Proceedings of the
44th IEEE Conference on Decision and Control, and
the European Control Conference 2005
Seville, Spain, December 12-15, 2005

Oscillatority Conditions for Nonlinear Systems with Delay

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Abstract— **Sufficient conditions for oscillatority in the sense of Yakubovich of a class of time delay nonlinear systems are proposed. Under proposed conditions upper and lower bounds for oscillation amplitude are given. Examples illustrating analytical results by computer simulation are presented for a model of testosterone dynamics and circadian oscillations.**

I. Introduction

MOST works on analysis or synthesis of nonlinear systems are devoted to studying stability-like behavior. Their typical results show that the motions of a system are close to a certain limit motion (limit mode) that either may exist in the system or may be created by a controller. Evaluating deflection of the system trajectory from a limit mode, one may obtain quantitative information about system behavior [9], [14].

During recent years an interest in studying more complex behavior of the systems related to oscillatory and chaotic modes has grown significantly [3], [5], [6], [10], [12]. An important and useful concept for studying irregular oscillations is that of "oscillatority" introduced by V.A.Yakubovich in 1973 [15]. Frequency domain conditions for oscillatority were obtained for Lurie systems, composed of linear and nonlinear parts [10], [15], [16]. However, when studying physical systems in many cases it is more natural to decompose the system description into two nonlinear parts. Extension of analysis and design methods for oscillations in such a class of systems was proposed in [3]. However, the presence of time delays in a system often leads to rise of oscillation [8], [13].

In this paper the proposed in [3] conditions of oscillatority are extended to nonlinear systems with time delay. Section 2 contains some useful auxiliary statements and definitions. Main definitions and oscillations existence conditions are presented in Section 3. Section 4 deals with examples of analytical calculations with computer simulations of proposed solutions for a model of testosterone dynamics and circadian oscillations.

II. Preliminaries

As usual, continuous function $\sigma: R_+ \to R_+$ ($R_+ = \{\tau \in R : \tau \geq 0\}$) is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\sigma(0) = 0$. It is said to belong to class $\mathcal{K}_\infty$ if it is additionally radially unbounded.

We will denote by $C^n[-\tau,0]$, $\tau > 0$ the Banach space of continuous functions $\varphi:[-\tau,0] \to R^n$ with the norm $\|\varphi\| = \sup_{-\tau\leq\varsigma\leq 0} |\varphi(\varsigma)|$, where $|\cdot|$ is the standard Euclidean norm. We will denote as $\mathcal{M}_{R^m}$ the set of all Lebesgue measurable functions $\mathbf{u}: R_+ \to R^m$ with property $\|\mathbf{u}\| < +\infty$, where

$$\|\mathbf{u}\| = \|\mathbf{u}\|_{[0,+\infty)},$$

$$\|\mathbf{u}\|_{[t_0,T)} = ess\sup\{|\mathbf{u}(t)|,\ t \in [t_0,T)\}.$$

Let model of system be described by functional differential equation:

$$d\,\mathbf{x}(t)/dt = \mathbf{f}(t, \mathbf{x}_\tau(t), \mathbf{u}(t)),\ t \geq 0 \qquad (1)$$

where $\mathbf{x} \in R^n$ is state vector, $\mathbf{x}_\tau \in C^n[-\tau,0]$; $\mathbf{u} \in \mathcal{M}_{R^m}$ is input vector; $\mathbf{f}: R_+ \times C^n[-\tau,0] \times R^m \to R^n$ is continuous with respect to the first argument and locally Lipschitz continuous function with respect to the rest ones, $\mathbf{f}(\cdot,0,0) = 0$. We will assume that all solutions of the system satisfy initial conditions

$$\mathbf{x}_\tau(0) = \mathbf{x}_0 \in C^n[-\tau,0].$$

It is well known by the fundamental theory of functional differential equations [2], that system (1) has a unique solution $\mathbf{x}(\mathbf{x}_0,\mathbf{u},t)$ satisfying initial condition $\mathbf{x}_0$, which is defined on some finite interval $[0,T)$. If $T = +\infty$ for every initial state $\mathbf{x}_0$ and $\mathbf{u} \in \mathcal{M}_{R^m}$, then system is called forward complete.

Lemma 1 [2]. *Let there exist functional* $V: R_+ \times C^n[-\tau,0] \to R_+$, *continuous with respect to the first argument and locally Lipschitz continuous with respect to the second one, and functions* $\alpha_i \in \mathcal{K}_\infty$, $i = \overline{1,4}$, *such, that*

$$\alpha_1(|\mathbf{x}(t)|) \leq V(t, \mathbf{x}_\tau(t)) \leq \alpha_2(|\mathbf{x}(t)|) + \alpha_3\left(\int_{t-\tau}^{t} \alpha_4(|\mathbf{x}(\varsigma)|)\,d\varsigma\right),$$

$$\partial V/\partial t + \partial V/\partial \mathbf{x}_\tau\, \mathbf{f}(t, \mathbf{x}_\tau(t), 0) \leq -\alpha_4(|\mathbf{x}(t)|) + M$$

for all $t \geq 0$ *and* $\mathbf{x}_0 \in C^n[-\tau,0]$, $M > 0$. *Then all solutions* $\mathbf{x}(\mathbf{x}_0,0,t)$ *of system (1) are uniformly bounded, i.e.*

$$|\mathbf{x}(\mathbf{x}_0,0,t)| \leq \alpha_1^{-1}\left(\alpha_2(B) + \alpha_3(\tau\alpha_4(B))\right),\ t \geq 0,$$

This work is partly supported by grant 05-01-00869 of Russian Foundation for Basic Research, by Russian Science Support Foundation and by Program of Presidium of Russian Academy of Science № 19.

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$B=\max\{|\mathbf{x}_0|,\alpha_4^{-1}(M)\}$, *and uniformly asymptotically bounded* ($R=\alpha_1^{-1}\left(\alpha_2(\alpha_4^{-1}(M))+\alpha_3(\tau M)\right)$):

$$\lim_{t\to+\infty}|\mathbf{x}(\mathbf{x}_0,0,t)|\le R. \quad \square$$

III. Oscillatority Conditions

At first, extending the result of [3] we give a precise definition of the term "oscillatority" for time delay systems.

Definition 1. *Solution* $\mathbf{x}(\mathbf{x}_0,0,t)$ *with* $\mathbf{x}_0\in C^n[-\tau,0]$ *of system (1) is called* $[\pi^-,\pi^+]$*-oscillation with respect to output* $\psi=\eta(\mathbf{x})$ *(where* $\eta:R^n\to R$ *is a continuous function) if system (1) is forward complete and*

$$\underline{\lim}_{t\to+\infty}\psi(t)=\pi^-;\ \overline{\lim}_{t\to+\infty}\psi(t)=\pi^+;\ -\infty<\pi^-<\pi^+<+\infty.$$

Solution $\mathbf{x}(\mathbf{x}_0,0,t)$ *is called* ***oscillating****, if there exist some output* ψ *and constants* π^-, π^+ *such, that* $\mathbf{x}(\mathbf{x}_0,0,t)$ *is* $[\pi^-,\pi^+]$*-oscillation with respect to the output* ψ. *System (1) with* $\mathbf{u}(t)\equiv 0$, $t\ge 0$ *is called* ***oscillatory****, if for almost all* $\mathbf{x}_0\in C^n[-\tau,0]$ *the solutions of the system* $\mathbf{x}(\mathbf{x}_0,0,t)$ *are oscillating.* □

Note that term "almost all solutions" is used to emphasize that generically system (1) has nonempty set of equilibrium points, thus, there exists a set of initial conditions with zero Lebesgue measure such, that the corresponding solutions are not oscillation.

The oscillatority property introduced in Definition 1 is defined for zero input and any initial conditions of system (1). The following property is a closely related characterization of the system behavior, extending the proposed above property to the case of non zero input and specific initial conditions [3], [6].

Definition 2. *Let* $\mathbf{u}\in\mathcal{M}_{R^m}$ *and* $\mathbf{x}_0\in C^n[-\tau,0]$ *be given such, that* $\mathbf{x}(\mathbf{x}_0,\mathbf{u},t)$ *is defined for all* $t\ge 0$. *The functions* $\chi^-_{\psi,\mathbf{x}_0}(\gamma)$, $\chi^+_{\psi,\mathbf{x}_0}(\gamma)$ *defined for* $0\le\gamma<+\infty$ *are called a lower and upper excitation indices of system (1) in point* $\mathbf{x}_0$ *with respect to output* $\psi=\eta(\mathbf{x})$ *(where* $\eta:R^n\to R$ *is a continuous function), if*

$$\left(\chi^-_{\psi,\mathbf{x}_0}(\gamma),\chi^+_{\psi,\mathbf{x}_0}(\gamma)\right)=\arg\sup_{(a,b)\in\mathcal{E}(\gamma)}\{b-a\}$$

$$\mathcal{E}(\gamma)=\left\{(a,b):\left(\begin{array}{l} a=\underline{\lim}_{t\to+\infty}\eta\left(\mathbf{x}(\mathbf{x}_0,\mathbf{u},t)\right),\\ b=\overline{\lim}_{t\to+\infty}\eta\left(\mathbf{x}(\mathbf{x}_0,\mathbf{u},t)\right)\end{array}\right)\right\}_{\|\mathbf{u}\|\le\gamma}.$$

Lower and upper excitation indices with respect to output ψ *for forward complete system (1) are defined as follows*

$$\chi^-_\psi(\gamma)=\inf_{\mathbf{x}_0\in C^n[-\tau,0]}\chi^-_{\psi,\mathbf{x}_0}(\gamma),$$

$$\chi^+_\psi(\gamma)=\sup_{\mathbf{x}_0\in C^n[-\tau,0]}\chi^+_{\psi,\mathbf{x}_0}(\gamma). \quad \square$$

Excitation indices characterize abilities of system (1) to perform forced or controllable oscillations caused by bounded inputs. It is clear that relations $\pi^-=\chi^-_\psi(0)$ and $\pi^+=\chi^+_\psi(0)$ are satisfied. For non zero inputs indices characterize maximum (over specified set of inputs $\|\mathbf{u}\|\le\gamma$) asymptotic amplitudes $\chi^+_\psi(\gamma)-\chi^-_\psi(\gamma)$ of ψ.

The sufficient conditions for oscillatority of system (1) are formulated in the following theorem.

Theorem 1. *Let system (1) have Lyapunov functional* $V:R_+\times C^n[-\tau,0]\to R_+$ *obeying inequalities from Lemma 1 and the origin of the system be locally unstable with region of repulsion* $|\mathbf{x}|\le r$, $0<r<R$. *Then system (1) is oscillatory, provided that set* $\Omega=\{\mathbf{x}:r\le|\mathbf{x}|\le R\}$ *does not contain equilibrium points of system (1) for* $\mathbf{u}(t)\equiv 0$, $t\ge 0$.

Proof. By Lemma 1 all solutions of system (1) with $\mathbf{u}(t)\equiv 0$, $t\ge 0$ are bounded and asymptotically converge to region where $|\mathbf{x}|\le R$. But inside the above set there exists a repulsive subset containing the origin, therefore set Ω is the global attractor in this case.

As it was supposed, Ω does not contain equilibrium points of the system. Then for almost all $\mathbf{x}_0\in C^n[-\tau,0]$ there exists an index i, $1\le i\le n$ such, that the solution is $[\pi^-,\pi^+]$-oscillation with respect to output $|x_i|$ with $0\le\pi^-<\pi^+<R$. Suppose that there is no such an output. It means that for all $1\le i\le n$ for every output $|x_i|$ the equality $\pi^-=\pi^+$ holds. However, the latter could be true only in equilibrium points, which are excluded from the set Ω by the theorem conditions. Therefore, for almost all initial conditions solutions have such oscillating outputs and system (1) is oscillatory by Definition 1. Note, that for different $\mathbf{x}_0\in C^n[-\tau,0]$ oscillations of the outputs $|x_i|$ are possible for different i, $1\le i\le n$. ■

Conditions of the above theorem are rather general and define the class of systems, which oscillatory behavior can be investigated by the approach. The systems should have attracting compact set in state space, which contains oscillatory movements of the systems. For such systems Theorem 1 provides useful tool for testing oscillating behavior and obtaining estimates of oscillations amplitude.

The Poincaré-Bendixson Theorem [11] provides another method to detect more stronger oscillating behavior in the system, like presence of limit cycles. However, Poincaré-Bendixson Theorem imposes additional restrictions on structure properties of system (1) and does not allow to investigate behavior of chaotic systems.

Remark 1. Note, that set Ω determines lower bound for value of π^- and upper bound for values of π^+ and $\pi^+ - \pi^- \le n R - r$. □

Remark 2. Like in [16] one can use linearization near the origin of system (1) to prove local instability of the system solutions. Instead of existence of Lyapunov functional V one can require just boundedness of the system solution $\mathbf{x}(t)$ with known upper bound obtained using an other approach not dealing with time derivative of Lyapunov functional analysis. □

Let us show a link between oscillatority and excitation indices.

Corollary 1. *Let for almost all initial conditions* $\mathbf{x}_0 \in C^n[-\tau, 0]$ *solutions* $\mathbf{x}(\mathbf{x}_0, \mathbf{k}(\mathbf{x}), t)$ *of the system (1) with control* $\mathbf{u} = \mathbf{k}(\mathbf{x})$, $\mathbf{k}(0) = 0$ *be* $[\pi^-, \pi^+]$*-oscillations with respect to output* $\psi = \eta(\mathbf{x})$:

$$\kappa_1(|\mathbf{x}|) \le \eta(\mathbf{x}) \le \kappa_2(|\mathbf{x}|),\ \mathbf{x} \in R^n,\ \kappa_1, \kappa_2 \in \mathcal{K}_\infty .$$

Then oscillation amplitude of system (1) adnits inequality

$$\pi^+ - \pi^- \le \chi_\psi^+(\gamma) - \chi_\psi^-(\gamma),$$

for $\gamma \ge \gamma^*$, *where* $\gamma^* = \sup_{\mathbf{x} \in \tilde{\Omega}} |\mathbf{k}(\mathbf{x})|$,

$$\tilde{\Omega} = \{\mathbf{x} : \kappa_2^{-1}(\pi^-) \le |\mathbf{x}| \le \kappa_1^{-1}(\pi^+)\} .$$

Proof. From oscillatority property with respect to output ψ solutions of the closed by feedback $\mathbf{k}$ system (1) are bounded for all $\mathbf{x}_0 \in C^n[-\tau, 0]$ (almost all solutions are oscillating, while others are equilibriums):

$$|\mathbf{x}(t)| \le P,\ P > 0,\ t \ge 0 .$$

Therefore input $\mathbf{u} = \mathbf{k}(\mathbf{x})$ is upper bounded by

$$\gamma = \sup_{|\mathbf{x}| \le P} |\mathbf{k}(\mathbf{x})|$$

and $\pi^+ - \pi^- \le \chi_\psi^+(\gamma) - \chi_\psi^-(\gamma)$. Here P is some positive constant calculated along solutions of closed loop system. Also solutions asymptotically converge to set $\tilde{\Omega}$ (that assumed to be non empty), where norm of control $\mathbf{k}$ is upper bounded by γ^*. Therefore, the statement follows from definitions 1 and 2 (excitation indices are not decreasing functions of γ). ■

Hence, to compute estimates of excitation indices it is enough to find some control $\mathbf{k}$ for system (1), which ensures oscillations existence in closed loop system.

IV. Applications

A. Delayed model of testosterone dynamics

Let us consider the following model of testosterone dynamics [13]:

$$\dot{R} = f\big(T(t-\tau_1)\big) - b_1 R;$$
$$\dot{L} = g_1 R - b_2 L;$$
$$\dot{T} = g_2 L(t-\tau_2) - b_3 T,$$

where L is luteinising hormone concentration; R is luteinising hormone releasing hormone concentration; T is concentration of testosterone in the blood; b_1, b_2, b_3, g_1 and g_2 are from R_+; $f : R_+ \to R_+$ is differentiable, bounded from above and monotone decreasing (during computer simulation we will use $f(T) = A/(K + T^2)$); τ_1 and τ_2 are non negative time delays. It is assumed, that the presence of R in the blood induces the secretion of L, which induces testosterone to be secreted in the testes. The testosterone in turn causes a negative feedback effect on the secretion of R. As it was proposed in [13] the presence of delay in this stable model leads to oscillations arising, see also [4] for additional results in this field. Let us apply proposed approach to the above system.

This model for monotone decreasing positive f has one unique equilibrium (R_0, L_0, T_0) being the solution of equations:

$$f(T_0) = \frac{b_1 b_2 b_3}{g_1 g_2} T_0;\ R_0 = \frac{b_2 b_3}{g_1 g_2} T_0;\ L_0 = \frac{b_3}{g_2} T_0 .$$

The instability property can be established based on linearization of testosterone model near the equilibrium:

$$\delta\dot{R} = f'(T_0)\delta T(t-\tau_1) - b_1 \delta R;$$
$$\delta\dot{L} = g_1 \delta R - b_2 \delta L;$$
$$\delta\dot{T} = g_2 \delta L(t-\tau_2) - b_3 \delta T,$$

where δR, δL and δT are deviations of R, L and T from the equilibrium respectively, f' derivative of f. The characteristic polynomial has form

$$(s + b_1)(s + b_2)(s + b_3) - g_1 g_2 f'(T_0) e^{-(\tau_1+\tau_2)s} = 0 .$$

For chosen during simulation parameters

$$A = 10,\ K = 2,\ b_1 = b_2 = b_3 = 1,\ g_1 = 10,\ g_2 = 50$$

the computation shows that for

$$\tau_1 + \tau_2 > 1.556$$

characteristic polynomial has roots with positive real parts. Thus, to apply Theorem 1 for the model we should find a Lyapunov functional satisfying Lemma 1, like this one:

$$V = 0.5 b_1^{-1}\left(a_R + b_2^{-2}(a_L + b_3^{-1} g_2^2) g_1^2\right) R^2 + 0.5 T^2 +$$
$$+ 0.5 b_2^{-1}(a_L + b_3^{-1} g_2^2) L^2 + 0.5 b_3^{-1} g_2^2 \int_{t-\tau_2}^{t} L(s)^2\, ds ,$$

where $a_R, a_L > 0$. Its time derivative for testosterone model admits upper estimate:

$$\dot{V} \le -0.5 a_R R^2 - 0.5 b_3 T^2 - 0.5 a_L L^2 + M ,$$

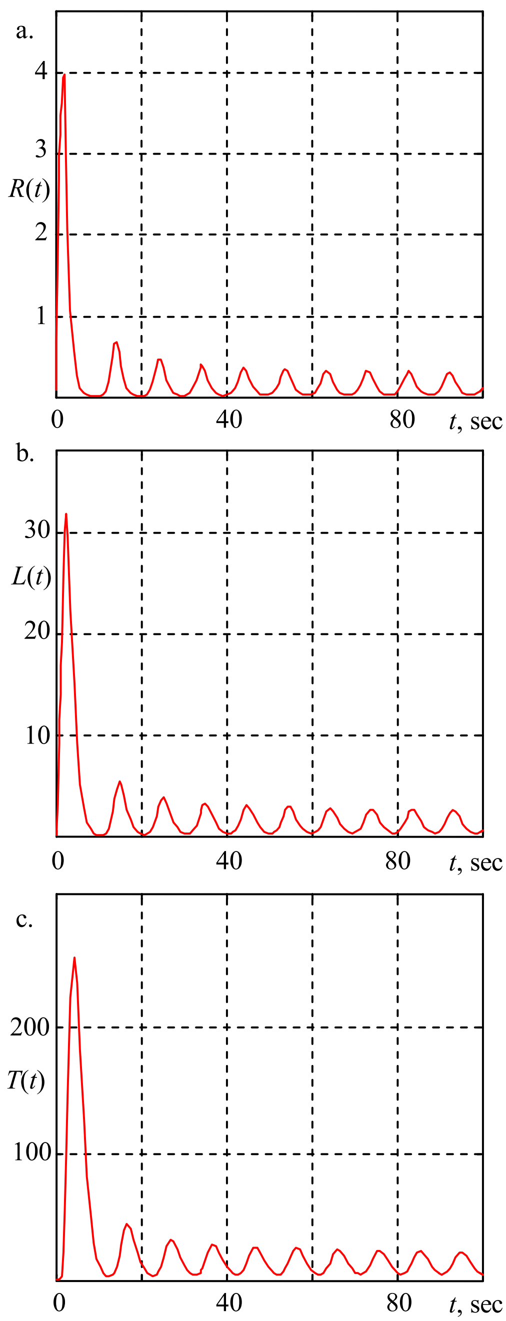

Fig. 1. Trajectories of testosterone model.

where $M = 0.5 b_1^{-2} \left(a_R + b_2^{-2} (a_L + b_3^{-1} g_2^2) g_1^2 \right) f_{\max}^2$ and $f_{\max} \geq f(T)$, $T \in R_+$. Therefore, system is oscillatory. It is worth to stress, that testosterone model with two time delays does not satisfy conditions of Poincaré-Bendixson Theorem [11]. The value of $\sqrt{M}$ serves as estimate on upper bounds of oscillation amplitude for state vector $\mathbf{x} = [R\, L\, T]^T$ (for $a_R = a_L = 1$, $\sqrt{M} \approx 1768$). The corresponding trajectory for $\tau_1 = \tau_2 = 1$ is presented in Fig. 1.

B. Circadian oscillations model

Let us consider circadian model from [1], [7] with time delays:

$$\dot{M} = v_s K_n / (K_n + P_N(t-\tau_1)^n) - v_m M / (k_m + M);$$
$$\dot{P}_0 = k_s M(t-\tau_2) - V_1 P_0/(K_1 + P_0) + V_2 P_1/(K_2 + P_1);$$
$$\dot{P}_1 = V_1 P_0/(K_1 + P_0) - V_2 P_1/(K_2 + P_1) - V_3 P_1/(K_3 + P_1) + V_4 P_2/(K_4 + P_2);$$
$$\dot{P}_2 = V_3 P_1/(K_3 + P_1) - V_4 P_2/(K_4 + P_2) - k_1 P_2 + k_2 P_N - v_d P_2/(k_d + P_2);$$
$$\dot{P}_N = k_1 P_2 - k_2 P_N,$$

where P_i, $i = \overline{0,2}$ are concentration degree of phosphorylation of PER protein; P_N indicates concentration of PER in nucleus; M is concentration of *per* mRNA; τ_1 and τ_2 are non negative time delays. The following values of all other parameters were chosen [1]:

$$V_1 = 3.2,\ V_2 = 1.58,\ V_3 = 5,\ V_4 = 2.5,\ k_1 = 1.9,$$
$$k_2 = 1.3,\ k_m = 0.5,\ k_d = 0.2,\ k_s = 0.38,\ v_s = 0.55,$$
$$v_d = 0.95,\ v_m = 0.65,\ n = 4,\ K_I = 1,$$
$$K_1 = K_2 = K_3 = K_4 = 2.$$

The description of functionality of above model can be found in [1] and [7]. In paper [1] it was mentioned without proof, that for $v_s = 0.5$ and for bigger values system with delays exhibits oscillations. Also it was proven in the paper [1] that this system has bounded solution (even with time delays) and unique equilibrium under some mild restrictions on values of model parameters (like chosen for computer simulation):

$$M^0 = 1.758,\ P_0^0 = 0.95,\ P_1^0 = 0.595,$$
$$P_2^0 = 0.474,\ P_N^0 = 0.693.$$

As it was mentioned in Remark 2, to establish global boundedness of the system trajectories it is possible to use any other approaches not dealing with Lyapunov functions analysis. For example, Proposition 3.1 and Theorem 1 from [1] help to establish global boundedness of the system trajectories here. So, to establish oscillatority property of the system we should investigate stability property of equilibrium. For linearized circadian model near the equilibrium the characteristic polynomial has form

$$s^5 + 7.559 s^4 + 15.484 s^3 + 9.226 s^2 + 1.463 s + 0.062 + 0.294 e^{-(\tau_1+\tau_2)s} = 0.$$

Computations show, that the equilibrium is unstable and oscillations exist for

$$\tau_1 + \tau_2 \geq 3.44.$$

For $\tau_1 = \tau_2 = 2$ the circadian system trajectory is shown in Fig. 2.

Note that the estimate $\tau_1 + \tau_2 \geq 3.44$ is better than the estimate $\tau_1 + \tau_2 = 100$ obtained by simulation in [1].

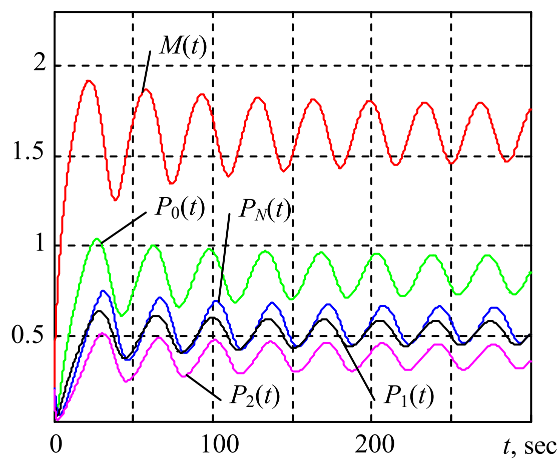

Fig. 2. Circadian model oscillations.

V. Conclusion

The paper presents definitions of oscillatority in the sense of Yakubovich and excitation indices for nonlinear dynamical systems with time delay (which models are described by functional differential equations). The sufficient conditions of oscillatority also are given as extension of result [3]. The good potentiality of proposed approach for detecting of oscillations arising and amplitudes bounds obtaining is demonstrated through examples of analytical design and computer simulation.

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