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A Stability Robustness Test for Systems with Linear Time-varying Uncertainties*

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Abstract— **In this paper, a novel stability robustness test for systems with linear time-varying uncertainties is introduced. The advantages of the stability robustness test lie in the fact that it can cope with multi-input multi-output, stable, or unstable systems whose linear time-varying uncertainties can be represented by a standard structured uncertainty model, and that the implementation of the test is straightforward. The tool used to provide the means of determining system stability robustness is the linear time-invariant ν-gap metric.**

I. Introduction

When a stabilizing controller is designed for a nominal plant, a desired objective is that the controller also succeeds in stabilizing the "true-life" system in the face of uncertainty [1]. Frequently, uncertainty is modelled as an unstructured perturbation to the nominal plant; classes of these uncertainties include additive uncertainty, input- or output-multiplicative uncertainty and input- or output-feedback uncertainty. The more general structured uncertainty model may be used when plants are subjected to multiple uncertainties, for example when the plant contains multiple unstructured uncertainties, or when the plant contains a number of uncertain parameters. The aim of this paper is to introduce a stability robustness test for systems with linear time-varying (LTV) uncertainties that can be represented with the standard structured uncertainty model.

The tool used to provide the means of determining system stability robustness is the LTI (linear time-invariant) ν-gap metric, defined in [2]. Regarding this metric, it was shown that, given a nominal LTI system and stabilizing LTI controller, that the same controller was guaranteed to stabilize a second LTI system, provided that the distance between the two plants, as measured by the ν-gap metric, was smaller than a number determined by the size of the nominal closed-loop transfer function matrix. It was also shown that if the distance between the nominal system and the second LTI system, as measured by the ν-gap metric, was not sufficiently small, then the second plant would be destabilized by some controller achieving a certain level of performance with the nominal system [2, Theorem 4.5]. Furthermore, it was shown that the LTI ν-gap metric is easy to compute.

In [3, Section 5], a simplified stability robustness test for single-input single-output systems with LTV output-multiplicative uncertainties was presented without proof. This test involved determining whether a certain LTI quantity was smaller than a number determined by the size of the nominal closed-loop transfer function matrix, called the generalized robust stability margin [4]. It was shown that if this LTI quantity was indeed the smaller of the quantities, then the controller that stabilized the nominal LTI system was also guaranteed to stabilize the system with the LTV uncertainty. It was then shown that it was possible to transform the problem of computing the LTI quantity into one of simply calculating a LTI ν-gap distance [3, Theorem 4]. This result is valid only for classes of multiplicative uncertainty.

The test introduced in this paper is a generalization of the kinds of results in [3, Section 5]. Now, multi-input multi-output systems with multiple uncertainty blocks (as opposed to one block) are considered, and the class of uncertainty is not restricted to the output-multiplicative family. The theoretical basis of the test is given in Section III. Again, it is shown that if a certain LTI quantity is smaller than the generalized robust stability margin achieved by a controller stabilizing the nominal LTI system, then robust internal stability is guaranteed for systems with LTV uncertainties.

Unlike in the case for the LTI quantity in [3], the more general, and therefore more complex, LTI quantity in this paper cannot be found by computing a LTI ν-gap distance. However, it is shown in Section IV that the quantity can, in fact, be determined by selecting a number, say β, as a guess to the LTI quantity, solving a collection of linear matrix inequality (LMI) feasibility problems, and implementing a bisectional line search over β. A numerical example is given in Section V.

Notation

The notation is standard. The space $\mathscr{L}_2(-\infty,\infty)$ consists of Lebesgue measurable functions with finite norm. $\mathscr{L}_2[0,\infty)$ is the subspace of $\mathscr{L}_2(-\infty,\infty)$ with functions zero for $t<0$. $\mathscr{R}$ denotes the set of proper real rational transfer function matrices. $\mathscr{L}_\infty$ is a Banach space of matrix- (or scalar-) valued functions that are essentially bounded on $j\mathbb{R}$. The Hardy space, $\mathscr{H}_\infty$, is the closed subspace of $\mathscr{L}_\infty$ with functions that

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are analytic and bounded in the open right-half plane (RHP), with norm denoted $\|\cdot\|_\infty$. In other words, $\mathscr{H}_\infty$ is the space of transfer functions of stable, LTI, continuous-time systems. $\mathscr{RH}_\infty$ denotes the subspace of $\mathscr{H}_\infty$ whose transfer function matrices are proper and real rational. The $\mathscr{L}_2$-induced norm for LTV operators will be denoted by $\|\cdot\|$ and corresponds to $\|\cdot\|_\infty$ for LTI systems.

For a general matrix $X = [x_{ij}] \in \mathbb{C}^{r\times s}$, X^* denotes the complex conjugate transpose $[\bar{x}_{ji}]$. For a transfer function matrix $X(s) \in \mathscr{R}^{r\times s}$, $X^*(s)$ is defined to mean $X(-s)^T$. The notation

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$$

refers to a system realization (A,B,C,D) and the notation $F_l(F,C) := F_{11} + F_{12}C(I-F_{22}C)^{-1}F_{21}$ and $F_u(F,E) := F_{22} + F_{21}E(I-F_{11}E)^{-1}F_{12}$ refers to the standard lower and upper linear fractional representations (LFTs), respectively, as shown in Fig. 1. $F_1 \star F_2$ denotes the interconnection of two LFTs known as the Redheffer star-product, as shown in Fig. 2. Well-posedness is assumed.

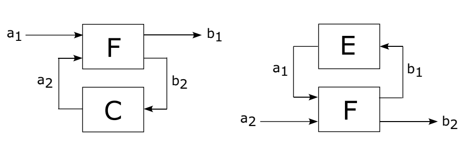

Fig. 1. Lower and upper LFTs respectively.

II. Mathematical preliminaries

In this section, a number of mathematical objects that are used often throughout the paper are defined. First, consider the following sets of LTV and LTI uncertainties. Let $\mathbf{\Delta}$ be the set of causal, stable, structured LTV uncertainties with $\mathscr{L}_2$-induced norm strictly less than one. By structured, it is meant that $\mathbf{\Delta}$ contains elements consisting of k blocks, not necessarily square, with each block denoted $\Delta^{q_1\times p_1}, \ldots, \Delta^{q_k\times p_k}$ and where $q_1 + \ldots + q_k = q$ and $p_1 + \ldots + p_k = p$. Let $\boldsymbol{\delta} := \{\delta : \delta \in \mathscr{RH}_\infty^{q\times p}, \|\delta\|_\infty < 1\}$. Similarly to [5], [6], define a set of constant diagonal matrix pairs sharing the same scalar coefficients as $\boldsymbol{D} := \{(D_l, D_r) : D_l = \mathrm{diag}(d_1 I_{q_1}, \ldots, d_k I_{q_k}), D_r = \mathrm{diag}(d_1 I_{p_1}, \ldots, d_k I_{p_k}), d_i \in \mathbb{R}_+\}$, and note that for each $D = (D_l, D_r) \in \boldsymbol{D}$, D_l and D_r commute with $\mathbf{\Delta}$ in the following way:

$$\Delta = D_l \Delta D_r^{-1} \;\forall\; \Delta \in \mathbf{\Delta}.$$

Next, consider a generalized system F that is partitioned as follows:

$$F = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix}, \tag{1}$$

where $F_{11} \in \mathscr{R}^{p\times q}$, $F_{12} \in \mathscr{R}^{p\times m}$, $F_{21} \in \mathscr{R}^{n\times q}$, $F_{22} \in \mathscr{R}^{n\times m}$. Let $P_0 := F_u(F,0)$ denote a nominal plant. Let a controller be $C \in \mathscr{R}^{m\times n}$.

Finally, let the interconnection of the plants P_0 and C, as shown in Fig. 3, be denoted by $[P_0, C]$. This interconnection is said to be internally stable if it is well-posed and if each of the four transfer functions mapping the signals v_1 and v_2 to y and u are stable; that is, they belong to $\mathscr{RH}_\infty$ in a LTI setting. Define the generalized robust stability margin, denoted $b_{P_0,C}$; the optimal generalized robust stability margin, $b_{opt}(P_0) := \sup_C b_{P_0,C}$; and the LTI ν-gap metric, denoted $\delta_\nu(P_0, P_1)$, as in [2].

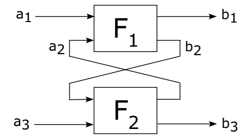

Fig. 2. Redheffer star product.

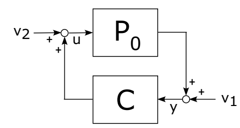

Fig. 3. Internal stability of $[P_0, C]$.

III. Main result

This section contains the result which forms the theoretical basis for the stability robustness test. In words, the result states that if a certain LTI quantity is smaller than the generalized robust stability margin determined from a nominal plant and a stabilizing controller, then the same controller will stabilize the system when subject to LTV uncertainty. Formally, this result is stated as Theorem 1 below. First, the definition of an induced realization is required.

Definition 1: Let a stabilizable and detectable realization[1] for $F \in \mathscr{R}^{(p+n)\times(q+m)}$ be given by

$$\left(\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right), \tag{2}$$

and let stabilizable and detectable realizations for $C \in \mathscr{R}^{m\times n}$ and $D_l^{-1}\delta D_r \in \mathscr{R}^{q\times p}$ be $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ and $(\check{A}, \check{B}, \check{C}, \check{D})$, respectively. The induced realization for

(a) $F_l(F,C)$ is the realization formed from the realizations for F and C, and is given by

$$\left(\begin{array}{c|c} A_\theta & B_\theta \\ \hline C_\theta & D_\theta \end{array}\right), \tag{3}$$

where $A_\theta := \begin{pmatrix} A+B_2\hat{D}C_2 & B_2\hat{C} \\ \hat{B}C_2 & \hat{A} \end{pmatrix}$, $B_\theta := \begin{pmatrix} B_1+B_2\hat{D}D_{21} \\ \hat{B}D_{21} \end{pmatrix}$, $C_\theta := \begin{pmatrix} C_1+D_{12}\hat{D}C_2 & D_{12}\hat{C} \end{pmatrix}$ and $D_\theta := D_{11} + D_{12}\hat{D}D_{21}$;

(b) $F_u(F, D_l^{-1}\delta D_r)$ is the realization formed from the realizations for F and $D_l^{-1}\delta D_r$, and is given by

$$\left(\begin{array}{c|c} A_\eta & B_\eta \\ \hline C_\eta & D_\eta \end{array}\right), \tag{4}$$

where $A_\eta := \begin{pmatrix} \check{A}+\check{B}(I-D_{11}\check{D})^{-1}D_{11}\check{C} & \check{B}(I-D_{11}\check{D})^{-1}C_1 \\ B_1(I-\check{D}D_{11})^{-1}\check{C} & A+B_1(I-\check{D}D_{11})^{-1}\check{D}C_1 \end{pmatrix}$, $B_\eta := \begin{pmatrix} \check{B}(I-D_{11}\check{D})^{-1}D_{12} \\ B_1(I-\check{D}D_{11})^{-1}\check{D}D_{12}+B_2 \end{pmatrix}$, $C_\eta := \begin{pmatrix} D_{21}(I-\check{D}D_{11})^{-1}\check{C} & D_{21}(I-\check{D}D_{11})^{-1}\check{D}C_1+C_2 \end{pmatrix}$ and $D_\eta := D_{21}(I-\check{D}D_{11})^{-1}\check{D}D_{12}$.

[1]The D_{22} term may be absorbed into C by a loop shifting argument (see [7, Section 4.6] for instance).

Theorem 1 (Main result): Let a generalized plant $F \in \mathscr{R}^{(p+n)\times(q+m)}$ be partitioned as in (1) and have a stabilizable and detectable realization as given by (2). Let $P_0 := F_u(F,0)$ be the nominal plant with an inherited realization (A,B_2,C_2) which is stabilizable and detectable[2], and let $C \in \mathscr{R}^{m\times n}$ be a stabilizing controller for P_0. Consider the uncertainty sets, $\boldsymbol{\Delta}$ and $\boldsymbol{\delta}$, and the set of constant diagonal matrix pairs, $\boldsymbol{D}$, as defined in Section II. Suppose that each induced realization for $F_u(F,\Delta)$ and $F_u(F,D_l^{-1}\delta D_r)$ is stabilizable and detectable. If

$$\inf_{D=(D_l,D_r)\in\boldsymbol{D}} \sup_{\delta\in\boldsymbol{\delta}} \delta_\nu(P_0,F_u(F,D_l^{-1}\delta D_r)) < b_{P_0,C}, \qquad (5)$$

then $[P_{LTV},C]$ is internally stable for all $\Delta\in\boldsymbol{\Delta}$, where $P_{LTV} := F_u(F,\Delta)$ is a time-varying uncertain plant.

The complete proof of the main result will be published elsewhere. The following proof sketch has been included to illustrate several key steps and to motivate the statement of some important results. The proof sketch proceeds as follows. From LTI ν-gap theory [2], it is known that if $\delta_\nu(P_0,F_u(F,D_l^{-1}\delta D_r)) < b_{P_0,C}$, then the closed-loop system $[F_u(F,D_l^{-1}\delta D_r),C]$ is internally stable. Next, as shown in Figs. 4 and 5, it is desired to transform the points of external signal injection from between the plant $F_u(F,D_l^{-1}\delta D_r)$ and controller C to between the uncertainty δ and the system $D_rF_l(F,C)D_l^{-1}$ such that internal stability is preserved. This is achieved via the following result.

Lemma 2: Consider a $F \in \mathscr{R}^{(p+n)\times(q+m)}$, a $C \in \mathscr{R}^{m\times n}$, a $\bar{\delta} \in \mathscr{R}^{q\times p}$ and a $D=(D_l,D_r)\in\boldsymbol{D}$, where $\boldsymbol{D}$ is the set as defined in Section II. Suppose that the induced realizations for $F_u(F,D_l^{-1}\bar{\delta}D_r)$ and $F_l(F,C)$ are stabilizable and detectable. Then $[F_u(F,D_l^{-1}\bar{\delta}D_r),C]$ is internally stable if and only if $[D_l^{-1}\bar{\delta}D_r,F_l(F,C)]$ is internally stable.

Proof: To be published elsewhere. ∎

[2]Such an assumption is a standard assumption in $\mathscr{H}_\infty$ control and is necessary and sufficient for F to be internally stabilizable via a controller connecting controller input y to plant input u.

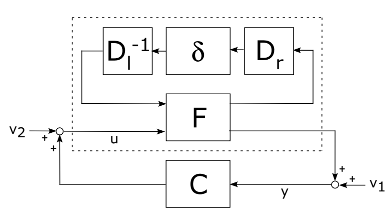

Fig. 4. Internal stability of $[F_u(F,D_l^{-1}\delta D_r),C]$.

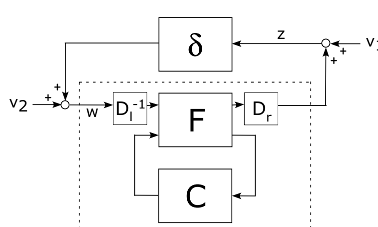

Fig. 5. Internal stability of $[\delta,D_rF_l(F,C)D_l^{-1}]$.

It follows that $[D_l^{-1}\bar{\delta}D_r,F_l(F,C)]$ is internally stable if and only if $[\bar{\delta},D_rF_l(F,C)D_l^{-1}]$ is internally stable since D_l and D_r are constant diagonal matrices. Note that Lemma 2 describes the more general case of a $\bar{\delta}\in\mathscr{R}^{q\times p}$ as opposed to the case $\delta\in\boldsymbol{\delta}$.

To sidetrack from the proof sketch of the main result for a moment, note that it is an important and interesting problem in its own right to investigate under what conditions the induced realizations for $F_u(F,D_l^{-1}\bar{\delta}D_r)$ and $F_l(F,C)$ are stabilizable and detectable, as is non-trivially supposed in Lemma 2. For the induced realization for $F_l(F,C)$, the answer is relatively straightforward. Let $P_0\in\mathscr{R}^{n\times m}$ and $C\in\mathscr{R}^{m\times n}$ have stabilizable and detectable realizations (A,B_2,C_2) (see Footnote 1) and $(\hat{A},\hat{B},\hat{C},\hat{D})$, respectively, and suppose that the nominal closed-loop system $[P_0,C]$ is internally stable. Then, by [8, Lemma 5.2], the matrix

$$\tilde{A} := \begin{pmatrix} A+B_2\hat{D}C_2 & B_2\hat{C} \\ \hat{B}C_2 & \hat{A} \end{pmatrix}$$

is Hurwitz. Now, let $F\in\mathscr{R}^{(p+n)\times(q+m)}$ have a stabilizable and detectable realization as given by (2) and consider the computation of the induced realization for $F_l(F,C)$ as given by (3). Since $A_\theta = \tilde{A}$, then A_θ is also Hurwitz, and so the induced realization for $F_l(F,C)$ is stabilizable and detectable.

Since it is not sensible to assume internal stability of the closed-loop system $[\delta,F_{11}]$, nor to assume that the inherited realization for F_{11} is stabilizable and detectable, a more complex result is required to give conditions under which the induced realization for $F_u(F,D_l^{-1}\bar{\delta}D_r)$ is stabilizable and detectable. Again consider (2) to be a stabilizable and detectable realization for F, and let $D_l^{-1}\bar{\delta}D_r\in\mathscr{R}^{q\times p}$ have a stabilizable and detectable realization $(\check{A},\check{B},\check{C},\check{D})$. The following state-space result gives conditions for the stabilizability and detectability of the induced realization for $F_u(F,D_l^{-1}\bar{\delta}D_r)$.

Theorem 3: Let $F\in\mathscr{R}^{(p+n)\times(q+m)}$ and $\bar{\delta}\in\mathscr{R}^{q\times p}$. Suppose that a stabilizable and detectable realization for F is given by (2). Then the induced realization for $F_u(F,D_l^{-1}\bar{\delta}D_r)$ where $D=(D_l,D_r)\in\boldsymbol{D}$, as defined in Definition 1, is stabilizable and detectable if and only if $\bar{n}(F_u(F,D_l^{-1}\bar{\delta}D_r)) = \bar{n}(T)$, where $\bar{n}(\cdot)$ denotes the number of closed RHP poles counted according to the usual notion of the Smith-McMillan decomposition and $T(s)$ denotes the transfer function matrix mapping $(v_2'\ v_1'\ u')'$ to $(w'\ z'\ y')'$, as shown in Figure 6. Furthermore, the induced realization for $F_u(F,D_l^{-1}\bar{\delta}D_r)$ is

(a) detectable if $\begin{pmatrix} A-\lambda I & B_1 \\ C_2 & D_{21}\end{pmatrix}$ has full column rank $\forall Re(\lambda)\geq 0$;

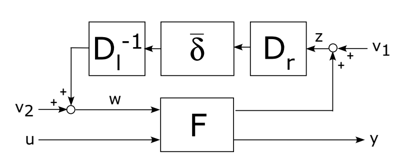

Fig. 6. Mapping of $(v_2'\ v_1'\ u')'$ to $(w'\ z'\ y')'$.

(b) stabilizable if $\begin{pmatrix} A-\lambda I & B_2 \\ C_1 & D_{12} \end{pmatrix}$ has full row rank $\forall Re(\lambda) \geq 0$.

Proof: Parts (a) and (b) of the second part of the theorem have been extracted from [8, Lemma 16.1] and are proved in that reference. The first part of the proof is to be published elsewhere. ■

Parts (a) and (b) of the final part of Theorem 3 are given because the necessary and sufficient condition stated in the earlier part of the theorem is dependent on $\bar{\delta}$, and can hence be difficult to check. This is as opposed to the sufficient conditions in (a) and (b), which are equivalent to requiring no unstable invariant zeros of the realizations

$$\left(\begin{array}{c|c} A & B_1 \\ \hline C_2 & D_{21} \end{array}\right) \text{ and } \left(\begin{array}{c|c} A & B_2 \\ \hline C_1 & D_{12} \end{array}\right),$$

for F_{21} and F_{12}, respectively.

Returning now to the proof sketch of Theorem 1, recall that it has been established that $[F_u(F, D_l^{-1}\delta D_r), C]$ is internally stable if and only if $[\delta, D_r F_l(F,C) D_l^{-1}]$ is internally stable, supposing that the induced realizations for $F_u(F, D_l^{-1}\delta D_r)$ and $F_l(F,C)$ are stabilizable and detectable. Note also that $F_l(F,C) \in \mathscr{RH}_\infty$ due to the suppositions of the theorem statement (and see [8, Lemma 12.2]). Then the small gain theorem (see [8, Theorem 9.1] for instance), which states that $[\delta, D_r F_l(F,C) D_l^{-1}]$ is internally stable for all $\delta \in \boldsymbol{\delta}$ if and only if $\|D_r F_l(F,C) D_l^{-1}\|_\infty \leq 1$, can be applied to give

$$\inf_{D=(D_l,D_r)\in\boldsymbol{D}} \|D_r F_l(F,C) D_l^{-1}\|_\infty \leq 1$$

when the infimum is appropriately carried through. It then remains to apply a time-varying small gain argument from [9, Section III.E] to obtain robust internal stability of $[\Delta, F_l(F,C)]$, and then to apply a time-varying version of Lemma 2 to obtain robust internal stability of $[F_u(F,\Delta), C]$. This concludes the proof sketch of the main result.

IV. On determining whether the inequality in the main result is satisfied

A procedure for computing the LTI quantity on the left-hand side (LHS) of (5) is now proposed. Let a LTI system R (dependent on a nominal system P_0 and a given number $\beta \in (0, b_{opt}(P_0))$) be introduced and defined as follows. Suppose that P_0 has a stabilizable and detectable realization $(A_{P_0}, B_{P_0}, C_{P_0})$. Let $X = X^* \geq 0$ be the stabilizing solution to the generalized control algebraic Riccati equation (GCARE)

$$A_{P_0}^* X + X A_{P_0} - X B_{P_0} B_{P_0}^* X + C_{P_0}^* C_{P_0} = 0$$

and $Z = Z^* \geq 0$ be the stabilizing solution to the generalized filtering algebraic Riccati equation (GFARE)

$$A_{P_0} Z + Z A_{P_0}^* - Z C_{P_0}^* C_{P_0} Z + B_{P_0} B_{P_0}^* = 0.$$

Define $R \in \mathscr{R}^{(m+n)\times(n+m)}$ as in [10], where R is invertible in $\mathscr{R}^{(n+m)\times(m+n)}$. A stabilizable and detectable realization for R^{-1} is

$$\left(\begin{array}{c|cc} A_{R^{-1}} & B_{R_1^{-1}} & B_{R_2^{-1}} \\ \hline C_{R_1^{-1}} & 0 & I \\ C_{R_2^{-1}} & \sqrt{\gamma^2-1}I & 0 \end{array}\right), \qquad (6)$$

where $A_{R^{-1}} := A_{P_0} - B_{P_0} B_{P_0}^* X - \gamma^2 \bar{Y} C_{P_0}^* C_{P_0}$, $B_{R_1^{-1}} := \frac{\gamma}{\sqrt{\gamma^2-1}}(I - YX)^{-1} B_{P_0}$, $B_{R_2^{-1}} := \gamma \bar{Y} C_{P_0}^*$, $C_{R_1^{-1}} := -\gamma C_{P_0}$, $C_{R_2^{-1}} := -\gamma B_{P_0}^* X$, $\bar{Y} := Y(I - XY)^{-1}$, $Y := \frac{1}{\gamma^2-1} Z$ and $\gamma := \frac{1}{\beta}$. The following result states that if a number of LMIs involving R^{-1} are feasible, then the LTI quantity on the LHS of (5) is upper bounded by the number $\beta \in (0, b_{opt}(P_0))$.

Theorem 4: Suppose $F \in \mathscr{R}^{(p+n)\times(q+m)}$ is a generalized system partitioned as in (1), with a stabilizable and detectable realization as given by (2), and suppose $P_0 := F_u(F,0)$ has the inherited realization (A, B_2, C_2) which is also stabilizable and detectable (see Footnote 2). Furthermore, suppose that $\boldsymbol{\delta}$ is the LTI uncertainty set, and that $\boldsymbol{D}$ is the set of constant diagonal matrix pairs, as defined in Section II; and that each induced realization for $F_u(F, D_l^{-1}\delta D_r)$ as defined in Definition 1 is stabilizable and detectable. Given a $\beta \in (0, b_{opt}(P_0))$, then

$$\inf_{D=(D_l,D_r)\in\boldsymbol{D}} \sup_{\delta\in\boldsymbol{\delta}} \delta_\nu(P_0, F_u(F, D_l^{-1}\delta D_r)) \leq \beta \qquad (7)$$

if $\exists D \in \boldsymbol{D} : \forall \omega \in \mathbb{R}\ \exists d_\omega \in \mathbb{R}_+ :$

$$J^*(j\omega) \begin{pmatrix} d_\omega^2 I_n & 0 \\ 0 & D_r^2 \end{pmatrix} J(j\omega) \leq \begin{pmatrix} d_\omega^2 I_m & 0 \\ 0 & D_l^2 \end{pmatrix},$$

where $J := R^{-1} \star \begin{pmatrix} P_0 & F_{21} \\ F_{12} & F_{11} \end{pmatrix}$, and R^{-1} is defined as in (6).

The proof of Theorem 4 will be published elsewhere. As in Section III, a sketch is provided as follows to illustrate the key steps and motivate the statement of some important results. The following result, which is a minor but important extension of [10, Proposition 1.1], is used to relate the LTI inequality (7) with a transfer function matrix stability and small gain concept.

Lemma 5: Given LTI systems $P_0, P_1 \in \mathscr{R}^{n\times m}$ and a number $\beta \in (0, b_{opt}(P_0))$, there exists a LTI system R (dependent on P_0 and β only), as defined above, such that $\delta_\nu(P_0, P_1) \leq \beta \Leftrightarrow S(s) \in \mathscr{RH}_\infty$ and $||F_l(R^{-1}, P_1)||_\infty \leq 1$, where $S(s)$ denotes the transfer function matrix mapping $(w'\ v_1'\ v_2')'$ to $(z'\ a_1'\ a_2')'$, as shown in Fig. 7.

Proof: As Lemma 5 is an extension of [10, Proposition 1.1], which states that $\delta_\nu(P_0, P_1) \leq \beta \Leftrightarrow F_l(R^{-1}, P_1) \in \mathscr{RH}_\infty$ and $||F_l(R^{-1}, P_1)||_\infty \leq 1$, it is only required to show that $F_l(R^{-1}, P_1) \in \mathscr{RH}_\infty$ if and only if $S(s) \in \mathscr{RH}_\infty$. This will be published elsewhere. ■

Now set P_1 in Lemma 5 to be $F_u(F, D_l^{-1}\delta D_r)$, as in (7). Then it is easy to show (to be published elsewhere) that $S(s) \in \mathscr{RH}_\infty$ if and only if $U(s) \in \mathscr{RH}_\infty$, where $U(s)$ denotes the transfer function matrix mapping $(w'\ v_1'\ v_2')'$ to $(z'\ b_1'\ b_2')'$, as shown in Fig. 8, where $G := \begin{pmatrix} 0 & D_r \\ I_n & 0 \end{pmatrix} J \begin{pmatrix} 0 & I_m \\ D_l^{-1} & 0 \end{pmatrix}$. The above now corresponds to a

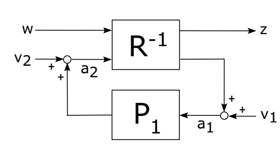

Fig. 7. Mapping of $(w'\ v_1'\ v_2')'$ to $(z'\ a_1'\ a_2')'$.

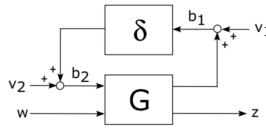

Fig. 8. Mapping of $(w'\ v_1'\ v_2')'$ to $(z'\ b_1'\ b_2')'$.

structured singular value problem [8, Chapter 11] (see Corollary 6 below) which is employed to reformulate the stability and small gain concept into one involving LMIs (since the structured singular value is equal to an infimum over a frequency dependent scaling factor as it has two uncertainty blocks). Correspondence with a structured singular value problem is possible due to the following result, which is a consequence of Lemma 5.

Corollary 6: Consider $J \in \mathscr{R}^{(n+p)\times(m+q)}$ as defined in Theorem 4. Then $J \in \mathscr{RH}_\infty$.

Proof: Follows from Lemma 5 by setting $P_1 = P_0$. To be published elsewhere. ∎

This concludes the proof sketch of Theorem 4. Finally, it remains to show how Theorem 4 is used to compute the LTI quantity on the LHS of (5), and hence determine internal stability of the LTV system $[P_{LTV}, C]$. Upon determination of the feasibility of the LMI constraints using Theorem 4 for some β, the next iteration of a bisectional line search may be implemented over the interval $(0, b_{opt}(P_0))$ to select the next test β. The direction in which the line search proceeds depends on the 'true' or 'false' result acquired by solving the LMI feasibility problem: a 'false' result suggests that a larger test β should be chosen; while a 'true' result indicates one can try a smaller test β. Consequently, the LTI quantity $\inf_{D=(D_l,D_r)\in\boldsymbol{D}} \sup_{\delta\in\boldsymbol{\delta}} \delta_\nu(P_0, F_u(F, D_l^{-1}\delta D_r))$ is achieved to within a sufficiently small pre-determined tolerance. Provided that the LTI quantity is smaller than the generalized robust stability margin $b_{P_0,C}$ achieved with some controller C that internally stabilizes the nominal plant, then internal stability of the system $[P_{LTV}, C]$, for all time-varying uncertainties $\Delta \in \boldsymbol{\Delta}$, is guaranteed.

A complete solution algorithm is provided as follows:

1) Set the bounds on possible β to be $\alpha_l = 0$ and $\alpha_r = b_{opt}(P_0)$. Set a sufficiently small tolerance $\varepsilon > 0$ for the iterative bisections with respect to finding β to end. Select an initial $\beta_0 = \alpha_r - \varepsilon$ and set $\beta_{feas} = b_{opt}(P_0)$. Set $i = 0$. Goto step 2.
2) Given a β_i, solve the convex optimization problem: "does there exist a $D \in D$ such that, for all $\omega \in \mathbb{R}$, there exist $d_\omega \in \mathbb{R}_+$ such that
$$J^*(j\omega)\begin{pmatrix} d_\omega^2 I_n & 0 \\ 0 & D_r^2 \end{pmatrix} J(j\omega) \leq \begin{pmatrix} d_\omega^2 I_m & 0 \\ 0 & D_l^2 \end{pmatrix},$$
where $J := R^{-1} \star \begin{pmatrix} P_0 & F_{21} \\ F_{12} & F_{11} \end{pmatrix}$, and R^{-1} is defined as in (6)". Note that each different d_ω can have a different value (with each different d_ω corresponding to a different frequency ω). Now,
   i) If the optimization problem is feasible, set $\beta_{feas} = \beta_i$ and $\beta_{i+1} = \frac{\alpha_l + \beta_i}{2}$. Update $\alpha_r = \beta_i$. Goto step 2iii.
   ii) If the optimization problem is not feasible, test if $\beta_{feas} - \beta_i \leq \varepsilon$. If yes, then end. If no, set $\beta_{i+1} = \frac{\beta_i + \alpha_r}{2}$. Update $\alpha_l = \beta_i$. Goto step 2iii.
   iii) Set $i = i + 1$ and goto step 2.

If $\beta_{feas} < b_{P_0,C}$, where C is some internally stabilizing controller, then $[P_{LTV}, C]$ is internally stable for all $\Delta \in \boldsymbol{\Delta}$. If not, internal stability of $[P_{LTV}, C]$ has not been determined (and a possibility if $\beta_{feas} \neq b_{opt}(P_0)$ is to choose a different controller to obtain a larger stability margin).

The convex optimization problem in Step 2 of the solution algorithm is easily solved using Matlab's LMI toolbox for instance. A numerical example follows in the next section.

V. Numerical Example

Consider a nominal system P_0 with state-space representation as given in Fig. 9. This data was obtained by implementing the $\mathscr{H}_\infty$ loop-shaping design procedure, described in [11], given the input from the example provided in [11]. Note that $b_{opt}(P_0) = 0.376$, which can be determined using Matlab's μ-Analysis and Synthesis Toolbox "ncfsyn" function, for instance.

Now consider the uncertain system shown in Fig. 10, where Δ_1 and Δ_2 represent output multiplicative and input feedback LTV uncertainties, respectively. Formally, the uncertain system shown in Fig. 10 is described by
$$P_{LTV} := (I + \varepsilon_1\Delta_1)P_0(I - \varepsilon_2\Delta_2)^{-1}, \tag{8}$$
where $\begin{pmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{pmatrix} \in \boldsymbol{\Delta}$ and $\varepsilon_1, \varepsilon_2 \in [0,1]$. Recall that output multiplicative uncertainty may typically represent output (sensor) errors or neglected high frequency dynamics, while input feedback uncertainty may represent low frequency parameter errors (see [8, Table 9.1]). Expressing (8) in the standard structured uncertainty form gives
$$F = \begin{pmatrix} 0 & \varepsilon_1\varepsilon_2 P_0 & \varepsilon_1 P_0 \\ 0 & \varepsilon_2 I & I \\ I & \varepsilon_2 P_0 & P_0 \end{pmatrix}.$$

The solution algorithm presented in this paper can be used to investigate stability robustness of the uncertain feedback interconnection. For this example, one hundred equally spaced frequency points on a logarithmic scale between $\omega = 10^{-4}$ and 10^4 rad/s were chosen for Step 2 of the algorithm and a tolerance of 0.001 was chosen for Step 1. First, stability robustness of the system subject only to output multiplicative uncertainty was investigated. Four hundred and one evenly spaced scaling factors ε_1 were chosen from between $[0,1]$ to represent different sizes of the uncertainty, while ε_2 was set fixed at zero. An algorithm output quantity β_{feas} (representative of the LTI quantity on the LHS of (5)) was produced for each of the 401 pairs of uncertainty scaling factors $(\varepsilon_1, \varepsilon_2)$. The results for when ε_1 ranged between $[0, 0.5]$ are shown in Fig. 11. For example, a size of $\varepsilon_1 = 0.4975$ resulted in a β_{feas} of 0.367, which is less than $b_{opt}(P_0) = 0.376$. This means that the interconnection $[P_{LTV}, C_s]$ subject to LTV output multiplicative uncertainties with scaling factors of

0	1.00	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
-23.8	-3.36	4.60	-0.239	11.8	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1.00	0	0	0	0	0	0	0	0	0	0	0	0	0	0
-16.8	-0.0248	22.8	-0.916	0	0	9.83	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	-50.0	63.2	0	0	112	-9590	-1530	-1250	0	0	0	0	-45300	0
0	0	0	0	0	-40.0	0	0	-142	12100	1930	1580	0	0	0	0	57400	0
0	0	0	0	0	0	-50.0	63.2	0	0	0	0	-58.0	10600	-145	-1320	0	-55500
0	0	0	0	0	0	0	-40.0	0	0	0	0	73.3	-13400	183	1670	0	70200
0	0	0	0	0	0	0	0	-0.00561	1.14	0.178	0.123	0	0	0	0	-15.9	0
0	0	0	0	0	0	0	0	-1.14	-258	-43.9	-74.7	0	0	0	0	-1360	0
0	0	0	0	0	0	0	0	-0.178	-43.9	-7.52	-13.9	0	0	0	0	-216	0
0	0	0	0	0	0	0	0	0.123	74.7	13.9	-43.9	0	0	0	0	177	0
0	0	0	0	0	0	0	0	0	0	0	0	-0.00271	1.52	-0.0158	-0.120	0	8.20
0	0	0	0	0	0	0	0	0	0	0	0	-1.52	-262	5.05	72.0	0	1500
0	0	0	0	0	0	0	0	0	0	0	0	0.0158	5.05	-0.117	-2.76	0	-20.5
0	0	0	0	0	0	0	0	0	0	0	0	-0.120	-72.0	2.76	-43.9	0	187
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Fig. 9. State-space model of P_0.

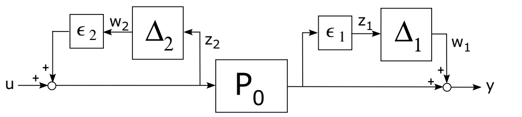

Fig. 10. Uncertain open-loop shaped plant.

size up to and including 0.4975, as described by (8), will be internally stable, given that C_s is a stabilizing controller that achieves $b_{P_0,C_s} > 0.367$. Note that the next (larger) scaling factor tested was $\varepsilon_1 = 0.5$, for which the algorithm produced an output $\beta_{feas} > b_{opt}(P_0)$ and so internal stability of $[P_0, C_s]$ subject to LTV output multiplicative uncertainties with $\varepsilon_1 > 0.4975$ was not concluded here.

Next, stability robustness of the feedback interconnection was tested with respect to when P_0 is subjected to both output multiplicative and input feedback LTV uncertainties. For example, when the scaling factors were set to $\varepsilon_1 = 0.35$ and $\varepsilon_2 = 0.38$, the algorithm produced a β_{feas} of 0.371, meaning that $[P_{LTV}, C_t]$ subject to both output multiplicative and input feedback LTV uncertainties of size 0.35 and 0.38, respectively, as described by (8), is guaranteed to be internally stable, given that C_t is a stabilizing controller that achieves $b_{P_0,C_t} > 0.371$.

VI. Conclusions

A new stability robustness test for systems with LTV uncertainties that can be described with the standard structured uncertainty model was introduced. The main theoretical result forming the basis of the test involved transforming the problem of LTV system stability robustness into a problem of determining whether a certain LTI quantity was smaller than the generalized robust stability margin achieved by a controller stabilizing a nominal plant. A solution algorithm that computed the LTI quantity was supplied, and was demonstrated with a numerical example.

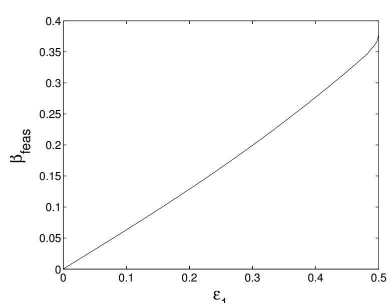

Fig. 11. The quantity β_{feas} with respect to the size of the output multiplicative uncertainty.

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