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Synchronization in Oscillator Networks: Switching Topologies and Non-homogeneous Delays

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Abstract— **We investigate the problem of synchronization in oscillator networks when the delay inherent in such systems is taken into account. We first investigate a general Kuramoto-type model with heterogeneous time delays, both with a complete network as well as a nearest neighbor interaction, for which we propose conditions for synchronization around a rotating frequency. Then, we turn our attention to the problem of synchronization when the topologies are allowed to change. We show that synchronization is possible in the presence of delay, using a common Lyapunov functional argument.**

I. Introduction

The phenomenon of *synchronization* [1] is one of the most intriguing in the physical world and shows the desire for nature to develop and maintain order of some sort. It is observed in biological oscillator networks such as cardiac pacemaker cells, as well as in ecological networks such as flashing fireflies and chorusing crickets. Beyond these are examples from various areas of engineering: for example in [2] the phenomenon was observed in semiconductor laser arrays.

From a design perspective, understanding how synchronization is achieved provides us with invaluable information for constructing large-scale systems for this purpose. For example, in leaderless coordination of multi-agent systems, a key issue is how to achieve consensus for arbitrary topologies, even when these change, a subject of many papers [3], [4], [5], [6]. The related issue of self-ordered particle motion was investigated in [7], [8]. Perhaps the most celebrated model for synchronization is the Kuramoto model, a system of structured ordinary differential equations, which has been used to explain how synchronization is achieved in many engineering, physical and biological systems. Details on the model can be found in [9], [10].

The original Kuramoto formulation was related to a network that is 'all-to-all', i.e. whose interconnection topology is one of a complete graph; the case of networks with arbitrary topologies was introduced and investigated in [3]. It was observed there too, that the system achieves synchronization for arbitrary topologies when the oscillators are identical.

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Kuramoto oscillator networks is the subject of this paper, but we include a feature that has been neglected in some of the earlier work on the subject. This is the presence of time delays in the oscillator interactions. Time delays in this framework can be used to model the effect of propagation of information in spatially large networks making the problem of achieving synchrony more difficult; for example in the phenomenon of flashing fireflies, one observes that synchronization is not perfect, just as in the case of a large crowd that is trying to sing the same song in synchrony. Previous work in this area has revealed that synchronization is possible for specific types of topologies, for homogeneous delays (i.e. identical for all interactions) and identical oscillators, under some specific assumptions. In this paper we will show that synchrony is possible for arbitrarily connected topologies, even in the presence of heterogeneous delays with non-identical oscillators under some assumptions.

Another aspect of the problem, which is also a subject of this paper, is whether synchronization can be ensured for *switching* topologies even if delays are present in the system. This question is somewhat related to the stability of systems for arbitrary switching with no chattering and with a finite dwell time, for which a sufficient but many times conservative condition is *quadratic stability* i.e. the existence of a common Lyapunov function for all the possible system instances (topologies) [11]. In a later part of this paper we present a proof of synchronization when switching occurs between connected topologies even if delays are included in the model description.

The paper is organized as follows. In Section II we present tools from Functional Differential Equation (FDE) theory that will be used in the rest of this paper, as well as some preliminaries in Algebraic Graph theory. In Section III we present our work on synchronization for a network with Kuramoto dynamics. In Section IV we investigate the effect that arbitrary switching between connected topologies has on synchronization, even when delays are taken into account. We conclude the paper in Section V.

II. Preliminaries

The effects of time-delay in system stability and performance has been a subject of intense research in the past few years [12], [13], and examples have traditionally been coming from population dynamics [14], and recently from network congestion control for the Internet [15]. In the latter problem, conditions ensuring stability of congestion control schemes for arbitrary topologies with delays at the linearized level have been proposed [16]; recently these have been extended to cover the nonlinear case [17], [18].

In this section we provide the theoretical basis that will be used in the sequel, and introduce some basic notations from algebraic graph theory.

A. General FDE Theory

Time delay systems are described by Functional Differential Equations. Let $C([a,b],\mathbb{R}^n)$ denote the Banach space of continuous functions mapping the interval $[a,b]$ to $\mathbb{R}^n$, with the topology of uniform convergence. For a function $\phi \in C([a,b],\mathbb{R}^n)$, we define the norm $|\phi| = \sup_{a\leq\theta\leq b}|\phi(\theta)|$. For any $\sigma > 0$, and any continuous function $\phi \in C([t_0 - a, t_0 - b+\sigma])$ and $t_0 \leq t \leq t_0+\sigma$, let $\phi_t \in C$ be a segment of ϕ defined by $\phi_t(\theta) = \phi(t+\theta), \theta \in [a,b]$. A Functional Differential Equation (FDE) of retarded type takes the form

$$\dot{x}(t) = f(t, x_t), \tag{1}$$

where $x(t) \in \mathbb{R}^n$ and $f : \mathbb{R}\times C \to \mathbb{R}^n$. An appropriate initial condition for such a differential equation is a function $x_{t_0} = \phi \in C$. For a $\sigma > 0$, a function x is a *solution on the interval* $[t_0 - a, t_0 - b + \sigma]$ if within this interval x is continuous and satisfies (1). Such a solution is known to exist and to be unique locally under specific assumptions, which can be found in, e.g [12]. Throughout the paper we assume that f satisfies these conditions. Just as in the case of Ordinary Differential Equations, we assume without loss of generality that there is a solution to (1) that satisfies $x^*(t) = 0$. Stability definitions for this steady-state can be found in [12].

Lyapunov theory for FDEs can be used to determine the stability properties of the equilibrium solutions. Lyapunov functions are functions of state; since the state itself in the case of FDEs is a function, the appropriate Lyapunov certificate is a functional, known as a *Lyapunov-Krasovskii functional*. In particular the following theorem is well known. Here, by $\dot{V}(t,\phi)$ we mean the upper right hand derivative of $V(t,\phi)$ along the solution of (1).

Theorem 1: [12] (*Lyapunov-Krasovksii*) Suppose $f : \mathbb{R}\times C \to \mathbb{R}^n$ in (1) maps $\mathbb{R}\times$ (bounded sets in C) into bounded sets in $\mathbb{R}^n$, and that $u, v, w : \mathbb{R}^+ \to \mathbb{R}^+$ are continuous nondecreasing functions, $u(s)$ and $v(s)$ are positive for $s > 0$, and $u(0) = v(0) = 0$. If there is a continuous function $V : \mathbb{R}\times C \to \mathbb{R}$ such that:

$$\begin{aligned} u(|\phi(0)|) \leq V(t,\phi) \leq v(|\phi|) \\ \dot{V}(t,\phi) \leq -w(|\phi(0)|) \end{aligned}$$

then the trivial solution of (1) is uniformly stable. If $w(s) > 0$ for $s > 0$, then it is uniformly asymptotically stable. If, in addition, $\lim_{s\to\infty} u(s) = \infty$, then it is globally uniformly asymptotically stable.

B. Autonomous FDEs

For autonomous functional differential equations, which take the form

$$\dot{x} = f(x_t), \tag{2}$$

with $f : C \to \mathbb{R}^n$ completely continuous, we consider a continuous function $V : C \to \mathbb{R}$. In this case, a LaSalle-type argument can be made when $\dot{V}$ is not negative definite to ensure attractivity to a positively invariant set, just as in the case of Ordinary Differential Equations. For this, define $V : C \to \mathbb{R}$ to be a Lyapunov function on a set G in C, if V be continuous on $\overline{G}$ (the closure of G) and $\dot{V} \leq 0$ on G. Let

$$S = \{\phi \in \overline{G} : \dot{V}(\phi) = 0\} \tag{3}$$

$$M = \begin{array}{l}\text{largest set in S that is invariant}\\ \text{with respect to (2)}\end{array} \tag{4}$$

Then we have the following theorem:

Theorem 2: [12] If V is a Lyapunov function on G and x_t is a bounded solution of (2) that remains in G, then x_t tends to M as $t \to \infty$.

We will be using the above theorem in the sequel.

C. Stability under arbitrary switching for systems of FDEs

Consider now a time-delay system comprised of a set of m systems of the form

$$\dot{x} = f_i(x_t), \quad i = 1,\ldots,m \tag{5}$$

We assume that 0 is a common steady state for all the above systems. The existence of a *common* Lyapunov function ensures the stability of the above system under arbitrary switching.

Theorem 3: Suppose $f_i : C \to \mathbb{R}^n$ in (5) are completely continuous for $i = 1,\ldots,m$ and that $u, v : \mathbb{R}^+ \to \mathbb{R}^+$ are continuous functions, $u(0) = v(0) = 0$ and $u(s) \to \infty$ as $s \to \infty$. If there is a continuous function $V : C \to \mathbb{R}$ such that:

$$\begin{aligned} V(\phi) \geq u(|\phi(0)|) \\ \dot{V}(\phi) \leq -v(|\phi(0)|) \end{aligned}$$

for all $i = 1,\ldots,m$ then the trivial solution of (5) is stable and every solution is bounded.

This theorem, in conjunction with the LaSalle type argument of the previous subsection will be used to prove synchronization under arbitrary switching for a system comprised of subsystems described by functional differential equations.

D. Algebraic Graph Theory

Throughout the paper we will be using the following notation to capture the topology of the network interactions. A graph $\mathbb{G} = (\mathcal{V},\mathcal{E})$ consists of a set of vertices $\mathcal{V}$ and a set of edges $\mathcal{E}$. We denote each vertex by $v_i \in \mathcal{V}$ for $i = 1,\ldots,N$, and each edge by $e = (v_i, v_j) \in \mathcal{E}$. All graphs in this paper are undirected. If $v_i, v_j \in \mathcal{V}$ and $(v_i, v_j) \in \mathcal{E}$, then v_i and v_j are neighbors. The valence of each vertex v_i, i.e. the number of its neighbors, is denoted by n_i. The valency matrix $\Delta(\mathbb{G})$ is an $N\times N$ diagonal matrix in which the (i,i) element is the valence of vertex i. If a graph is regular, then $n_i = n$ and $\Delta(\mathbb{G}) = nI$ where $I \in \mathbb{R}^{N\times N}$ denotes the identity matrix. A path of length r from vertex v_i to v_j is a sequence of $r+1$ distinct vertices starting from v_i

and ending at v_j so that consecutive vertices are neighbors. A graph $\mathbb{G}$ is said to be connected if there is a path between any two vertices in it. The adjacency matrix $A(\mathbb{G}) = [a_{ij}]$ of an (undirected) graph is an $N \times N$ symmetric matrix such that $a_{ij} = 1$ if v_i and v_j are neighbors, and $a_{ij} = 0$ otherwise.

The matrix

$$L(\mathbb{G}) = \Delta(\mathbb{G}) - A(\mathbb{G})$$

is called the Laplacian of G, and it has the properties that it is symmetric and singular (the row sums of L are zero). The algebraic multiplicity of its zero eigenvalue is equal to the number of connected components in the graph. The N-dimensional eigenvector associated to the zero eigenvalue is the vector of ones.

III. The Delayed Kuramoto model

In this section we provide the model we will be working with, previous results and our contribution.

Consider a set of N coupled oscillators with phases $\theta_i \in [0, 2\pi]$ and natural frequencies ω_i. The phase of each oscillator θ_i is associated to a vertex $v_i \in \mathcal{V}$ of an underlying undirected graph $\mathbb{G}$ with no loops and adjacency matrix A.

The original Kuramoto model, proposed by Kuramoto [10] in 1975 was for a complete graph (the 'all-to-all' case), and took the form:

$$\dot{\theta}_i = \omega_i + \frac{K}{N}\sum_{j=1}^{N} \sin(\theta_j - \theta_i)$$

where K is the coupling strength between the oscillators, assumed to be the same. An excellent review paper on the Kuramoto model and how synchronization emerges as a property of the system is [9]. The stability of this system when an underlying graph structure is imposed, i.e. when it is transformed to

$$\dot{\theta}_i = \omega_i + \frac{K}{N}\sum_{j=1}^{N} A_{ij}\sin(\theta_j - \theta_i) \qquad (6)$$

was the subject of an earlier paper [19], where a network of identical oscillators was proven stable for arbitrary topologies. The coupling strength K plays an important role when the oscillators are not the same, i.e. the ω_i differ — below a certain coupling strength, there is no synchronization. For the case of non-identical oscillators, lower and upper bounds on the strength of the coupling strength K to ensure synchrony were established in [19].

In this paper we consider a time-delayed version of the above system. We introduce a time delay τ_{ij} in the coupling between two vertices v_i and v_j if these are neighbors. This models the finite time in the propagation of phase information from oscillator to oscillator. We assume that these delays are inhomogeneous (i.e. unequal and non-commensurate), which is one of the main contributions of this paper. In particular we consider the following delayed Kuramoto dynamics at the i-th oscillator:

$$\dot{\theta}_i = \omega_i + \frac{K}{N}\sum_{j=1}^{N} A_{ij}\sin(\theta_j(t-\tau_{ij}) - \theta_i(t)) \qquad (7)$$

Most available results on the above system can be found in the physics literature: In [20] and [21] the above system was studied in the case of regular graphs with $\tau_{ij} = \tau$, and synchronization criteria were established. In [22] the general connected graph case was considered, in which the model was modified to

$$\dot{\theta}_i = \omega_i + \frac{K}{n_i}\sum_{j=1}^{N} A_{ij}\sin(\theta_j(t-\tau_{ij}) - \theta_i(t)) \qquad (8)$$

where n_i is the number of neighbors to node i, again for the case in which $\tau_{ij} = \tau$. Other phenomena were also observed, which are attributed to the presence of multi-stabilities. For example, a phenomenon known as 'time-delay-induced oscillator death' was asserted analytically [23] and was later observed experimentally [24]. This phenomenon is due to the fact that other equilibria may 'appear' as parameters change. Further results behind weakly connected oscillators in the presence of interaction delay have been given in [25].

We center our attention to the case in which all the oscillators synchronize, i.e. $\theta_i(t) = \theta(t)$ for all i. We are interested in the case of uniform rotations, i.e. $\theta(t) = \Omega t + \alpha$. For this to be achievable for system (7) self-consistency relations of the following form need to be imposed:

$$\Omega = \omega_i - \frac{K}{N}\sum_{j} A_{ij}\sin(\Omega\tau_{ij}) \qquad (9)$$

for all i. We assume that the oscillators ω_i and the coupling strength K are chosen judiciously so that there exists a compatible solution to the above equations. Now let us consider the dynamics of oscillator θ_i in this rotating frame. To do this, we write $\theta_i = \Omega t + \phi_i(t)$ to get:

$$\begin{aligned}\dot{\phi}_i(t) \quad &= \quad \frac{K}{N}\sum_{j=1}^{N} A_{ij}\sin(-\Omega\tau_{ij} + \phi_j(t-\tau_{ij}) - \phi_i(t)) \\ &\quad + \frac{K}{N}\sum_{j=1}^{N} A_{ij}\sin(\Omega\tau_{ij})\end{aligned}$$

Linearizing about the steady state $\phi_i(t) = \alpha$ we have:

$$\dot{\phi}_i(t) = \frac{K}{N}\sum_{j=1}^{N} G_{ij}(\phi_j(t-\tau_{ij}) - \phi_i(t)) \qquad (10)$$

where $G_{ij} = A_{ij}\cos(\Omega\tau_{ij})$. Note that this system does not have a unique equilibrium, but rather a *subspace* of equilibria, given by $\phi_i(t) = \alpha$, $i = 1, \ldots, N$. This subspace is one-dimensional in the case in which the underlying graph topology is connected, a property of the Laplacian dynamics of the undelayed system.

We have the following result for the linearization:

Theorem 4: Consider a network of N oscillators with dynamics given by (10) and with an underlying topology of a connected graph. If

$$\cos(\Omega\tau_{ij}) > 0$$

for all i, j for which $A_{ij} = 1$ and Ω solves (9), then

$$\lim_{t\to\infty} \phi_i(t) = \alpha, \quad i = 1, \ldots, N$$

for some constant α, i.e. the oscillators will synchronize.

Proof: Consider a Lyapunov-Krasovskii functional comprised of two terms:

$$\begin{aligned} V(\phi) &= V_1(\phi) + V_2(\phi) \\ V_1(\phi) &= \frac{1}{2}\sum_{i=1}^{N} \phi_i^2(t) \\ V_2(\phi) &= \frac{1}{2}\frac{K}{N}\sum_{i=1}^{N}\sum_{j=1}^{N} G_{ij}\int_{-\tau_{ij}}^{0} \phi_j^2(t+\xi)d\xi \end{aligned}$$

Note that $V > 0$. Consider the first term in the Lyapunov functional. Differentiating,

$$\begin{aligned} \dot{V}_1 &= \frac{K}{N}\sum_{i=1}^{N}\sum_{j=1}^{N} G_{ij}\left(\phi_j(t-\tau_{ij}) - \phi_i(t)\right)\phi_i(t) \\ &= -\frac{K}{N}\sum_{i=1}^{N}\sum_{j=1}^{N} G_{ij}\phi_i^2(t) \\ &\quad + \frac{K}{N}\sum_{i=1}^{N}\sum_{j=1}^{N} G_{ij}\phi_i(t)\phi_j(t-\tau_{ij}) \end{aligned}$$

The second term in the Lyapunov functional differentiates to

$$\dot{V}_2 = \frac{1}{2}\frac{K}{N}\sum_{i=1}^{N}\sum_{j=1}^{N} G_{ij}\phi_j^2(t) - \frac{1}{2}\frac{K}{N}\sum_{i=1}^{N}\sum_{j=1}^{N} G_{ij}\phi_j^2(t-\tau_{ij})$$

Therefore we have:

$$\dot{V} = -\frac{1}{2}\frac{K}{N}\sum_{i=1}^{N}\sum_{j=1}^{N} G_{ij}(\phi_i(t) - \phi_j(t-\tau_{ij}))^2 \leq 0,$$

so $\dot{V} \leq 0$. The set S defined by (3) is the set of all functions $\phi \in C([-\max_{i,j}\tau_{ij}, 0], \mathbb{R}^N)$ such that $\dot{V} = 0$. This is only true when $\phi_i = \phi_j = \alpha$. To see this, note that $\dot{V} = 0$ implies that $\phi_i(t) = \phi_j(t-\tau_{ij})$ for any i that is adjacent to j. Since the graph is connected and the delays are finite, the result follows immediately. The only invariant set for the system's equations, the set M (see Equation (4)) corresponds to $\phi_i = \alpha$, a constant. Using a version of LaSalle theorem for time-delayed systems (Theorem 2), we conclude that $\phi_i \to \alpha$ as $t \to \infty$. Therefore the oscillators synchronize. ■

Note that in the proof we did not require that the graph be regular, nor that the delays be homogeneous which were key assumptions in [21].

It may be counter-intuitive that the terms in the vector field (10) were scaled down by the total number of nodes in the network N. A more interesting case would be the one in which they are scaled down only by the number of neighbors to node i. This is the system given by Equation (8). The linearization about the uniform rotation state $\theta_i = \Omega t + \alpha$, where Ω satisfies

$$\Omega = \omega_i - \frac{K}{n_i}\sum_{j} A_{ij}\sin(\Omega\tau_{ij}) \tag{11}$$

is

$$\dot{\phi}_i(t) = \frac{K}{n_i}\sum_{j=1}^{N} G_{ij}(\phi_j(t-\tau_{ij}) - \phi_i(t)) \tag{12}$$

and $G_{ij} = A_{ij}\cos(\Omega\tau_{ij})$ as before. For this system, we have the following result:

Theorem 5: Consider a network of N oscillators with dynamics given by (12) and with an underlying topology of a connected graph. If

$$\cos(\Omega\tau_{ij}) > 0$$

for all i, j for which $A_{ij} = 1$ and Ω solves (11), then

$$\lim_{t\to\infty} \phi_i(t) = \alpha, \quad i = 1, \ldots, N$$

for some constant α, i.e. the oscillators will synchronize.

The proof of this theorem is similar to the proof of Theorem 4.

Proof: Consider a Lyapunov-Krasovskii functional comprised of two terms:

$$\begin{aligned} V(\phi) &= V_1(\phi) + V_2(\phi) \\ V_1(\phi) &= \frac{1}{2}\sum_{i=1}^{N} n_i\phi_i^2(t) \\ V_2(\phi) &= \frac{1}{2}K\sum_{i=1}^{N}\sum_{j=1}^{N} G_{ij}\int_{-\tau_{ij}}^{0} \phi_j^2(t+\xi)d\xi \end{aligned}$$

Note that $V > 0$. Consider the first term in the Lyapunov functional. Differentiating,

$$\begin{aligned} \dot{V}_1 &= K\sum_{i=1}^{N}\sum_{j=1}^{N} G_{ij}\left(\phi_j(t-\tau_{ij}) - \phi_i(t)\right)\phi_i(t) \\ &= -K\sum_{i=1}^{N}\sum_{j=1}^{N} G_{ij}\phi_i^2(t) \\ &\quad + K\sum_{i=1}^{N}\sum_{j=1}^{N} G_{ij}\phi_i(t)\phi_j(t-\tau_{ij}) \end{aligned}$$

The second term in the Lyapunov functional differentiates to

$$\dot{V}_2 = \frac{1}{2}K\sum_{i=1}^{N}\sum_{j=1}^{N} G_{ij}\phi_j^2(t) - \frac{1}{2}K\sum_{i=1}^{N}\sum_{j=1}^{N} G_{ij}\phi_j^2(t-\tau_{ij})$$

Therefore we have

$$\dot{V} = -\frac{1}{2}K\sum_{i=1}^{N}\sum_{j=1}^{N} G_{ij}(\phi_i(t) - \phi_j(t-\tau_{ij}))^2 \leq 0.$$

We can use a similar argument as in the proof of the previous theorem to get attractivity to a set, i.e. to show that the oscillators synchronize about the steady-state $\phi_i = \alpha$. ■

We would like to make a few remarks about the above systems. First, the stability proof uses Lyapunov-Krasovskii functionals that prove stability *independent of delay* even though the criteria for stability are delay-dependent. This is because even though a system of the form

$$\dot{x} = -ax(t) - bx(t-\tau)$$

is asymptotically stable for all delays if $a > 0$ and $|b| < a$, in our case, the coefficients a and b are static functions of the delay themselves, and this gives rise to the delay-dependent criterion. The 'scaling' of the gains by the delay size was judiciously used in the case of network congestion control for the Internet to render a delay-dependent stability condition delay-independent [16].

For comparison purposes, both systems (7) and (8) with *homogeneous* delays $\tau_{ij} = \tau$ and identical oscillators $\omega_i = \omega$ were shown to be stable under the assumption that $\cos \Omega\tau > 0$ in [21] for regular graphs and in [26] for general graphs. In both cases the parameter Ω solves the 'self-consistency' relation

$$\Omega = \omega - K\sin(\Omega\tau). \tag{13}$$

In our case, we allow for heterogeneous delays, and networks that need not be regular. The problem that may arise with the introduction of heterogeneous delays in the network, is that Equations (9) or (11) may not have solutions, unless the coupling strength K and the oscillator frequencies ω_i are tuned appropriately.

We would like to stress that synchronization is independent of the network topology. A question that arises, is whether the same properties hold for the system when the topology changes, which is a more complicated issue. In the next section we present a result that shows how the system given by (12) with homogeneous delays can switch in an arbitrary way between topologies with regular graph structures, in a way that synchronization is ensured.

IV. Switching topologies

The issue of coordination under changing topologies has been investigated in [3] [6] [4] [27], for the case in which the system does not have any delays that make the state-space infinite-dimensional. But even if the time-delays are ignored, the problem of ensuring stability in this case is difficult. Switching arbitrarily among a set of possible topologies of size N can be regarded as a problem of establishing stability for a switching system with an unknown switching rule.

In the area of switching systems, the (conservative) condition of *quadratic stability* has been used to ensure that a system comprised of m subsystems of the form $\dot{x} = A_i x$, $i = 1,\ldots,m$ is stable under arbitrary switching; the conditions in this case require the existence of a *common* Lyapunov function $V = x^T P x$, $P > 0$ so that $A_i^T P + PA_i < 0$ for all $i = 1,\ldots,m$. This argument is many times inconclusive, as this criterion is conservative. This conservativeness was observed in [3], where a different approach had to be taken to conclude coordination.

This said, one can imagine the difficulties faced when trying to prove stability for arbitrary switching when delays are taken into account. This is the aim of this section, in which we consider a model that allows synchronization to be ensured for Kuramoto networks with delays, and arbitrary switching between connected, regular graphs. In other words, we assume that switching occurs through a piecewise constant switching signal $\sigma : [0,\infty) \to P$ where $P = \{1,\ldots,m\}$ denotes the indices of m regular graphs $\mathbb{G}_m$ of degree n with N vertices so that $\mathbb{G}_i$, $i = 1,\ldots,m$ are connected. Associated to graph $\mathbb{G}_k$ is an adjacency matrix $A^{(k)}$. The switching signal is assumed to be non-chattering, with a dwell time $\tau_D > 0$, i.e. consecutive discontinuities of σ are separated by τ_D. The switching signal may however have an infinite period of persistence, i.e. $\sigma \in S_{\text{weak-dwell}}$ — for definitions see [28].

A technicality which we are faced with is that in the presence of heterogeneous delays and changing graph structures the Ω that solves the various consistency conditions may be different. For this reason, in this section we assume that we are dealing with identical oscillators with homogeneous time delays over regular graphs. In this case, the self-consistency equation is given by (13). Each network has the following dynamics:

$$\dot{\phi}_i = \frac{K}{n}\sum_{j=1}^{N} G_{ij}^{(k)}(\phi_j(t-\tau)-\phi_i(t)), \quad k = 1,\ldots,m. \tag{14}$$

where $G_{ij}^{(k)} = A_{ij}^{(k)}\cos(\Omega\tau)$. We have the following result:

Theorem 6: Consider a piecewise constant switching signal $\sigma : [0,\infty) \to P$ where $P = \{1,\ldots,m\}$ is the index set of graphs $\{\mathbb{G}_1,\ldots,\mathbb{G}_m\}$ with N vertices that are connected and regular, and $\sigma \in S_{\text{weak-dwell}}$. To each vertex v_i assign the dynamics given by Equation (14). Then

$$\lim_{t\to\infty} \phi_i(t) = \alpha, \quad i = 1,\ldots,N$$

for some constant α, i.e. the oscillators will synchronize.

Proof: We will show that the following is a common Lyapunov functional:

$$V = \frac{1}{2}\sum_{i=1}^{N}\phi_i^2 + \frac{K\cos(\Omega\tau)}{2}\sum_{i=1}^{N}\int_{-\tau}^{0}\phi_i^2(t+\xi)d\xi \tag{15}$$

Note that $V > 0$ and is independent of the topology of the network. The derivative of this, when the topology is given by graph $\mathbb{G}_k$ is:

$$\dot{V} = \frac{K}{n}\sum_{i=1}^{N}\sum_{j=1}^{N} G_{ij}^{(k)}(\phi_j(t-\tau)-\phi_i(t))\phi_i(t) + \frac{K\cos(\Omega\tau)}{2}\sum_{i=1}^{N}(\phi_i^2(t)-\phi_i^2(t-\tau))$$

$$
\begin{aligned}
&= -\frac{K\cos(\Omega\tau)}{2}\sum_{i=1}^{N}\left(\phi_i^2(t)+\phi_i^2(t-\tau)\right)\\
&\quad+\frac{K\cos(\Omega\tau)}{n}\sum_{i=1}^{N}\sum_{j=1}^{N}A_{ij}^{(k)}\phi_j(t-\tau)\phi_i(t)\\
&= -\frac{K\cos(\Omega\tau)}{2n}\sum_{i=1}^{N}\sum_{j=1}^{N}A_{ij}^{(k)}\left(\phi_j(t-\tau)-\phi_i(t)\right)^2
\end{aligned}
$$

Note that $\dot{V} \leq 0$ for all graphs $\mathbb{G}_i$, so as switching occurs V is non-increasing. For each graph $\mathbb{G}_i$, the set S defined by (3) is the set of all functions $\phi \in C([-\tau, 0], \mathbb{R}^N)$ such that $\dot{V} = 0$. This is only true when $\phi_i = \phi_j = \alpha$. Since we assumed that switching happens between connected graphs and $\sigma \in S_{\text{weak-dwell}}$, the only invariant set for the system's equations, the set M (see Equation (4)) is one-dimensional and corresponds to $\phi_i = \alpha$, a constant. Using a version of LaSalle theorem for time-delayed systems (Theorem 2), we conclude that $\phi_i \to \alpha$ as $t \to \infty$. Therefore the oscillators synchronize, even if the topology changes. ∎

Remark 7: The connectivity assumption in the above theorem can probably be relaxed to the case in which the set $\{\mathbb{G}_1, \ldots, \mathbb{G}_m\}$ is *jointly* connected, i.e. if the graph $\mathbb{G}$ with vertex set $\mathcal{V}$ and edge set $\mathcal{E}$ equaling the union of the edge sets of all of the graphs in the collection is connected [29]. This ensures that there is a time interval $[a, b]$ within which all oscillators are 'linked together' by the sequence of graphs encountered in it. This will be the focus of future research.

V. Conclusions

In this paper we have considered the problem of synchronization of coupled oscillator networks in the Kuramoto framework with heterogeneous communication delays and not necessarily regular graphs and have derived conditions that ensures it. We have manifested that synchronization can be achieved even if the oscillators are non-identical, if we allow heterogeneous delays in the networks. Moreover, we have established synchronization for Kuramoto networks with switching topologies even in the presence of delay.

The synchronization of the delayed Kuramoto models may not be global in the nonlinear case, as in general the solution of the consistency Equations (9) or (11) may not be unique. The uniqueness of the solution to the consistency equations will depend on the value of K - and so will the stability properties of these uniform rotation states.

A nonlinear result similar to the one obtained in the case of network congestion control would be desirable, at least in the case in which there is a unique attracting set. This will be the focus of future research.

References

[1] S. Strogatz, *SYNC: The emerging science of spontaneous order*. Hyperion Press, New York, 2003.

[2] G. Kozyreff, A. G. Vladimirov, and P. Mandel, "Global coupling with time delay in an array of semiconductor lasers," *Physical Review Letters*, vol. 85, no. 18, pp. 3809–3812, 2000.

[3] A. Jadbabaie, J. Lin, and A. Stephen Morse, "Coordination of groups of mobile autonomous agents using nearest neighbor rules," *IEEE Transactions on Automatic Control*, vol. 48, no. 6, pp. 988–1001, 2003.

[4] L. Moreau, "Leaderless coordination via bidirectional and unidirectional time-dependent communication," in *Proceedings of the IEEE Conference on Decision and Control*, 2003.

[5] R. Olfati-Saber and R. M. Murray, "Consensus protocols for networks of dynamic agents." in *Proceedings of the American Control Conference*, 2003.

[6] ——, "Consensus problems in networks of agents with switching topology and time-delays," *IEEE Transactions On Automatic Control*, vol. 49, pp. 1520–1533, 2004.

[7] T. Vicsek, "A question of scale," *Nature*, vol. 411, p. 421, 2001.

[8] T. Vicsek, A. Czirók, E. Ben-Jacob, I. Cohen, and O. Shochet, "Novel type of phase transition in a system of self-driven particles," *Physical Review Letters*, vol. 75, no. 6, pp. 1226–1229, 1995.

[9] S. H. Strogatz, "From Kuramoto to Crawford: exploring the onset of synchronization in populations of coupled oscillators," *Physica D*, vol. 143, pp. 1–20, 2000.

[10] Y. Kuramoto, *Chemical Oscillations, Waves and Turbulence*. Dover, 2003.

[11] M. Johansson and A. Rantzer, "Computation of piecewise quadratic Lyapunov functions for hybrid systems," *IEEE Transactions on Automatic Control*, vol. 43, no. 4, pp. 555–559, 1998.

[12] J. K. Hale and S. M. V. Lunel, *Introduction to Functional Differential Equations*, ser. Applied Mathematical Sciences (99). Springer-Verlag, 1993.

[13] V. Kolmanovskii and A. Myshkis, *Introduction to the Theory and Applications of Functional Differential Equations*. Kluwer Academic Publishers, 1999.

[14] Y. Kuang, *Delay Differential Equations with Applications in Population Dynamics*, ser. Mathematics in Science and Engineering (191). Academic Press, 1993.

[15] R. Srikant, *The Mathematics of Internet Congestion Control*. Birkhäuser, 2003.

[16] G. Vinnicombe, "On the stability of end-to-end congestion control for the Internet," Cambridge University, CUED/F-INFENG/TR.398, Tech. Rep., December 2000.

[17] L. Ying, G. Dullerud, and R. Srikant, "Global stability of Internet congestion controllers with heterogeneous delays," in *Proceedings of the American Control Conference*, 2004.

[18] A. Papachristodoulou, "Global stability analysis of a TCP/AQM protocol for arbitrary networks with delay," in *Proceedings of the IEEE Conference on Decision and Control*, 2004.

[19] A. Jadbabaie, N. Motee, and M. Barahona, "On the stability of the Kuramoto model of coupled nonlinear oscillators," in *Proceedings of the American Control Conference*, 2004.

[20] M. K. Stephen Yeung and S. H. Strogatz, "Time delay in the Kuramoto model of coupled oscillators," *Physical Review Letters*, vol. 82, no. 3, pp. 648–651, 1999.

[21] M. G. Earl and S. H. Strogatz, "Synchronization in oscillator networks with delayed coupling: A stability criterion," *Physical Review E*, vol. 67, 036204, no. 3, 2003.

[22] C. Li and G. Chen, "Synchronization in general complex dynamical networks with coupling delays," *Physica A*, vol. 343, pp. 263–278, 2004.

[23] D. V. Ramana Reddy, A. Sen, and G. L. Johnston, "Time delay induced death in coupled limit cycle oscillations," *Physical Review Letters*, vol. 80, no. 23, pp. 5109–5112, 1998.

[24] ——, "Experimental evidence of time-delay-induced death in coupled limit-cycle oscillations," *Physical Review Letters*, vol. 85, no. 16, pp. 3381–3384, 2000.

[25] E. M. Izhikevich, "Phase models with explicit time delays," *Physical Review E*, vol. 58, no. 1, pp. 905–908, 1998.

[26] C. Li, H. Xu, X. Liao, and J. Yu, "Synchronization in small-world oscillator networks with coupling delays," *Physica A*, vol. 335, pp. 359–364, 2004.

[27] D. Angeli and P.-A. Bliman, "Stability of leaderless multi-agent systems. Extension of a result by Moreau," 2004, oai:arXiv.org:math/0411338.

[28] J. P. Hespanha, "Uniform stability of switched linear systems: Extensions of LaSalle's invariance principle," *IEEE Transactions on Automatic Control*, vol. 49, no. 4, pp. 470–482, 2004.

[29] L. Moreau, "Stability of continuous-time distributed consensus algorithms," 2004, arXiv:math.OC/0409010.