Robust analogues of stability with respect to set or output were formulated in [13] and [9], [27], [28] in IOS framework (as extension of ISS property for systems with output).

Although set stabilization algorithms are demanded in practical applications, the corresponding part of theory dealing with control design is developed only for CLF approach [6], feedback linearization [7], [30] and passification [19], [20]. Paper [14] analyzes robust properties of energy level stabilization control for a pendulum. To our best knowledge, no results in set stabilization by backstepping are available.

In this paper we attempt to provide such result. Also we investigate robust properties of proposed control laws basing on IOS theory. In Section 2 preliminary results and definitions are summarized. Section 3 contains problem statement and control design results for stabilization with respect to set. In the second part of Section 3 robust properties of proposed controls are also analyzed. The problem of robust energy level stabilization for a pendulum is considered in Section 4.

II. Preliminaries

Let us consider a nonlinear dynamical system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x},\mathbf{u}),\ \mathbf{y} = \mathbf{h}(\mathbf{x}), \tag{1}$$

where $\mathbf{x} \in R^n$ is state vector; $\mathbf{u} \in R^m$ is input vector; $\mathbf{y} \in R^p$ is output vector; $\mathbf{f}$ and $\mathbf{h}$ are locally Lipschitz continuous vector functions, $\mathbf{h}(0) = 0$, $\mathbf{f}(0,0) = 0$. Euclidean norm will be denoted as $|\mathbf{x}|$, and $\|\mathbf{u}\|_{[t_0,t]}$ denotes the L_∞^m norm of the input ($\mathbf{u}(t)$ is measurable and locally essentially bounded function $\mathbf{u} : R_+ \to R^m$, $R_+ = \{\tau \in R : \tau \ge 0\}$):

$$\|\mathbf{u}\|_{[t_0,T]} = ess\sup\{|\mathbf{u}(t)|,\ t \in [t_0,T]\},$$

if $T = +\infty$ then the notation $\|\mathbf{u}\|$ will be used. We will denote as $\mathcal{M}_{R^m}$ the set of all such Lebesgue measurable inputs $\mathbf{u}$ with property $\|\mathbf{u}\| < +\infty$ and $\mathcal{M}_\Omega$ will be the set of inputs $\mathbf{u}$ such that $\mathbf{u}(t) \in \Omega \subset R^m$ for almost all $t \ge 0$, where Ω is a compact set. For initial state $\mathbf{x}_0$ and input $\mathbf{u} \in \mathcal{M}_{R^m}$ let $\mathbf{x}(t,\mathbf{x}_0,\mathbf{u})$ be the unique maximal solution of (1) (we will use notation $\mathbf{x}(t)$ if all other arguments of solution are clear from the context; $\mathbf{y}(t,\mathbf{x}_0,\mathbf{u}) = \mathbf{h}(\mathbf{x}(t,\mathbf{x}_0,\mathbf{u}))$), which is defined on some finite interval $[0,T)$; if $T = +\infty$ for every initial state $\mathbf{x}_0$ and $\mathbf{u} \in \mathcal{M}_{R^m}$, then system is called forward complete.

This work is partly supported by grant 05-01-00869 of Russian Foundation for Basic Research, by Russian Science Support Foundation and by Program of Presidium of Russian Academy of Science № 19.

Authors are with the Control of Complex Systems Laboratory, Institute of Problem of Mechanical Engineering, Bolshoi av., 61, V.O. , St-Petersburg, 199178 Russia (efde@mail.rcom.ru, alf@control.ipme.ru).

0-7803-9568-9/05/$20.00 ©2005 IEEE

System (1) has unboundedness observability (UO) property, if for each state $\mathbf{x}_0$ and input $\mathbf{u} \in \mathcal{M}_{R^m}$ such that $T < +\infty$ necessarily

$$\limsup_{t \to T} |\mathbf{y}(t, \mathbf{x}_0, \mathbf{u})| = +\infty .$$

Boundedness of UO output means forward completeness. The necessary and sufficient conditions for forward completeness and UO properties were investigated in [1]. Distance in R^n from given point $\mathbf{x}$ to set $\mathcal{A}$ is denoted as $|\mathbf{x}|_{\mathcal{A}} = \text{dist}(\mathbf{x}, \mathcal{A}) = \inf_{\boldsymbol{\eta} \in \mathcal{A}} |\mathbf{x} - \boldsymbol{\eta}|$ and $|\mathbf{x}|_0 = |\mathbf{x}|$ is standard Euclidean norm.

As usual, continuous function $\sigma : R_+ \to R_+$ belongs to class $\mathcal{K}$ if it is strictly increasing and $\sigma(0) = 0$; it belongs to class $\mathcal{K}_\infty$ if it is additionally radially unbounded; and continuous function $\beta : R_+ \times R_+ \to R_+$ is from class $\mathcal{KL}$, if it is from class $\mathcal{K}$ for the first argument for any fixed second, and it is strictly decreasing to zero by the second argument for any fixed first one.

It is said that system (1) has bounded-input-bounded-state (BIBS) property, if inequality

$$|\mathbf{x}(t, \mathbf{x}_0, \mathbf{u})| \le \max\{\vartheta(|\mathbf{x}_0|), \vartheta(\|\mathbf{u}\|)\},\ t \ge 0$$

holds for some function $\vartheta \in \mathcal{K}$ and for all $\mathbf{x}_0 \in R^n$ and $\mathbf{u} \in \mathcal{M}_{R^m}$. Another one characterization of (1) is global stability modulo output (GSMO) property:

$$|\mathbf{x}(t, \mathbf{x}_0, \mathbf{u})| \ge \nu(|\mathbf{y}(t, \mathbf{x}_0, \mathbf{u})|),\ t \in [0, \tilde{T}) \Rightarrow$$

$$|\mathbf{x}(t, \mathbf{x}_0, \mathbf{u})| \le \max\left\{\mu(|\mathbf{x}_0|), \mu\left(\|\mathbf{u}\|_{[0,\tilde{T})}\right)\right\},\ t \in [0, \tilde{T}),$$

where functions $\nu \in \mathcal{K}_\infty$, $\mu \in \mathcal{K}$ and $\mathbf{x}_0 \in R^n$, $\mathbf{u} \in \mathcal{M}_{R^m}$, $\tilde{T} < T$. If $\mathbf{u} \in \mathcal{M}_\Omega$, then the term $\mu\left(\|\mathbf{u}\|_{[0,\tilde{T})}\right)$ can be dropped in the last inequality. It is worth to note, that GSMO property and boundedness of the output ensure BIBS. Additionally, if set $\mathcal{Z} = \{\mathbf{x} : \mathbf{h}(\mathbf{x}) = 0\}$ is compact, then system possesses GSMO property. Symbol $DV(\mathbf{x})\mathbf{F}(\cdot)$ will be used for directional derivative of function V with respect to vector field $\mathbf{F}$ if function V is differentiable and for Dini derivative if function V is Lipschitz continuous.

D e f i n i t i o n 1 [9], [27]. *A UO system (1) is IOS, if there exist* $\beta \in \mathcal{KL}$ *and* $\gamma \in \mathcal{K}$ *such, that*

$$|\mathbf{y}(t, \mathbf{x}_0, \mathbf{u})| \le \beta(|\mathbf{x}_0|, t) + \gamma(\|\mathbf{u}\|),\ t \ge 0$$

holds for all $\mathbf{x}_0 \in R^n$ *and* $\mathbf{u} \in \mathcal{M}_{R^m}$. □

D e f i n i t i o n 2 [9]. *For system (1), a smooth function* V *and a function* $\lambda : R^n \to R_+$ *are called respectively an IOS-Lyapunov function and auxiliary modulus if there exist* $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ *such that inequalities*

$$\alpha_1(|\mathbf{h}(\mathbf{x})|) \le V(\mathbf{x}) \le \alpha_2(|\mathbf{x}|) \qquad (2)$$

hold; there exist $\chi \in \mathcal{K}$ *and* $\alpha_3 \in \mathcal{KL}$ *such, that*

$$V(\mathbf{x}) > \chi(|\mathbf{u}|) \Rightarrow DV(\mathbf{x})\mathbf{f}(\mathbf{x}, \mathbf{u}) \le -\alpha_3(V(\mathbf{x}), \lambda(\mathbf{x})) \quad (3)$$

for all $\mathbf{x}_0 \in R^n$ *and all* $\mathbf{u} \in \mathcal{M}_{R^m}$, *and there exists some* $\delta \in \mathcal{K}$ *such that for any* $T \ge 0$

$$V(\mathbf{x}(t, \mathbf{x}_0, \mathbf{u})) > \chi(|\mathbf{u}(t)|),\ t \in [0, T) \Rightarrow$$

$$\lambda(\mathbf{x}(t, \mathbf{x}_0, \mathbf{u})) \le \max\{\delta(|\mathbf{x}_0|), \delta(\|\mathbf{u}\|)\}. \qquad □$$

In [28] IOS-Lyapunov functions was introduced for BIBS system (1), in this case one can use $|\mathbf{x}|$ as auxiliary modulus λ.

T h e o r e m 1 [9]. *Suppose that system (1) is UO. The following are equivalent for the system:*

– it is IOS;

– it admits an IOS-Lyapunov function. □

If input $\mathbf{u}$ takes values in compact set $\Omega \subset R^m$, then there is another characterization of output stability.

D e f i n i t i o n 3 [9], [27]. *A forward complete system (1) with inputs from* $\mathcal{M}_\Omega$ *is UOS, or uniformly output stable, if there exists* $\beta \in \mathcal{KL}$ *such, that*

$$|\mathbf{y}(t, \mathbf{x}_0, \mathbf{u})| \le \beta(|\mathbf{x}_0|, t),\ t \ge 0$$

for all $\mathbf{x}_0 \in R^n$ *and* $\mathbf{u} \in \mathcal{M}_\Omega$. □

D e f i n i t i o n 4 [9], [28]. *For system (1), a smooth function* V *and a function* $\lambda : R^n \to R_+$ *are called respectively an UOS-Lyapunov function and auxiliary modulus if there exist* $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ *so that (2) holds and there exists* $\alpha_3 \in \mathcal{KL}$ *such that*

$$DV(\mathbf{x})\mathbf{f}(\mathbf{x}, \mathbf{u}) \le -\alpha_3(V(\mathbf{x}), \lambda(\mathbf{x})) \qquad (4)$$

is satisfied for all $\mathbf{x}_0 \in R^n$ *and all* $\mathbf{u} \in \mathcal{M}_\Omega$, *and* λ *is locally Lipschitz on the set* $\{\mathbf{x} : V(\mathbf{x}) > 0\}$ *and* $\lambda(\mathbf{x}(t, \mathbf{x}_0, \mathbf{u})) \le \lambda(\mathbf{x}_0)$. □

T h e o r e m 2 [9], [28]. *Suppose that system (1) is forward complete and* $\mathbf{u} \in \mathcal{M}_\Omega$. *The following are equivalent for the system:*

– it is UOS;

– it admits an UOS-Lyapunov function. □

In work [6] the corresponding CLF formulations are given for the tasks of IOS and UOS stabilization.

L e m m a 1. *Let system (1) with inputs* $\mathbf{u} \in \mathcal{M}_\Omega$ *be UO and there exist a continuously differentiable function* $V : R^n \to R_+$, *which admits (2) and with* $\alpha_0 \in \mathcal{K}$ *for all* $\mathbf{x} \in R^n$ *and* $\mathbf{u} \in \mathcal{M}_\Omega$

$$\partial V / \partial \mathbf{x}\ \mathbf{f}(\mathbf{x}, \mathbf{u}) \le -\alpha_0(|\mathbf{y}|).$$

If function $\eta(\mathbf{x}, \mathbf{u}) = \partial \mathbf{h} / \partial \mathbf{x}\ \mathbf{f}(\mathbf{x}, \mathbf{u})$ *is bounded for bounded values of* $V(\mathbf{x})$ *and* $\mathbf{u} \in \mathcal{M}_\Omega$, *then system is UOS.* ■

Proofs of the lemma and all theorems are excluded due to space limitation. It is said that system (1) is V-detectable [19] with respect to continuous function $V: R^n \to R_+$ on set $\mathcal{X} \subseteq R^n$, if for all $\mathbf{x}_0 \in R^n$ and $\mathbf{u}=0$ the following property holds:

$$\mathbf{x}(t,\mathbf{x}_0,0) \in \mathcal{X}, \ \forall t \ge 0 \ \Rightarrow \ \lim_{t\to+\infty} V\big(\mathbf{x}(t,\mathbf{x}_0,0)\big)=0 .$$

Sufficient conditions for system (1) to be V-detectable are presented in work [19].

III. Main Results

Let us consider the model of the plant

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x},\mathbf{z},\mathbf{d}_1), \ \mathbf{y}=\mathbf{h}(\mathbf{x}), \qquad (5)$$

$$\dot{\mathbf{z}} = \mathbf{u}+\mathbf{d}_2, \qquad (6)$$

where $\mathbf{x}\in R^n$ is state vector of system (5). The system should be stabilized with respect to set $\mathcal{Z}=\{\mathbf{x}:\mathbf{h}(\mathbf{x})=0\}$ defined by zero level of output vector $\mathbf{y}\in R^p$; $\mathbf{z}\in R^m$ is state vector of system (6); $\mathbf{u}\in R^m$ is control vector; $\mathbf{d}_1\in R^{q_1}$, $\mathbf{d}_2\in R^{q_2}$ are vectors of external disturbances, $\mathbf{d}=(\mathbf{d}_1,\mathbf{d}_2)\in R^q$, $q=q_1+q_2$. Functions $\mathbf{h}: R^n\to R^p$ and $\mathbf{f}: R^{n+m+q_1}\to R^n$ are locally Lipschitz continuous, $\mathbf{f}(0,0,0)=0$. As usual [12] we assume that there exists some continuously differentiable feedback control law $\mathbf{k}: R^n\to R^m$ such, that system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x},\mathbf{k}(\mathbf{x})+\mathbf{e},\mathbf{d}_1) \qquad (7)$$

possesses UOS or IOS properties with respect to output $\mathbf{y}$ and input $\mathbf{d}_1$ for $\mathbf{e}=0$, where variable $\mathbf{e}$ corresponds to "virtual" control realization error $\mathbf{e}=\mathbf{z}-\mathbf{k}(\mathbf{x})$. More precise definition of control $\mathbf{k}$ properties will be described later. Taking into account control $\mathbf{k}$ it is necessary to design control $\mathbf{u}=\mathbf{U}(\mathbf{x},\mathbf{z})$, which provides UOS or IOS property with respect output $\mathbf{y}$ and input $\mathbf{d}$ for overall system (5), (6). It is the standard backstepping control problem reformulated with respect to the output.

Assume that relations $\iota_1(|\mathbf{x}|_{\mathcal{Z}}) \le \mathbf{h}(\mathbf{x}) \le \iota_2(|\mathbf{x}|_{\mathcal{Z}})$ are satisfied for $\iota_1,\iota_2\in\mathcal{K}_\infty$. Therefore, stability with respect to set $\mathcal{Z}$ in the sense of [13] and UOS with respect to output $\mathbf{y}$ (see definition 3) will be equivalent. The above relations mean that convergence to zero and boundedness of output function imply the corresponding convergence to zero and boundedness of the distance to set $\mathcal{Z}$ and vice versa. Here we assume that both state vectors $\mathbf{x}$ and $\mathbf{z}$ are available for measurements.

A. UOS stabilization

Let $\mathbf{d}(t)\equiv 0$, $t\ge 0$. The following assumptions formulate requirements to "virtual" control $\mathbf{k}$ in this case.

Assumption 1. *There exist continuously differentiable functions $V: R^n\to R_+$ and $\mathbf{k}: R^n\to R^m$ such, that*

$$\alpha_1(|\mathbf{h}(\mathbf{x})|) \le V(\mathbf{x}) \le \alpha_2(|\mathbf{x}|),\ \alpha_1,\alpha_2\in\mathcal{K}_\infty \qquad (8)$$

and one of the following properties holds for all $\mathbf{x}\in R^n$, $\mathbf{y}\in R^p$, and $\mathbf{e}\in R^m$:

1. *System (7) is UO and*
$\partial V/\partial\mathbf{x}\ \mathbf{f}(\mathbf{x},\mathbf{k}(\mathbf{x}),0) \le -\alpha_3(V(\mathbf{x}))$, $\alpha_3\in\mathcal{K}$;
2. *System (7) is UO and*
$\partial V/\partial\mathbf{x}\ \mathbf{f}(\mathbf{x},\mathbf{k}(\mathbf{x}),0) \le -\alpha_4(|\mathbf{y}|)$, $\alpha_4\in\mathcal{K}$ *and*
$\partial\mathbf{h}/\partial\mathbf{x}\ \mathbf{f}(\mathbf{x},\mathbf{k}(\mathbf{x})+\mathbf{e},0) \le \mu_1(V)+\mu_2(|\mathbf{e}|)$, $\mu_1,\mu_2\in\mathcal{K}$; (9)
3. *System (7) is GSMO or BIBS and*
$\partial V/\partial\mathbf{x}\ \mathbf{f}(\mathbf{x},\mathbf{k}(\mathbf{x}),0) \le -\alpha_5(|\boldsymbol{\psi}(\mathbf{x})|)$, $\alpha_5\in\mathcal{K}$

and system (7) is V-detectable into set $\mathcal{X}=\{\mathbf{x}:\boldsymbol{\psi}(\mathbf{x})=0\}$, $\boldsymbol{\psi}: R^n\to R^l$ is a continuous function. □

Assumption 2. *There exist continuously differentiable functions $V: R^n\to R_+$ and $\mathbf{k}: R^n\to R^m$ such, that (8) is satisfied and one the following properties holds for all $\mathbf{x}\in R^n$ and $\mathbf{e}\in R^m$:*

1. *System (7) is UO and for $\alpha_6\in\mathcal{K}_\infty$ and $\sigma_1\in\mathcal{K}$*
$\partial V/\partial\mathbf{x}\ \mathbf{f}(\mathbf{x},\mathbf{k}(\mathbf{x})+\mathbf{e},0) \le -\alpha_6(V(\mathbf{x}))+\sigma_1(|\mathbf{e}|)$;
2. *System (7) is GSMO or BIBS and*
$\partial V/\partial\mathbf{x}\ \mathbf{f}(\mathbf{x},\mathbf{k}(\mathbf{x})+\mathbf{e},0) \le -\alpha_7(|\boldsymbol{\psi}(\mathbf{x})|)+\sigma_2(|\mathbf{e}|)$

and system (7) is V-detectable into set $\mathcal{X}$, $\alpha_7\in\mathcal{K}_\infty$ and $\sigma_2\in\mathcal{K}$, $\boldsymbol{\psi}: R^n\to R^l$ is a continuous function. □

In the first assumption the control $\mathbf{k}$ provides only UOS property or its analogues for system (7), while the second assumption establishes IOS-like property for system (7).

Note that opposite to classical works [12] right hand side of equation (5) depends on $\mathbf{z}$ in a nonlinear fashion.

Assumption 3. *There exists continuous function $\mathbf{r}: R^{n+m}\to R^m$ s., t. for all $\mathbf{x}\in R^n$, $\mathbf{z},\mathbf{z}'\in R^m$, $\mathbf{d}_1\in R^{q_1}$*

$$\frac{\partial V}{\partial\mathbf{x}}\mathbf{f}(\mathbf{x},\mathbf{z},\mathbf{d}_1) - \frac{\partial V}{\partial\mathbf{x}}\mathbf{f}(\mathbf{x},\mathbf{z}',\mathbf{d}_1) \le \mathbf{r}(\mathbf{x},\mathbf{z}')^T(\mathbf{z}-\mathbf{z}'). \ □$$

The last assumption is satisfied for standard case when $\mathbf{f}(\mathbf{x},\mathbf{z},\mathbf{d}_1)=\mathbf{f}_0(\mathbf{x},\mathbf{d}_1)+\mathbf{G}(\mathbf{x})\mathbf{z}$, where $\mathbf{f}_0$ and $\mathbf{G}_0$ are locally Lipschitz continuous vector and matrix functions of appropriate dimensions. In this case $\mathbf{r}(\mathbf{x})=\partial V/\partial\mathbf{x}\ \mathbf{G}(\mathbf{x})$. Also the last assumption is valid for example if $\mathbf{f}(\mathbf{x},\mathbf{z},\mathbf{d}_1)=\mathbf{f}_0(\mathbf{x},\mathbf{d}_1)+\mathbf{g}(\mathbf{x},\mathbf{z})$, where $\mathbf{g}$ is a continuously differentiable vector function dependent on variable $\mathbf{z}$ in concave fashion, then $\mathbf{r}(\mathbf{x},\mathbf{z})=\partial V/\partial\mathbf{x}\ \partial\mathbf{g}/\partial\mathbf{z}$.

Theorem 3. *Let assumptions 1 and 3 hold. Then system (5), (6) with control*

$$\mathbf{u}=\partial\mathbf{k}/\partial\mathbf{x}\ \mathbf{f}(\mathbf{x},\mathbf{z},0)-\mathbf{r}(\mathbf{x},\mathbf{k}(\mathbf{x}))-\varphi(\mathbf{z}-\mathbf{k}(\mathbf{x})) \qquad (10)$$

possesses UOS property, $\varphi: R^m\to R^m$ is a continuous function, $\mathbf{z}^T\varphi(\mathbf{z})\ge\kappa(|\mathbf{z}|)$ for all $\mathbf{z}\in R^m$, $\kappa\in\mathcal{K}$. ■

Remark 1. Note, that the third part of assumption 1 covers the second one under weakening supposition, that for some $\mu_1, \mu_2 \in \mathcal{K}$ and all $\mathbf{x} \in R^n$, $\mathbf{e} \in R^m$ estimate

$$\partial \boldsymbol{\psi}/\partial \mathbf{x}\ \mathbf{f}(\mathbf{x}, \mathbf{k}(\mathbf{x}) + \mathbf{e}, 0) \le \mu_1(V) + \mu_2(|\mathbf{e}|)$$

holds instead of GSMO or BIBS properties. □

Theorem 4. *Let assumption 2 hold. Then system (5), (6) with control*

$$\mathbf{u} = \partial \mathbf{k}/\partial \mathbf{x}\ \mathbf{f}(\mathbf{x}, \mathbf{z}, 0) - \varphi(\mathbf{z} - \mathbf{k}(\mathbf{x})) \qquad (11)$$

possesses UOS property, where $\varphi : R^m \to R^m$ *is a continuous function,* $\mathbf{z}^T \varphi(\mathbf{z}) \ge \kappa(|\mathbf{z}|)$ *for all* $\mathbf{z} \in R^m$, $\kappa \in \mathcal{K}$ *and* $\kappa(s) \ge \tilde{\kappa}(s) + \max\{\sigma_1(s), \sigma_2(s)\}$, $\tilde{\kappa} \in \mathcal{K}$. ■

Let us compare results of theorems 3 and 4 or control laws (10) and (11). Theorem 3 needs additional assumption 3, but it starts from UOS control $\mathbf{k}$. Theorem 4 assumes IOS properties for control $\mathbf{k}$ and it does not require any structural properties of function $\mathbf{f}$ stated in assumption 3, but in (11) additional restriction on growth of φ is needed. Using terminology from paper [3] one may say that control (10) realizes so-called "cancellation backstepping", while control (11) corresponds to so-called "$L_g V$-backstepping". Also control (11) is close to algorithms proposed in book [11] and it is a part of control (10).

B. IOS stabilization

In this section we will consider problem of IOS stabilization of system (5), (6) and some other variants of robustification of controls (10) and (11), when closed loop system possesses integral variants of IOS property. This new property introduced in the appendix and is called integral ISS (iISS) with respect to set by analogy with [2], [24]. The main result of the appendix is formulated in theorem A1. The result of theorem A1 can be used to prove robustness with respect to integrally bounded disturbances of passification based controls [7], [17], [18]–[20] (as it will be done in the next section for the example).

Assumption 4. *There exist continuously differentiable functions* $V : R^n \to R_+$ *and* $\mathbf{k} : R^n \to R^m$ *such, that (8) is satisfied and one the following properties holds for all* $\mathbf{x} \in R^n$, $\mathbf{e} \in R^m$ *and* $\mathbf{d}_1 \in R^{q_1}$:

1. *System (7) is UO and for* $\alpha_8 \in \mathcal{K}_\infty$ *and* $\sigma_3 \in \mathcal{K}$

$$\partial V/\partial \mathbf{x}\ \mathbf{f}(\mathbf{x}, \mathbf{k}(\mathbf{x}), \mathbf{d}_1) \le -\alpha_8(V(\mathbf{x})) + \sigma_3(|\mathbf{d}_1|);$$

2. *System (7) is UO and for* $\alpha_9 \in \mathcal{K}_\infty$ *and* $\sigma_4 \in \mathcal{K}$

$$\partial V/\partial \mathbf{x}\ \mathbf{f}(\mathbf{x}, \mathbf{k}(\mathbf{x}) + \mathbf{e}, \mathbf{d}_1) \le -\alpha_9(V(\mathbf{x})) + \sigma_4(|\mathbf{e}|) + \sigma_4(|\mathbf{d}_1|);$$

3. *Set* $\mathcal{Z}$ *is compact and*

$$\partial V/\partial \mathbf{x}\ \mathbf{f}(\mathbf{x}, \mathbf{k}(\mathbf{x}) + \mathbf{e}, \mathbf{d}_1) \le -\alpha_{10}(|\boldsymbol{\psi}(\mathbf{x})|) + \sigma_5(|\mathbf{e}|) + \sigma_5(|\mathbf{d}_1|)$$

and system (7) is V*-detectable into set* $\mathcal{X}$, $\alpha_{10} \in \mathcal{K}_\infty$ *and* $\sigma_5 \in \mathcal{K}$, $\boldsymbol{\psi} : R^n \to R^l$ *is a continuous function.* □

This assumption incorporates assumptions 1 and 2 for control (10) (the first part) and control (11) (the last two parts).

Assumption 5. *There exists a continuous function* $b : R^{n+m} \to R_+$ *such, that for all* $\mathbf{x} \in R^n$ *and* $\mathbf{d}_1, \mathbf{d}_1' \in R^{q_1}$ *with* $\lambda \in \mathcal{K}$

$$\left| \frac{\partial \mathbf{k}}{\partial \mathbf{x}} \mathbf{f}(\mathbf{x}, \mathbf{z}, \mathbf{d}_1) - \frac{\partial \mathbf{k}}{\partial \mathbf{x}} \mathbf{f}(\mathbf{x}, \mathbf{z}, \mathbf{d}_1') \right| \le b(\mathbf{x}, \mathbf{z}) \lambda(|\mathbf{d}_1 - \mathbf{d}_1'|). \ □$$

If disturbance $\mathbf{d}$ is present, then we should handle nonlinear dependence of function $\mathbf{f}$ in (5) on vector $\mathbf{d}_1$.

Theorem 5. *Let the first part of assumption 4 and assumptions 3,5 hold. Then system (5), (6) with control*

$$\begin{aligned}\mathbf{u} = \partial \mathbf{k}/\partial \mathbf{x}\ \mathbf{f}(\mathbf{x}, \mathbf{z}, 0) - \mathbf{r}(\mathbf{x}, \mathbf{k}(\mathbf{x})) - \varphi(\mathbf{z} - \mathbf{k}(\mathbf{x})) - \\ - 0.5\left(1 + b(\mathbf{x}, \mathbf{z})^2\right)(\mathbf{z} - \mathbf{k}(\mathbf{x}))\end{aligned} \qquad (12)$$

is IOS provided that $\varphi : R^m \to R^m$ *is a continuous function,* $\mathbf{z}^T \varphi(\mathbf{z}) \ge \kappa(|\mathbf{z}|)$ *for all* $\mathbf{z} \in R^m$, $\kappa \in \mathcal{K}_\infty$.

Let the second part of assumption 4 and assumption 5 hold. Then system (5), (6) with control

$$\begin{aligned}\mathbf{u} = \partial \mathbf{k}/\partial \mathbf{x}\ \mathbf{f}(\mathbf{x}, \mathbf{z}, 0) - \varphi(\mathbf{z} - \mathbf{k}(\mathbf{x})) - \\ - 0.5\left(1 + b(\mathbf{x}, \mathbf{z})^2\right)(\mathbf{z} - \mathbf{k}(\mathbf{x}))\end{aligned} \qquad (13)$$

possesses IOS property, where $\varphi : R^m \to R^m$ *is a continuous function,* $\mathbf{z}^T \varphi(\mathbf{z}) \ge \kappa(|\mathbf{z}|)$ *for all* $\mathbf{z} \in R^m$, $\kappa \in \mathcal{K}_\infty$, $\kappa(s) \ge \tilde{\kappa}(s) + \sigma_4(s)$, $\tilde{\kappa} \in \mathcal{K}_\infty$.

Let the third part of assumption 4 and assumption 5 hold, and additionally for all $\mathbf{x} \in R^n$, $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$

$$\alpha_1(|\mathbf{h}(\mathbf{x})|) \le V(\mathbf{x}) \le \alpha_2(|\mathbf{h}(\mathbf{x})|).$$

Then system (5), (6) with control (13) is iISS with respect to set $\mathcal{A} = \{\mathbf{x}, \mathbf{z} : \mathbf{x} \in \mathcal{Z}, \mathbf{z} = \mathbf{k}(\mathbf{x})\}$ *with* $\varphi : R^m \to R^m$ *be a continuous function,* $\mathbf{z}^T \varphi(\mathbf{z}) \ge \kappa(|\mathbf{z}|)$ *for all* $\mathbf{z} \in R^m$, $\kappa \in \mathcal{K}$ *and* $\kappa(s) \ge \tilde{\kappa}(s) + \sigma_5(s)$, $\tilde{\kappa} \in \mathcal{K}$. ■

Controls (12) and (13) are robust modifications of controls (10) and (11) respectively. The difference between them consists in presence of additional feedback with respect to error $\mathbf{e} = \mathbf{z} - \mathbf{k}(\mathbf{x})$ with functional gain $1 + b(\mathbf{x}, \mathbf{z})^2$. In fact one can hide this gain under sign of function φ. Therefore the price of robustification of controls (10) and (11) is more stronger requirement the growth rate for function φ.

IV. Control of a Pendulum with Actuator

Let us consider the following example of system (5), (6):

$$\dot{x}_1 = x_2;\ \ \dot{x}_2 = -\theta x_2 - \omega^2 \sin(x_1) + z + d_1; \qquad (14)$$

$$\dot{z} = u + d_2, \qquad (15)$$

where subsystem (14) corresponds to the forced pendulum ($x_1 \in R$ is the angle and $x_2 \in R$ is the velocity); θ is friction coefficient and ω is pendulum frequency; system (15) models some actuator presence in the input of the pendulum; $\mathbf{d} = col(d_1\, d_2) \in R^2$ is external disturbances vector. It is well known that in the absence of friction (i.e. $\theta=0$) the function $H(x_1,x_2)=0.5x_2^2+\omega^2(1-\cos(x_1))$ describes the full energy of the pendulum. The problem is to stabilize the desired level H^* of energy function H. In this case we can introduce the output function for system (14) as $y=H(x_1,x_2)-H^*$, which points out to a compact set $\mathcal{Z}$. To UOS stabilization of system (14) it is possible to use well known control law [7], [19]:

$$\mathbf{k}(x_1,x_2)=-k\,x_2\,y+\theta x_2,$$

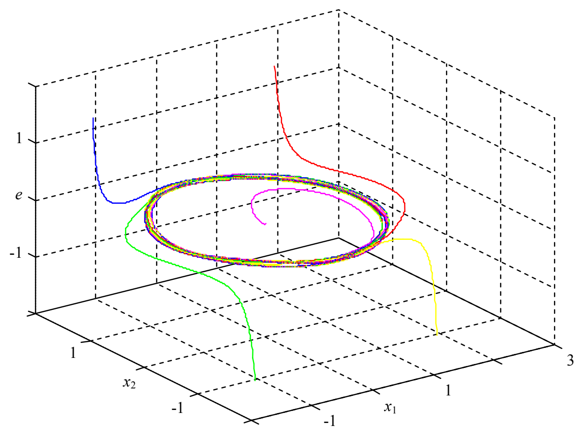

Fig. 1. Trajectories of system (14) – (16)

which for positive definite, radially unbounded with respect to set $\mathcal{Z}$ and differentiable Lyapunov function $V=0.5\,y^2$ provides the estimate for its time derivative

$$\dot V\le-(k-0.5)x_2^2\,y^2+d_1^2+e^2,\ k>0.5.$$

First of all note, that according to result of theorem A1 system (14) with control $\mathbf{k}$ ($e=0$) has iISS property with respect to set $\mathcal{Z}$, that is a new result too. Therefore, all conditions of the third part of assumption 4 are satisfied with $\psi(x_1,x_2)=x_2\,y$, $\alpha_{10}(s)=(k-0.5)s^2$ and $\sigma_5(s)=s^2$; set $\mathcal{X}=\{x_1,x_2:x_2=0\cup y=0\}$ and in paper [19] it was established V-detectability property of the system on this set. Time derivative of control $\mathbf{k}$ has form

$$\dot{\mathbf{k}}(x_1,x_2)=-k\,\dot x_2\,y-k\,x_2\,\dot y+\theta\dot x_2=(\theta-k\,y)(-\theta x_2-$$
$$-\omega^2\sin(x_1)+z+d_1)-k\,x_2(-\theta x_2^2+x_2\,z+x_2\,d_1)$$

and assumption 5 holds in this case for $b(x_1,x_2,z)=\theta-$ $-k\,y-k\,x_2^2$. Thus, according to theorem 5 control

$$u=-\left(K+0.5+0.5b(x_1,x_2,z)^2\right)(z-\mathbf{k}(x_1,x_2))+$$
$$+k\,x_2^2(z-\theta x_2)-(k\,y-\theta)\left(z-\theta x_2-\omega^2\sin(x_1)\right), \quad (16)$$

here $\varphi(\mathbf{z})=K\mathbf{z}$, $K>0$ provides for system (14), (15) iISS with respect to set $\mathcal{Z}$ property. The trajectories of system (14)–(16) for $\theta=\omega=1$, $K=1$, $k=2$, $H^*=1$, $d_1(t)=d_2(t)=0.1\sin(2t)+0.1$ are shown in Fig. 1 in coordinates (x_1,x_2,e), $e=z-\mathbf{k}(x_1,x_2)$. As it possible to conclude from Fig. 1 all trajectories of the system converge to small neighborhood of set $\mathcal{Z}$ in plane (x_1,x_2) while e converges to vicinity of zero, the size of neighborhood is proportional to amplitude of disturbance $\mathbf{d}$.

V. Conclusion

The paper presents the extension of backstepping technique for problem of stabilization with respect to set or UOS stabilization. Robust version of this result for IOS stabilization is also proposed. Introduced in the appendix new iISS property with respect to set helps to establish robust properties of passification controls in tasks of energy level stabilization.

Appendix

Here we will consider a dynamical nonlinear system

$$\dot{\mathbf{x}}=\mathbf{f}(\mathbf{x},\mathbf{u}), \quad (A.1)$$

where $\mathbf{x}\in R^n$ is state space vector, $\mathbf{u}\in R^m$ is control or disturbing input, Lebesgue measurable and essentially bounded function of time, that is $\mathbf{u}\in\mathcal{M}_{R^m}$; $\mathbf{f}$ is locally Lipschitz continuous vector field in R^n, $\mathbf{f}(0,0)=0$. Then $\mathbf{x}(t,\mathbf{x}_0,\mathbf{u})$ denotes the system solution with initial value $\mathbf{x}_0\in R^n$ and given $\mathbf{u}\in\mathcal{M}_{R^m}$ defined at the least locally. It is assumed, that system (A.1) has an uniformly globally asymptotically stable (GAS) set $\mathcal{A}$ for $\mathbf{u}\equiv0$ in the sense of definitions from [14]. Then according to that work the system possesses corresponding smooth Lyapunov function $V:R^n\to R_{\ge0}$ with respect to the set $\mathcal{A}$:

$$\alpha_1(|\mathbf{x}|_{\mathcal{A}})\le V(\mathbf{x})\le\alpha_2(|\mathbf{x}|_{\mathcal{A}}),\ \frac{\partial V}{\partial\mathbf{x}}\mathbf{f}(\mathbf{x},0)\le-\alpha_3(|\mathbf{x}|_{\mathcal{A}}), \quad (A.2)$$

where α_1, α_2, α_3 are from class $\mathcal{K}_\infty$; function V is positive definite and radially unbounded with respect to set $\mathcal{A}$. A positive definite function $\tilde V:R^n\to R_{\ge0}$ is called semi-proper [2], if there exist radially unbounded positive definite function $V:R^n\to R_{\ge0}$ and function $\pi\in\mathcal{K}$, such, that $\tilde V(\mathbf{x})=\pi\circ V(\mathbf{x})$. The following result is a slight extension of Proposition 2.5 proposed in [2].

L e m m a A 1. *System (A.1) is GAS with respect to compact set* $\mathcal{A}$ *with zero input* $\mathbf{u}$ *if and only if there exist smooth semi-proper function* $\tilde V:R^n\to R_{\ge0}$, *a function* $\sigma\in\mathcal{K}$ *and continuous positive definite function* $\rho:R_{\ge0}\to R_{\ge0}$, *such, that*

$$\frac{\partial\tilde V}{\partial\mathbf{x}}\mathbf{f}(\mathbf{x},\mathbf{u})\le-\rho(|\mathbf{x}|_{\mathcal{A}})+\sigma(|\mathbf{u}|), \quad (A.3)$$

for all $\mathbf{x}\in R^n$ *and* $\mathbf{u}\in R^m$. ■

Proofs are omitted due to space limitation. In work [2] several characterizations of integral input-to-state stability (iISS) property [21] were introduced. Here we present a simple development of that results for case of compact sets.

Definition A1. *Forward complete system (A.1) is called iISS with respect to closed invariant set* $\mathcal{A}$ *if there exist functions* $\alpha \in \mathcal{K}_\infty$, $\gamma \in \mathcal{K}$ *and* $\beta \in \mathcal{KL}$, *such, that for any* $\mathbf{x}_0 \in R^n$ *and all* $\mathbf{u} \in \mathcal{M}_{R^m}$, *the solution* $\mathbf{x}(t, \mathbf{x}_0, \mathbf{u})$ *is defined for all* $t \geq 0$ *and inequality holds:*

$$\alpha\left(\left|\mathbf{x}(t,\mathbf{x}_0,\mathbf{u})\right|_{\mathcal{A}}\right) \leq \beta\left(\left|\mathbf{x}_0\right|_{\mathcal{A}}, t\right) + \int_0^t \gamma\left(\left|\mathbf{u}(\tau)\right|\right) d\tau, \; t \geq 0 . \; \square$$

Definition A2. *Continuously differentiable function* $W : R^n \to R_{\geq 0}$ *is called an iISS Lyapunov function with respect to closed invariant set* $\mathcal{A}$ *for system (A.1) if there exist functions* $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, $\sigma \in \mathcal{K}$ *and continuous positive definite function* α_3 *s., t. for all* $\mathbf{x} \in R^n$, $\mathbf{u} \in R^m$:

$$\alpha_1\left(\left|\mathbf{x}\right|_{\mathcal{A}}\right) \leq W(\mathbf{x}) \leq \alpha_2\left(\left|\mathbf{x}\right|_{\mathcal{A}}\right),$$

$$\partial W / \partial \mathbf{x}\, \mathbf{f}(\mathbf{x},\mathbf{u}) \leq -\alpha_3\left(\left|\mathbf{x}\right|_{\mathcal{A}}\right) + \sigma\left(\left|\mathbf{u}\right|\right). \quad \square$$

If in above definition impose $\alpha_3 \in \mathcal{K}_\infty$, then one can recover W as ISS or IOS Lyapunov function with respect to set $\mathcal{A}$ from papers [25], [28]. The following theorem presents only sufficient conditions for system (A.1) to be iISS with respect to compact set, while the main result in [2] provides complete equivalent characterizations for iISS property with respect to the origin.

Theorem A1. *System (A.1) is iISS with respect to compact set* $\mathcal{A}$ *if one of the conditions is fulfilled*:

1. *The system has iISS Lyapunov function with respect to set* $\mathcal{A}$.

2. *Set* $\mathcal{A}$ *is GAS for the system with zero input* $\mathbf{u}$ *and for system (A.1) there exists continuously differentiable, positive definite and radially unbounded with respect to* $\mathcal{A}$ *function* $U : R^n \to R_{\geq 0}$ *and for all* $\mathbf{x} \in R^n$, $\mathbf{u} \in R^m$

$$\partial U / \partial \mathbf{x}\, \mathbf{f}(\mathbf{x},\mathbf{u}) \leq \nu\left(\left|\mathbf{u}\right|\right), \; \nu \in \mathcal{K} . \quad \blacksquare$$

References

[1] Angeli, D. and E.D. Sontag, "Forward completeness, unboundedness observability, and their Lyapunov characterizations". *Systems and Control Letters*, **38**, 1999, pp. 209 – 217.

[2] Angeli D., Sontag E.D. and Y. Wang, "A characterization of integral input to state stability". *Systems and Control Letters*, **38**, 1999, pp. 209 – 217.

[3] Arcak M., Seron M., Braslavsky J. and P. Kokotović, "Robustification of backstepping against input unmodeled dynamics". *IEEE Trans. Aut. Contr*., **45**, 2000, pp. 1 – 6.

[4] Artstein Z. "Stabilization with relaxed control". *Nonlinear Analysis*, **7**, 1983, pp. 1163–1173.

[5] Efimov D. V. "A condition of clf existence for affine systems". *Proc. of 41th IEEE Conference on Decision and Control*, Las Vegas, 2002.

[6] Efimov, D.V. "Universal formula for output asymptotic stabilization". *Proc. of 15th IFAC World Congress 2002*, T-We-M 07 4, Barselona, Spain, 2002.

[7] Fradkov A.L., Miroshnik I.V. and V.O. Nikiforov, *Nonlinear and adaptive control of complex systems*. Kluwer, 1999, p. 528.

[8] Isidori A. *Nonlinear control systems: An Introduction.* 2nd ed. — Berlin: Springer-Verlag, 1989, p. 478.

[9] Ingalls, B. and Y. Wang, "On input-to-output stability for systems not uniformly bounded". *In Proc. NOLCOS'01*, Saint-Petersburg, Russia, 2001.

[10] Kokotovirć, P.V. and M. Arcak, "Constructive nonlinear control: a historical perspective". *Automatica*, **37**, 2001, pp. 637 – 662.

[11] Kolesnikov A.A. *Synergetic control theory*. Energoatomizdat, Moscow, 1994, p. 344. (in Russian)

[12] Krstić M., Kanellakopoulos I. and P.V. Kokotović, *Nonlinear and Adaptive Control Design*. Wiley & Sons, Inc., 1995, p. 563.

[13] Lin Y., Sontag E.D. and Y. Wang, "A Smooth Converse Lyapunov Theorem for Robust Stability". *SIAM J. Control and Optimization*, **34**, 1996, pp. 124 – 160.

[14] Polushin I.G., Fradkov A.L., Putov V.V. and K.A. Rogov, "Energy control of one-degree-of-freedom oscillators in precense of bounded force disturbance". *Proc. of ECC99*, paper F1017–3 , 1999, p. 6.

[15] Rumyantsev, V.V. and A.S. Oziraner, *Stability and stabilization of motion with respect to part of variables.* Moscow: Nauka, 1987, 263 p. (in Russian)

[16] Sepulchre R, Jankovic M and P. Kokotović (1997). *Constructive nonlinear control*. Springer-Verlag, NY.

[17] Seron M., Hill D.J., Fradkov A.L. "Nonlinear adaptive control of feedback passive systems". *Automatica*, **31**, 1995, pp. 1053 – 1060.

[18] Shiriaev, A.S. and C. Canudas-de-Wit, "Virtual constraints: a tool for orbital stabilization of nonlinear systems theory". *Proc. 6th IFAC Symposium NOLCOS 2004*, Stuttgart, 2004, pp. 1355 – 1360.

[19] Shiriaev, A.S. and A.L. Fradkov, "Stabilization of invariant sets for nonlinear non-affine systems". *Automatica*, **36**, 2000, pp. 1709 – 1715.

[20] Shiriaev, A.S. and A.L. Fradkov, "Stabilization of invariant sets for nonlinear systems with application to control of oscillations". *International Journal of Robust and Nonlinear Control*, **11**, 2001, pp. 215 – 240.

[21] Skjetne R, Teel A.R. and P.V. Kokotović, "Nonlinear Maneuvering with Gradient Optimization". *Proc. of CDC 2001*, paper FrA11-1 , 2001, pp. 3926–3931.

[22] Sontag E.D. "A "universal" constraction of Arstein's theorem on nonlinear stabilization". *Systems & Control Letters*, **12**, 1989, pp. 542 – 550.

[23] Sontag E.D. "Smooth stabilization implies coprime factorization". *IEEE Trans. Aut. Contr*., **34**, 1989, pp. 435 – 443.

[24] Sontag E.D. "Comments on integral variants of ISS". *Systems and Control Letters*, **34**, 1998, pp. 93 – 100.

[25] Sontag, E.D. and Y. Wang, "Various results concerning set input-to-state stability". *Proc. IEEE CDC 95*, IEEE Publications, 1995, pp. 1330 – 1335.

[26] Sontag, E.D. and Y. Wang, "On characterization of the input to state stability property". *Systems and Control Letters*, **24**, 1995, pp. 351 – 359.

[27] Sontag, E.D. and Y. Wang, "Notions of input to output stability". *Systems and Control Letters*, **38**, 1999, pp. 235 – 248.

[28] Sontag, E.D. and Y. Wang, "Lyapunov characterizations of input to output stability". *SIAM Journal on Control and Optimization*, **39**, 2001, pp. 226–249.

[29] Teel A.R. "Using saturation to stabilize a class of single-input partially linear composite systems". *In Prep. of the 2nd IFAC Nonlinear Control Systems Design Symposium*, Bordeaux, France, 1992, p. 224–229.

[30] Vorotnikov V.I. *Partial Stability and Control.* Boston e.t.c.: Birkhauser, 1998, p. 442.