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Robust Model Reference Adaptive Control of Parabolic and Hyperbolic Systems with Spatially-varying Parameters

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Abstract— **A long-standing problem of the start-up instability in the model-reference adaptive control of distributed parameter systems caused by setting the initial controller parameter values sufficiently far from the ideal ones, unknown a priori, is solved for a class of systems. The latter include systems modeled by parabolic and hyperbolic partial differential equations (PDEs) with spatially varying parameters. The stabilizing direct model reference adaptive control (MRAC) laws are synthesized using Lyapunov redesign. The controller uses plant state and for hyperbolic case, additionally, its time derivative. The key feature of the approach proposed is the elimination from the control laws of the plant state spatial derivatives that could give rise to the closed loop system ill-posedness. The approach also prevents closed-loop system instability by keeping the gains for plant state and, in the hyperbolic case - state and its time derivative, negative under arbitrary initial controller parameter setting.**

I. Introduction

One of the main drawbacks of the adaptive control laws (cf. [1]) for distributed parameter systems (DPS) has been the possibility of the quick unpredictable algorithm blow-up at the start-up. This feature characterizes MRAC laws for DPS proposed in [2], [3], [6], and references therein. It was conjectured that this drawback is caused by the use of the second order spatial derivatives of the output in the MRAC laws feedback path. Effort to robustify these control laws through the spatial differentiation order reduction of the output was made in [2], where the reference input was limited to a constant. The latter work, however, has not produced algorithms capable of avoiding the blow-up possibility. This problem is solved in the present work for a class of DPS described by parabolic and hyperbolic PDEs with spatially-varying coefficients. For brevity, one dimensional case is considered. Extension to higher dimensions can be carried out by imposing more regularity on coefficients, reference input, domain, and initial conditions. The paper has the following structure: the systems to be controlled are described in the Introduction. An infinite-dimensional adaptive control law synthesis for these system is carried out in Section II. This section demonstrates the start-up instability of the MRAC laws of the type presented in [2], [3], [6], and then introduces a novel direct MRAC structure that eliminates this problem both for the parabolic and the hyperbolic cases. Numerical simulations demonstrating the stable algorithm performance are presented in Section III. Finally, conclusions are given in Section IV.

For open and bounded domain $U \subset R$, consider a plant represented by a parabolic PDE

$$u_t = (a(x)u_x)_x + b(x)u + f,\ (x,t) \in U_T,$$

$$\begin{aligned} u(x,0) &= g(x), \qquad x \in U, \\ u(x,t) &= \alpha(x), \qquad x \in \partial U, \end{aligned} \tag{1}$$

where $U_T := U \times (0,T]$, $\nu \geq a(x) \geq \varepsilon > 0$ and $b(x) \leq 0$ are unknown spatially varying parameters, $f(x,t)$ is a control input, and $(\cdot)_x$ and $(\cdot)_t$ denote partial derivatives with respect to x and t, respectively. The reference model is in the same form as the plant:

$$v_t = (a_1(x)v_x)_x + b_1(x)v + r,\ (x,t) \in U_T,$$

$$\begin{aligned} v(x,0) &= g_1(x), \qquad x \in U, \\ v(x,t) &= \alpha(x), \qquad x \in \partial U, \end{aligned} \tag{2}$$

where $\nu_1 \geq a_1(x) \geq \varepsilon_1 > 0$ and $b_1(x) \leq 0$ are the prespecified spatially varying parameters, and $r(x,t)$ is a reference input. The boundary conditions for the model are assumed to be equal to those of the plant. Since the non-zero boundary problem can be changed into the zero boundary one, $\alpha(x) = 0$ is considered hereafter.

Correspondingly, a plant modeled by a hyperbolic PDE is described by

$$\begin{aligned} u_{tt}(x,t) &= (a(x)u_x(x,t))_x + b(x)u(x,t) \\ &\quad + c(x)u_t(x,t) + f(x,t),\ (x,t) \in U_T, \end{aligned}$$

$$\begin{aligned} u(x,0) &= g(x), \qquad u_t(x,0) = h(x),\ x \in U, \\ u(x,t) &= \alpha(x), \qquad x \in \partial U, \end{aligned} \tag{3}$$

where $\nu \geq a(x) \geq \varepsilon > 0$, $b(x) \leq 0$ and $c(x) \leq 0$, with the reference model given by

$$\begin{aligned} v_{tt}(x,t) &= (a_1(x)v_x(x,t))_x + b_1(x)v(x,t) \\ &\quad + c_1(x)v_t(x,t) + r(x,t),\ (x,t) \in U_T, \end{aligned}$$

$$\begin{aligned} v(x,0) &= g_1(x), \qquad v_t(x,0) = h_1(x),\ x \in U, \\ v(x,t) &= \alpha(x), \qquad x \in \partial U, \end{aligned} \tag{4}$$

where $\nu_1 \geq a_1(x) \geq \varepsilon_1 > 0$, $b_1(x) \leq 0$, and $c_1(x) \leq 0$ are spatially varying parameters, and $r(x,t)$ is a reference input.

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Again, the same boundary condition is assumed for the plant and the model, and $\alpha(x) = 0$ hereafter.

The following definitions will be used subsequently.

Definition 1 [5]: A Lyapunov function is a continuos real-valued function V on D such that

$$\dot{V}(x) \equiv \limsup_{\Delta t \to 0^+} \frac{1}{\Delta t}\{V(x(t+\Delta t)) - V(x(t))\} \leq 0 \quad (5)$$

for all $x \in D$.

To cope with a broader class of initial data and input functions a weak solution is defined.

Definition 2 [4, p. 352]: A function $u \in L^2(0,T;H_0^1(U))$, with $u' \in L^2(0,T;H^{-1}(U))$, is said to be a weak solution of the parabolic initial/boundary-value problem (1) provided

$$\int_U u'vdx = \int_U -au_xv_x + buvdx + \int_U fvdx \quad (6)$$

for each $v \in H_0^1(U)$ and a.e. time $0 \leq t \leq T$, and

$$u(0) = g, \quad (7)$$

where $H^{-1}(U)$ is a dual space of $H_0^1(U)$.

Definition 3 [4, p. 379]: A function $u \in L^2(0,T;H_0^1(U))$, with $u' \in L^2(0,T;L^2(U))$, $u'' \in L^2(0,T;H^{-1}(U))$, is said to be a weak solution of the hyperbolic initial/boundary-value problem (3) provided

$$\int_U u''vdx = \int_U -au_xv_x + buv + cu_tvdx + \int_U fvdx \quad (8)$$

for each $v \in H_0^1(U)$ and a.e. time $0 \leq t \leq T$, and

$$u(0) = g,\ u'(0) = h. \quad (9)$$

II. Controller design

For the purpose of designing a control law that uses the reference input, and the plant and the model states, assume that:

i) the plant equation structure is known but the parameters are unknown;

ii) the initial condition of the plant can be unknown;

iii) the plant state $u(x,t)$ is measured and control input $f(x,t)$ is applied for every x, t.

First, well-definedness of the closed loop system is demonstrated. The term "*well-definedness"* characterizes the system equation that has a solution in some sense. It is then proved that the given controller makes the plant output follow that of the reference model under a given reference input, $r(x,t)$. Define

$$\begin{aligned}
e(x,t) &= v(x,t) - u(x,t),\ \eta_{a1}^*(x) = a_1(x) - a(x),\\
\eta_{a2}^*(x) &= (a_1(x) - a(x))_x,\\
\eta_b^*(x) &= b_1(x) - b(x),\ \eta_c^*(x) = c_1(x) - c(x),\\
\xi_{a1}(x,t) &= \eta_{a1}(x,t) - \eta_{a1}^*(x),\\
\xi_{a2}(x,t) &= \eta_{a2}(x,t) - \eta_{a2}^*(x),\\
\xi_b(x,t) &= \eta_b(x,t) - \eta_b^*(x),\\
\xi_c(x,t) &= \eta_c(x,t) - \eta_c^*(x), \quad (10)
\end{aligned}$$

where $\eta_{a1,a2,b,c}$ will be used as controller parameters, $e(x,t)$ is the output error, and $\xi_{a1,a2,b,c}(x,t)$ are the parameter errors.

A. Parabolic case

The basic structure of adaptive control laws in [2] and [6] is as follows. Let us rewrite the system and the reference model as

$$u_t = Lu + f,\ v_t = L_1v + r. \quad (11)$$

If plant operator L is known, control input f can be chosen as

$$f = L_1u - Lu + r, \quad (12)$$

resulting in

$$u_t = Lu + L_1u - Lu + r = L_1u + r. \quad (13)$$

If the closed loop system, i.e., in this case, the operator of the reference model L_1, has a dissipation term, the difference between u and v caused by their possible initial condition mismatch decays, and u follows v asymptotically. When L is unknown and the reference model does not have a dissipating term, L has to be identified and correction has to be made in f, respectively. The problem with control form like

$$f = \tilde{L}(t)u + r \quad (14)$$

is that the closed loop system can be ill-posed and unstable depending on how $\tilde{L}(t)$ evolves. For illustration, the algorithm [6] is simulated. Using (1) and (2) with the same parameters as in [6]

$$\begin{aligned}
a(x) &= 0.1 + 0.2\sin(\pi x),\\
a_1(x) &= 0.5,\ \eta_a(x,0) = 0.1, \quad (15)
\end{aligned}$$

the simulation shows that tracking is achieved when the control in [6] is used. When initial controller parameter is set, however, as

$$\eta_a(x,0) = -1, \quad (16)$$

increasing the possibility of making the closed loop system ill-posed, plant state blows up in a short time.

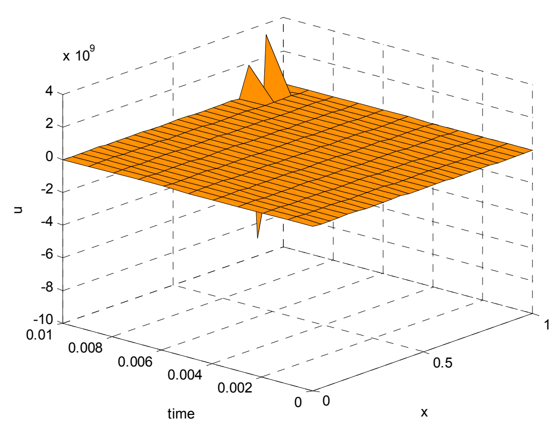

Fig. 1. Plant state u of the closed loop system: start-up instability.

To fix this problem consider controller

$$f = r + \varepsilon_0 e + \eta_{a1} v_{xx} + \eta_{a2} v_x + \eta_b v \tag{17}$$

with the parameter update laws

$$\begin{aligned} \dot{\eta}_{a1} &= \dot{\xi}_{a1} = \varepsilon_{a1} e v_{xx}, \\ \dot{\eta}_{a2} &= \dot{\xi}_{a2} = \varepsilon_{a2} e v_x, \\ \dot{\eta}_b &= \dot{\xi}_b = \varepsilon_b e v, \end{aligned} \tag{18}$$

where proportional gain $\varepsilon_0 > 0$ determines the dependence of control input on state error, and adaptation gains $\varepsilon_{a1,a2,b}$ determine the size of parameter update.

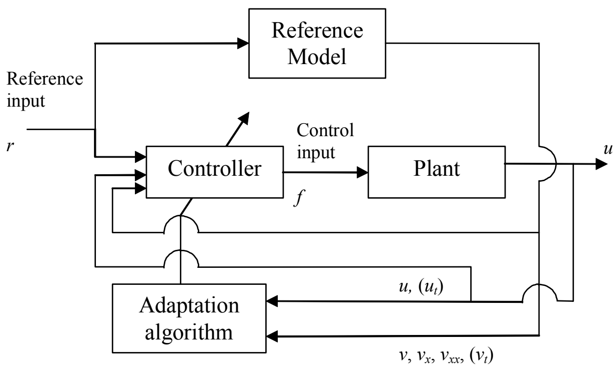

Fig. 2. Schematics of MRAC of parabolic case. (hyperbolic case uses u_t and v_t additionally)

Note that control input does not use the spatial derivative of the plant state.

B. Well-definedness of the closed loop system for parabolic case

The system will be referred to as well-defined if it has a weak solution as in Definition 2. The following lemma will be used to prove that the closed system is well-defined under proper conditions.

Lemma 1: Suppose $a,\ b \in C^1(\bar{U})$, $f(x,t) \in L^\infty(U_T)$, $d(x,0) \in C^1(\bar{U})$, and $g(x) \in L^2(U)$. Then there exists a unique weak solution $u(x,t)$ of

$$u_t = (a(x)u_x)_x + b(x)u - d(x,t)f(x,t),$$

$$\begin{aligned} \frac{\partial d(x,t)}{\partial t} &= \varepsilon u f(x,t),\ (x,t) \in U_T, \\ u(x,0) &= g(x), \qquad x \in U \end{aligned} \tag{19}$$

where $\nu \geq a(x) \geq \theta > 0$, $b(x) \leq 0$, and $\varepsilon > 0$. The proof is omitted due to space limitations.

For well-definedness of the closed loop system, the following conditions are assumed on the reference model and the reference input to guarantee the necessary regularity of v:

$$\text{(A1)} \left\{ \begin{array}{c} a_1,\ b_1 \in C^\infty(\bar{U}),\ g_1 \in H^3(U), \\ \frac{\partial^k r}{\partial t^k} \in L^2(0,T;H^{2-2k}(U)), \qquad (k=0,\ 1) \\ \text{with compatibility conditions} \\ g_1 \in H_0^1(U), \\ r(\cdot,0) + (a_1 g_{1x})_x + b_1 g_1 \in H_0^1(U), \\ \frac{d}{dt} r(\cdot,0) + (a_1 h_{1x})_x + b_1 h_1 \in H_0^1(U). \end{array} \right.$$

If (A1) is satisfied, by Theorem 6 [4, p. 365] the reference model has a weak solution ($v \in L(0,T;H^4(U))$) such that for $C_1,\ C_2 < \infty$,

$$\frac{\partial}{\partial x} v_{xx}^2 = 2 v_{xx} v_{xxx} \text{ for a.e. } U_T, \tag{20}$$

which is justified by Theorem 5 [4, p. 280] and

$$\begin{aligned} \left| v_{xx}^2 \right| &\leq C_1 + 2\int_U |v_{xx} v_{xxx}|\, dx \\ &\leq C_1 + \int_U v_{xx}^2 dx + \int_U v_{xxx}^2 dx < C_2 \text{ for a.e. } U_T. \end{aligned}$$

Similarly,

$$\left| v^2 \right| < C_2,\ \left| v_x^2 \right| < C_2 \text{ for a.e. } U_T. \tag{21}$$

(A1) will be assumed in Proposition 1 to guarantee that the forcing terms $-\xi_{a1} v_{xx} - \xi_{a2} v_x - \xi_b v$ of the state error equation below satisfy $L^\infty(U_T)$ boundedness.

Proposition 1: If the controller input and the controller parameter update law are given as (18), and conditions (A1) and (A2) on plant parameters, initial conditions, and control parameter initial conditions are satisfied, where

$$\text{(A2)} \left\{ \begin{array}{c} a,\ b \in C^1(\bar{U}), \\ g(x) \in H_0^1(U),\ h(x) \in L^2(U), \\ \xi_{a1}(x,0),\ \xi_{a2}(x,0),\ \xi_b(x,0) \in C^1(\bar{U}), \end{array} \right.$$

then the closed loop system is well defined.

Proof: Lemma 1 can be used to prove well-definedness of the error equation. Since

$$u = v - e, \tag{22}$$

the closed loop system is well defined. ■

C. Properties of the control algorithm for parabolic case

Let us now prove that the controller proposed makes the plant output follow that of the reference model.

Theorem 1: If the control input (17) and controller parameter update laws (18) are defined for (1) and (2), and (A1), (A2) are satisfied, the L^2 norm of the output error $e(x,t)$ goes to zero asymptotically.

Proof: The following function will be used as a candidate Lyapunov function

$$\begin{aligned} V &= \frac{1}{2}(e,e) + \frac{1}{2\varepsilon_{a1}}(\xi_{a1},\xi_{a1}) \\ &\quad + \frac{1}{2\varepsilon_{a2}}(\xi_{a2},\xi_{a2}) + \frac{1}{2\varepsilon_b}(\xi_b,\xi_b), \end{aligned} \tag{23}$$

where $(\cdot,\cdot)$ is an inner product in real $L^2(U)$ space. Time derivative of V is

$$\dot{V} = (e_t, e) + \frac{1}{\varepsilon_{a1}}(\dot{\xi}_{a1}, \xi_{a1}) + \frac{1}{\varepsilon_{a2}}(\dot{\xi}_{a2}, \xi_{a2}) + \frac{1}{\varepsilon_b}(\dot{\xi}_b, \xi_b). \quad (24)$$

Substituting the equation for e_t

$$\begin{aligned}\dot{V} &= ((ae_x)_x, e) + ((b-\varepsilon_0)e, e) \\ &\quad -(\xi_{a1}v_{xx}, e) - (\xi_{a2}v_x, e) - (\xi_b v, e) \\ &\quad +\frac{1}{\varepsilon_{a1}}(\dot{\xi}_{a1}, \xi_{a1}) + \frac{1}{\varepsilon_{a2}}(\dot{\xi}_{a2}, \xi_{a2}) \\ &\quad +\frac{1}{\varepsilon_b}(\dot{\xi}_b, \xi_b). \end{aligned} \quad (25)$$

Using integration by parts with $e = 0$ on ∂U, and $\dot{\xi}_{a1,a2,b}$ defined above

$$\dot{V} = -(ae_x, e_x) + ((b-\varepsilon_0)e, e). \quad (26)$$

Because $e(x,t) \in H_0^1(U)$ for a.e. t by Proposition 1 and $b \le 0$ is assumed, there exists $C > 0$, independent of $e(x,t)$, such that

$$\dot{V} < -\varepsilon_0(1+C)(e, e). \quad (27)$$

Thus, V is indeed a Lyapunov function in $D = \{e(\cdot,t) \in H_0^1(U),\ \text{a.e. } t\}$. Following [5, p. 84], since V is a nonnegative nonincreasing function of t, there exists $l < \infty$ such that

$$\lim_{t\to\infty} V(t) = l. \quad (28)$$

If $l = 0$,

$$\lim_{t\to\infty} \|e(t)\|_{L^2} = 0. \quad (29)$$

If $l > 0$, as t goes to ∞, each term in V either converges to some constant or keeps fluctuating in a bounded interval while maintaining $V = l$. To cover both cases, suppose

$$\lim_{t\to\infty} \sup \|e(t)\|_{L^2} = m \neq 0. \quad (30)$$

Then

$$\lim_{t\to\infty} \inf \dot{V}(t) = -\varepsilon_0(1+C)m^2, \quad (31)$$

which contradicts the fact that $\lim_{t\to\infty} V(t) = l > 0$. Since $\|e(t)\|_{L^2}$ is nonnegative

$$\lim_{t\to\infty} \|e(t)\|_{L^2} = 0. \quad (32)$$

■

Remark 1: Since Theorem 1 requires either $e = 0$ or $e_x = 0$ on ∂U, under the proper compatibility conditions this controller will work for Neumann boundary or mixed boundary cases as long as the system and the model have the same boundary values.

D. Hyperbolic case

Let us now set the controller input to

$$f(x,t) = r + \varepsilon_0 e + e_t + \eta_{a1}v_{xx} + \eta_{a2}v_x + \eta_b v + \eta_c v_t, \quad (33)$$

and introduce the controller parameter update laws as

$$\begin{aligned}\dot{\eta}_{a1} &= \dot{\xi}_{a1} = \varepsilon_{a1}(e_t + e)v_{xx}, \\ \dot{\eta}_{a2} &= \dot{\xi}_{a2} = \varepsilon_{a2}(e_t + e)v_x, \\ \dot{\eta}_b &= \dot{\xi}_b = \varepsilon_b(e_t + e)v,\ \dot{\eta}_c = \dot{\xi}_c = \varepsilon_c(e_t + e)v_t. \end{aligned} \quad (34)$$

E. Well-definedness of the closed loop system for hyperbolic case

The system will be referred to as well-defined if it has a weak solution as in Definition 3. The following lemma will be used to prove that the closed loop system is well defined under proper conditions.

Lemma 2: Suppose $a,\ b,\ c \in C^1(\bar{U})$, $f(x,t) \in L^\infty(U_T)$, $h(x) \in L^2(U)$, $d(x,0) \in C^1(\bar{U})$, and $g(x) \in H_0^1(U)$. Then there exists a unique weak solution $u(x,t)$ of

$$u_{tt} = (a(x)u_x)_x + b(x)u + c(x)u_t - d(x,t)f(x,t),$$

$$\begin{aligned}\frac{\partial d(x,t)}{\partial t} &= \varepsilon(u + u_t)f(x,t),\ (x,t) \in U_T, \\ u(x,0) &= g(x), \qquad u_t(x,0) = h(x),\ x \in U \end{aligned} \quad (35)$$

where $\nu \ge a(x) \ge \theta > 0$, $b(x), c(x) \le 0$, $\varepsilon > 0$. The proof is omitted due to space limitations.

The following remarks about the weak solution of Lemma 2 will be used further.

Remark 2: Referring to the proof of Theorem 4.1 [7, p. 162], if

$$\text{(A3)}\ f_t \in L^\infty(U_T),$$

then $u_{xt} = u_{tx} \in L^\infty(0,T;L^2(U))$, $u_{tt} \in L^\infty(0,T;L^2(U))$.

Remark 3: If u is a weak solution of Lemma 8 and (A3) holds,

$$u_t \in L^2(0,T;H_0^1(U)),\ u_{tt} \in L^2(0,T;L^2(U)) \quad (36)$$

by the remark [4, p. 380] and $C(0,T;H^{-1}(U)) \subseteq L^2(0,T;H^{-1}(U))$,

$$u_t \in L^2(0,T;H_0^1(U)),\ u_{tt} \in L^2(0,T;H^{-1}(U)) \quad (37)$$

by Theorem 3 [4, p. 287], and

$$\frac{d}{dt}\|u_t\|_{L^2}^2 = 2(u_{tt}, u_t) \text{ for a.e. } 0 \le t \le T. \quad (38)$$

Also by Corollary 4.1 [7, p. 164],

$$u_x \in L^2(0,T;H_0^1(U)),\ u_{xt} \in L^2(0,T;L^2(U)). \quad (39)$$

Similarly, $\frac{d}{dt}\|u_x\|_{L^2}^2 = 2(u_{xt}, u_t) = 2(u_{tx}, u_t)$ for a.e. $0 \le t \le T$.

For well-definedness of the closed loop system, the following conditions are assumed on the reference model and reference input to guarantee the necessary regularity of v:

$$
\text{(A4)}\begin{cases} a_1,\ b_1,\ c_1 \in C^\infty(\bar{U}), \\ g_1 \in H^4(U),\ h_1 \in H^3(U), \\ \frac{\partial^k r}{\partial t^k} \in L^2(0,T;H^{3-k}(U)) \qquad (k=0,...,3) \\ \text{with compatibility conditions} \\ g_1 \in H_0^1(U), h_1 \in H_0^1(U), \\ r(\cdot,0) + (a_1 g_{1x})_x + b_1 g_1 + c_1 h_1 \in H_0^1(U), \\ \frac{d}{dt} r(\cdot,0) + (a_1 h_{1x})_x + b_1 h_1 \in H_0^1(U). \end{cases}
$$

If (A4) is satisfied, by Theorem 6 [4, p. 391] the reference model has a weak solution ($v \in L^\infty(0,T;H^4(U))$) such that for $C_1,\ C_2 < \infty$,

$$
\frac{\partial}{\partial x} v_{xxt}^2 = 2 v_{xxt} v_{xxxt} \text{ for a.e. } U_T, \tag{40}
$$

which is justified by Theorem 5 [4, p. 280] and

$$
\begin{aligned} \left| v_{xxt}^2 \right| &\leq C_1 + 2\int_U |v_{xxt} v_{xxxt}|\, dx \\ &\leq C_1 + \int_U v_{xxt}^2 dx + \int_U v_{xxxt}^2 dx < C_2 \text{ for a.e. } U_T. \end{aligned}
$$

Similarly,

$$
\left| v_{xt}^2 \right| < C_2, \left| v_t^2 \right| < C_2, \left| v_{tt}^2 \right| < C_2 \text{ for a.e. } U_T. \tag{41}
$$

(A4) will be assumed in Proposition 2 to make the forcing terms, $-\xi_{a1} v_{xx} - \xi_{a2} v_x - \xi_b v - \xi_c v_t$ of the state error equation below satisfy (A3).

Note that when applying Theorem 6 [4, p. 391], $c_1(x) v_t(x,t)$ term does not affect the validity of the theorem. The term is energy dissipating with time invariant coefficient that does not cause any additional complication for the existence or the regularity of a weak solution, which can be checked through the energy estimate in [4, p. 381] and subsequent argument.

Proposition 2: If the controller input and the controller parameter update laws are given as (34) and (34), conditions (A4) and (A5) on plant parameters, initial conditions, and controller parameter initial conditions are satisfied, where

$$
\text{(A5)}\begin{cases} a,\ b,\ c \in C^1(\bar{U}), \\ g(x) \in H_0^1(U),\ h(x) \in L^2(U), \\ \xi_{a1}(x,0),\ \xi_{a2}(x,0),\ \xi_b(x,0),\ \xi_c(x,0) \in C^1(\bar{U}), \end{cases}
$$

then the closed loop system is well defined.

Proof: Lemma 2 can be used to prove well-definedness of error equation. Since

$$
u = v - e, \tag{42}
$$

the closed loop system is well defined. ■

F. Properties of the control algorithm for hyperbolic case

Let us now prove that the proposed controller makes the system output follow that of the reference model.

Theorem 2: If the control input (33) and controller parameter update laws (34) are defined for (3) and (4), and (A4), (A5) are satisfied, the L^2 norm of the output error $e(x,t)$ goes to zero asymptotically.

Proof: The same argument as in the proof of Theorem 1 completes the proof using the following Lyapunov function

$$
\begin{aligned} V &= \frac{1}{2}(e_t + e, e_t + e) + \frac{1}{2}(a e_x, e_x) - \frac{1}{2}(b e, e) \\ &\quad - \frac{1}{2}(c e, e) + \frac{\varepsilon_0}{2}(e, e) \\ &\quad + \frac{1}{2\varepsilon_{a1}}(\xi_{a1}, \xi_{a1}) + \frac{1}{2\varepsilon_{a2}}(\xi_{a2}, \xi_{a2}) \\ &\quad + \frac{1}{2\varepsilon_b}(\xi_b, \xi_b) + \frac{1}{2\varepsilon_c}(\xi_c, \xi_c). \end{aligned} \tag{43}
$$

■

The comment in Remark 1 also holds for this control scheme.

Remark 4: The controller actually guarantees almost everywhere convergence of tracking error, e to zero.

Proof: Since $e(x,t) \in H_0^1(U)$ for a.e. t,

$$
e^2(x,t) = \int_0^x 2 e e_x dx \text{ for a.e.} U_T, \tag{44}
$$

and by Hölder inequality

$$
e^2(x,t) \leq 2 \left\| e \right\|_{L^2(U)} \left\| e_x \right\|_{L^2(U)}. \tag{45}
$$

Since by Theorem 2, $\left\| e_x \right\|_{L^2(U)}$ is bounded (actually goes to zero) for a.e. t, L^2 convergence implies convergence almost everywhere. ■

Remark 5: For $U \subseteq R^d$, $d \geq 2$, establishing the a.e. convergence of the tracking error from L^2 convergence may requires more regularity of the solution and a different control scheme that guarantees L^2 convergence of the higher order spatial derivatives of tracking error.

III. Simulation

A. Parabolic case

The proposed control algorithm is simulated on the system (1) and the model (2) defined earlier with the following parameters. Plant and model parameters are defined for the domain $U = (0,1)$ as

$$
\begin{aligned} a &= 0.1 + 0.2\sin(\pi x), \qquad a_1 = 0.5, \\ b &= -0.5 - 0.4\sin(\pi x), \qquad b_1 = -0.1. \end{aligned} \tag{46}
$$

Initial conditions are

$$
g(x) = \sin(\pi x),\ g_1(x) = 2\sin(\pi x), \tag{47}
$$

with the homogeneous Dirichlet boundary and the reference input

$$
r(x,t) = 20(\sin(2\pi x) + \sin(\pi x))(1 + \sin(\pi t)). \tag{48}
$$

Fig. 3 shows asymptotic tracking when controller (17) is used with the adaptive algorithm (18). Initial controller parameters are defined as $\eta_{a1}(x,0) = -1, \eta_{a2}(x,0) = -1, \eta_b(x,0) = 0$.

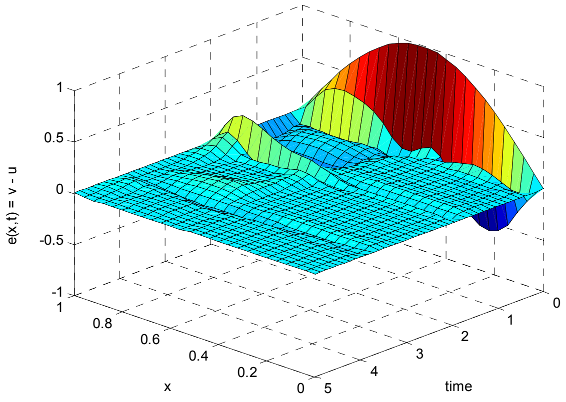

Fig. 3. State error when adaptive control (17) is used.

B. Hyperbolic case

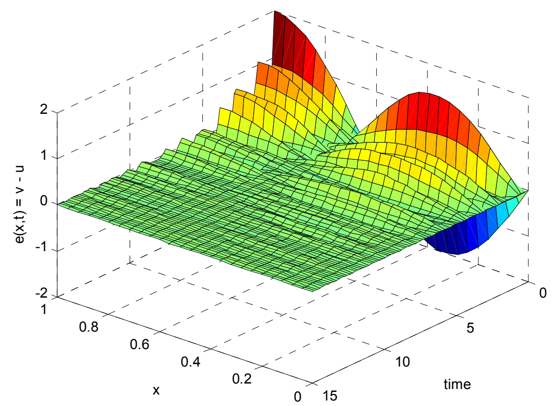

Fig. 4. State error for the hyperbolic case with mixed boundary conditions.

The control algorithm for hyperbolic case is tested on the system (3) and the model (4) with the modified boundary conditions. The plant and the model parameters are defined for the domain $U = (0, 1)$ as

$$\begin{aligned} a &= a_1 = 1, \ c = c_1 = 0. \\ b &= -1.2 - \sin(2\pi x), \qquad b_1 = -1.2, \end{aligned} \tag{49}$$

Initial conditions are

$$\begin{aligned} u(0,t) &= v(0,t) = 0, \ u_x(1,t) = v_x(1,t) = 0, \\ g(x) &= \sin(\frac{3\pi x}{2}), \ g_1(x) = -\sin(\frac{3\pi x}{2}), \\ h(x) &= h_1(x) = 0, \end{aligned} \tag{50}$$

with homogeneous Dirichlet boundary at $x = 0$, Neumann - at $x = 1$, reference input

$$r(x,t) = 15\sin(6\pi t)\sin(4\pi x), \tag{51}$$

and zero initial controller parameters. Asymptotic tracking is shown in Fig. 4 when (33) and (34) are used.

IV. Conclusion

It is shown that the use of the feedback signals u_{xx} in MRAC of DPS can result in the closed loop system instability. The latter could be avoided in this case only through utilizing the prior knowledge about the plant parameters to set the controller parameters close to the perfect tracking case. This, however, significantly limits the applicability of such control laws. To address this problem, MRAC algorithms that do not use spatial derivative of plant state have been proposed for parabolic (1) and hyperbolic (3) plants and demonstrated to provide stability under arbitrary initial controller parameter setting. It has been also shown that Neumann and mixed boundary conditions do not pose any additional problems for the algorithms proposed.

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